i.e.,

$$\sum_{k=1}^{N} g_{jk} A_k = E_j, \qquad j = 1, 2, \cdots N, \tag{5}$$

where

$$g_{jk} = (\phi_j, \phi_k) \tag{6}$$

and

$$E_i = (\phi_i, E). \tag{7}$$

One refers to the coefficient matrix in (5) as the Gram matrix formed from vectors $\phi_1, \phi_2, \cdots, \phi_N$. One denotes this matrix by

$$G(\phi_1, \phi_2, \cdots, \phi_N) = \{g_{jk}\} \tag{8}$$

where g_{jk}, the element in row j and column k of G, is defined in (6). Furthermore, the minimum value of Δ^2 in (3), i.e., the value of Δ^2 obtained when the A_k from (5) are substituted into (3), is given (see, for example, Achieser [5]) by the following ratio of determinants:

$$\Delta_{\min}^2 = \frac{\det G(\phi_1, \phi_2, \cdots, \phi_N, E)}{\det G(\phi_1, \phi_2, \cdots, \phi_N)}. \tag{9}$$

Since the parameters A_k do not occur in (9) one can now remove the restriction that the nonlinear parameters are known and select these parameters, for example by the method of steepest descent [6], so as to minimize the ratio of the determinants in (9). In view of (3) this minimum ratio is then the smallest least square error achievable in approximating

$$E(x) \text{ by } \sum_{k=1}^{N} A_k \phi_k.$$

In order to achieve this minimum, the A_k are selected as the solution of the linear equations (5).

If the experimental data varies in reliability one can take this into account by introducing weights $w_i > 0$, i.e., by replacing (1) by

$$(f, g) = \sum_{i=1}^{M} f^*(x_i) g(x_i) w_i. \tag{10}$$

Furthermore, if the experimental data is taken at all values of the independent variable x in an interval $a \geq x \geq b$, then the appropriate definition of the inner production is

$$(f, g) = \int_a^b f^*(x) g(x) w(x)\, dx \tag{11}$$

where $w(x) > 0$ is a weight function which accounts for the varying reliability of the data. Equations (2)–(9) are then also valid in terms of the more general definition of the inner production in (10) or (11).

Of course, in special cases the procedure can be simplified. For example, consider the approximation of $E(x)$ by $Ae^{-\alpha x}e^{-j\beta x}$ where the nonlinear parameters α and β are real. This can be rephrased as the approximation of $\log E(x)$ by $\log A - \alpha x - j\beta x$. One is then led to two independent approximation problems in each of which the parameters occur linearly. Hence in each of these problems the classical method of least squares embodied in (5) can be employed. The first problem is to approximate $\log |E|$ by $\log |A| - \alpha x$. This determines $\log |A|$ and α from the measured values of $|E|$ via (5). More precisely, since

$$\log |A| - \alpha x = A_1 \phi_1(x) + A_2 \phi_2(x) \tag{12}$$

where

$$\phi_1(x) = 1, \qquad \phi_2(x) = -x$$
$$A_1 = \log |A|, \qquad A_2 = \alpha.$$

Hence from (6)

$$g_{11} = M, \qquad g_{22} = \sum_{i=1}^{M} x_i^2$$

$$g_{12} = g_{21} = -\sum_{i=1}^{M} x_i$$

and from (7)

$$E_1 = \sum_{i=1}^{M} \log |E(x_i)|,$$

$$E_2 = -\sum_{i=1}^{M} x_i \log |E(x_i)|.$$

Define

$$\overline{f(x)} = \frac{1}{M} \sum_{i=1}^{M} f(x_i). \tag{13}$$

Then $\log |A|$ and α are determined from (5) which here become

$$\log |A| - \alpha \bar{x} = \overline{\log |E(x)|}$$
$$\bar{x} \log |A| - \alpha \overline{x^2} = \overline{x \log |E(x)|}. \tag{14}$$

The second problem is to approximate $\arg E(x)$ by $\arg A - \beta x$. This determines $\arg A$ and β from the measured values of $\arg E(x)$ via (5) in a manner analogous to (12) and (13). Unfortunately, a separate analysis of the data on amplitude $|E(x)|$ and phase $\arg E(x)$ is not in general possible when more than one mode is present. One is then forced to employ the analysis discussed above in connection with (9). When $|E(x)|$ alone is measured then an analysis of the data in a least square sense as in the above manner for the multimode case does not appear feasible.

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Impedance of a Spherical Probe in a Magnetoplasma

In a recent paper, Balmain [1] used quasi-static theory to compute the impedance of a short dipole in a magnetoplasma and also discussed the use of a scaling procedure which transforms the free space equations into magnetoplasma equations. It will be shown here how such a scaling procedure can be used to derive an approximate, closed-form expression for the impedance of a small, spherical probe immersed in a cold magnetoplasma.

The basic impedance formula may be set up using some results from elementary electrostatics. The energy necessary to assemble a charge Q on a conducting body with capacitance C is

$$W = \frac{1}{2} \frac{Q^2}{C}. \tag{1}$$

If the charge has density ρ and if the conductor potential is V, the energy is also given by

$$W = \frac{1}{2} \int \rho V\, dv. \tag{2}$$

Thus the input impedance of the conducting body for slowly varying sinusoidal fields may be expressed as

$$Z_{\text{in}} = \frac{1}{j\omega C} = \frac{1}{j\omega Q^2} \int \rho V\, dv. \tag{3}$$

For a spherical probe, the charge ρ is spread in a thin layer over the surface of the sphere. If the probe is small, the potential V may be approximated by the potential of a point charge of Q coulombs located at the center of the sphere.

The potential of a point charge in a magnetoplasma may be obtained by scaling. If primes are used to indicate free space coordinates an appropriate scaling is given by

$$x' = \sqrt{K_0 K'}\, x, \qquad y' = \sqrt{K_0 K'}\, y, \qquad z' = K' z. \tag{4}$$

The quantities K' and K_0 are the diagonal elements of the permittivity tensor and are given by

$$K' = 1 - \frac{XU}{U^2 - Y^2}, \qquad K_0 = 1 - \frac{X}{U} \tag{5}$$

in which

$$X = \omega_N^2/\omega^2, \qquad Y = \omega_H/\omega,$$
$$U = 1 - jZ = 1 - j\nu/\omega.$$

The quantities ω_N, ω_H, and ν are, respectively, the electron plasma, cyclotron, and collision frequencies. Scaling of the free space point charge potential may be carried out as follows:

$$V = \frac{Q}{4\pi\epsilon_0 r'} = \frac{Q}{4\pi\epsilon_0 \sqrt{x'^2 + y'^2 + z'^2}}$$
$$= \frac{Q}{4\pi\epsilon_0 \sqrt{K_0 K' x^2 + K_0 K' y^2 + K'^2 z^2}}. \tag{6}$$

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The following definitions,

$$x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi,$$

$$z = r\cos\theta, \qquad m^2 = 1 - \frac{K_0}{K'},$$

make it possible to express (6) as

$$V = \frac{Q}{4\pi\epsilon_0 rK'\sqrt{1 - m^2\sin^2\theta}} \tag{7}$$

which is the required potential of a point charge in a magnetoplasma.

In order to calculate the impedance, the charge density ρ must be determined. The total charge of Q coulombs is spread in a thin layer over the surface of the sphere which has radius R. If the actual charge distribution is approximated by a uniform distribution, then the charge density may be expressed as

$$\rho = \frac{Q}{4\pi R^2}\delta(r - R) \tag{8}$$

in which δ is the Dirac delta. Substitution of (7) and (8) into (3) permits the volume integration to be carried out, giving

$$j8\pi\epsilon\omega_0 K'RZ_{\text{in}} = \int_0^\pi \frac{\sin\theta\, d\theta}{\sqrt{1 - m^2\sin^2\theta}} = \left[-\frac{1}{m}\ln\{m\cos\theta + \sqrt{1 - m^2\sin^2\theta}\}\right]_0^\pi.$$

Thus the impedance formula reduces to

$$Z_{\text{in}} = \frac{1}{j\omega 8\pi\epsilon_0 RK'm}\ln\frac{1+m}{1-m} \tag{9}$$

in which

$$m = \sqrt{1 - \frac{K_0}{K'}}.$$

In the above expression, the logarithm must be calculated using the formula

$$\ln w = \ln|w| + j\arg w$$

in which

$$-\pi < \arg w < \pi.$$

The influence of an ion sheath may be estimated by representing it as a free space gap of thickness S separating the probe from the uniform plasma. Thus if the probe radius is R, the radius of the sheath edge is $R+S$. Under these conditions the probe impedance consists of the sheath impedance in series with the plasma impedance, the latter being approximated by (9) with R replaced by $R+S$. If the quantity T is defined as

$$T = \frac{1}{2K'm}\ln\frac{1+m}{1-m},$$

then the probe impedance with an ion sheath present is given by

$$Z_{\text{in}} = \frac{1}{j\omega 4\pi\epsilon_0}\left(\frac{1}{R} - \frac{1}{R+S}\right) + \frac{T}{j\omega 4\pi\epsilon_0(R+S)} = \frac{1}{j\omega 4\pi\epsilon_0 R}\cdot\frac{S+RT}{R+S}. \tag{10}$$

Under lossless conditions ($Z\to 0$) the above formula has a positive real part when K_0/K' is negative. This anomalous resistance has been noticed by Kaiser [2] and Balmain [1] in their studies of dipole impedance and it arises whenever the quasi-static differential equation is hyperbolic. The impedance is entirely reactive when the differential equation is elliptic, that is, when K_0/K' is positive. Under hyperbolic conditions the anomalous resistance arises from the imaginary part of the logarithm whose sign must be determined by evaluating the logarithm for a small value of Z and then taking the limit as $Z\to 0$.

Another probe effect of interest is the "resonance rectification" effect in which an RF probe exhibits a direct current peak at a frequency which for isotropic plasmas is below the plasma frequency. It is believed that the peak in direct current occurs near the minimum in RF impedance. This minimum may be regarded as a "series resonance" due to the series connection of a capacitive sheath region and an inductive plasma region (refer to Crawford [3] and Dote and Ichimiya [4] for further discussion). Inspection of (10) indicates that, under lossless conditions, a series resonance (zero in impedance) can occur if T is negative. The factor T is dominated by K' which is negative in the frequency range

$$\omega_H < \omega < \sqrt{\omega_H^2 + \omega_H^2}. \tag{11}$$

Thus rectified current peaks are to be expected mainly within this frequency range.

It must be emphasized that the spherical probe model assumed is highly idealized. In practice the fields would be distorted by the presence of the connecting wires and the nearby reference electrode. In addition, the sheath-plasma interface would not be spherical but would be distorted by the magnetic field. Another important factor is the size of the probe; its radius would have to be much larger than a Debye length to avoid effects arising from the nonzero temperature of the plasma [5].

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On the Variability of Microwave Transmission Time over Tropospheric Paths

A preliminary study of the variability of transmission time for microwaves propagated over fixed tropospheric paths at White Sands Missile Range (WSMR), New Mexico, was recently completed. This investigation marks the first serious attempt at WSMR to examine the detailed structure of tropospherically induced variations in radio-distance measurements over long paths at low elevation angles. A knowledge of the statistics of these variations is important in examining refraction errors in data from phase measuring tracking systems, in developing methods for the minimization of these errors, and in examining the uncertainties in phase references (used in bistatic tracking systems) resulting from their transmission over tropospheric links.

Under this study, observations of radio range variations were made with the Electrotape Model DM-20,[1] a surveying instrument built by the Cubic Corporation, modified to permit the continuous recording of the phase of the 7.492427 Mc/s ranging frequency used to frequency modulate one of the nine selectable X-band carriers. A pair of the instruments were operated sequentially over three paths of lengths 38.45 km, 50.54 km, and 57.03 km, having a common lower terminus and elevation angles of 2.4, 31.7, and 28.2 milliradians, respectively. Data were recorded during the spring months of 1964 for periods ranging from one to five hours.

The analysis of the data was carried out by the Central Radio Propagation Laboratory (CRPL) of the National Bureau of Standards (NBS), Boulder, Colo., following the procedure outlined below. Recent changes, not described here, have been made to this procedure and include prewhitening of the data and an improved method for obtaining smoothed estimates of the power spectral density function.[2] n equally spaced readings are made of a continuous recording of the phase variations (from an arbitrary zero reference) of the ranging signal over the two-way path, and are used to compute the autocovariance function according to

$$R(\tau) = \frac{1}{n-k}\sum_{i=1}^{i=n-k}[\Delta R(t_i+\tau) - \overline{\Delta R}]\cdot[\Delta R(t_i) - \overline{\Delta R}] \tag{1}$$

where $\tau = k\delta (k=0, 1, \cdots, m)$, δ = sampling interval (in seconds), m = number of time displacements (or lags), and

$$\overline{\Delta R} = \frac{1}{n}\sum_{i=1}^{i=n}\Delta R(t_i).$$

The power spectral density function is then

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[1] This publication in no way constitutes an endorsement by the U. S. Government of the instrument capability claimed by the manufacturer.

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