ECE424F MICROWAVES

Homework #2

1. A voltage generator V_g has an internal impedance $Z_g = R_g + jX_g$ and provides power to a lossless transmission line of characteristic impedance Z_o and propagation constant β. The line has a length l and is terminated to a load impedance Z_L.

a.) Prove that the real power delivered to the load impedance can be expressed as:

$$P = P_A \frac{\left(1-|\Gamma_G|^2\right)\left(1-|\Gamma_L|^2\right)}{\left|1-\Gamma_L\Gamma_G e^{-2j\beta l}\right|^2}$$

where $P_A = |V_g|^2 / (8R_g)$ is the available power from the source and Γ_G, Γ_L are the reflection coefficients at the source and load respectively.

b.) Deduce an expression for the delivered power when the load is matched to the line.

c.) Deduce an expression for the delivered power when the generator is matched to the line.

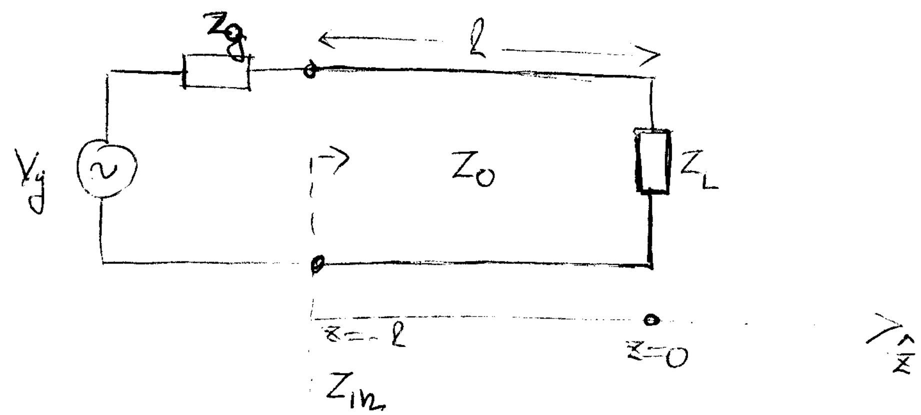
d.) Starting from the expression found in (a) above, prove that in the case of a conjugate matched line $P = P_A$.

2. Problem 2.13 in textbook.

3. Problem 2.22 in textbook.

4. An air filled transmission line 250m long, operating at 2.86 MHz, is terminated to a load impedance of 200Ω. The line characteristics are $Z_o = 300\Omega$ and $\alpha = 4\times10^{-4}$ Np/m. The line is fed at its input by a voltage generator $V_g = 30\text{V}$. Compute the power delivered to the loaded line at input, the power delivered to the load, the power lost in the line and the power reflected from the load.

Problem ① :



(a)

From the input voltage divider, $V(-l) = \dfrac{V_g Z_{in}}{Z_{in} + Z_g}$ . . . (1)

But $V(-l) = V_0^+ e^{j\beta l}\left(1 + \Gamma_L e^{-2j\beta l}\right)$ . . . (2)

- Therefore from (1), (2) : $V_0^+ = \dfrac{V(-l)\, e^{-j\beta l}}{\left(1 + \Gamma_L e^{-2j\beta l}\right)} = V_g \dfrac{Z_{in}}{Z_{in} + Z_g} \dfrac{e^{-j\beta l}}{\left(1 + \Gamma_L e^{-2j\beta l}\right)}$

Consider, $\dfrac{Z_{in}}{Z_{in} + Z_g} = \dfrac{1}{1 + Z_g/Z_{in}}$ . . . (3)

where $Z_{in} = Z_0 \dfrac{1 + \Gamma_L e^{-2j\beta l}}{1 - \Gamma_L e^{-2j\beta l}}$ (4)

Hence $1 + \dfrac{Z_g}{Z_{in}} = 1 + \dfrac{Z_g}{Z_0} \dfrac{\left(1 - \Gamma_L e^{-2j\beta l}\right)}{\left(1 + \Gamma_L e^{-2j\beta l}\right)} = \dfrac{Z_0\left(1 + \Gamma_L e^{-2j\beta l}\right) + Z_g\left(1 - \Gamma_L e^{-2j\beta l}\right)}{Z_0\left(1 + \Gamma_L e^{-2j\beta l}\right)}$ (5)

So, $V_0^+ = V_g e^{-j\beta l} \dfrac{Z_0 \cancel{\left(1 + \Gamma_L e^{-2j\beta l}\right)}}{Z_0\left(1 + \Gamma_L e^{-2j\beta l}\right) + Z_g\left(1 - \Gamma_L e^{-2j\beta l}\right)} \dfrac{1}{\cancel{\left(1 + \Gamma_L e^{-2j\beta l}\right)}}$

Finally $$V_0^+ = V_g \frac{Z_0 e^{-j\beta l}}{(Z_0+Z_g) - \Gamma_L e^{-2j\beta l}(Z_g - Z_0)} = \frac{V_g Z_0}{(Z_0+Z_g)} \frac{e^{-j\beta l}}{(1-\Gamma_L\Gamma_G e^{-2j\beta l})} \quad \text{- - - - (6)}$$

where $\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0}$, $\Gamma_G = \frac{Z_g - Z_0}{Z_g + Z_0}$

Power delivered to the load:

$$P = \frac{|V_0^+|^2}{2Z_0}\left(1 - |\Gamma_L|^2\right) \quad \text{- - - - - (7)}$$

From (6), (7) :
$$P = |V_g|^2 \frac{Z_0}{2|Z_0+Z_g|^2} \frac{\left(1-|\Gamma_L|^2\right)}{|1-\Gamma_L\Gamma_G e^{-2j\beta l}|^2} \quad (8)$$

Consider, $$1 - |\Gamma_G|^2 = \mathrm{Re}\left\{(1-\Gamma_G)(1+\Gamma_G^*)\right\} \quad \text{- - - - - (9)}$$

But $$1-\Gamma_G = 1 - \frac{Z_g - Z_0}{Z_g + Z_0} = \frac{2Z_0}{Z_g+Z_0}$$

and $$1+\Gamma_G = 1 + \frac{Z_g - Z_0}{Z_g + Z_0} = \frac{2Z_g}{Z_g+Z_0}$$

Hence (9) yields : $$1-|\Gamma_G|^2 = \mathrm{Re}\left\{\frac{2Z_0}{Z_g+Z_0}\frac{2Z_g^*}{(Z_g+Z_0)^*}\right\} = \frac{4Z_0}{|Z_g+Z_0|^2}\mathrm{Re}(Z_g)$$

i.e., $$\frac{1}{|Z_0+Z_g|^2} = \left(1-|\Gamma_G|^2\right)\frac{1}{Z_0}\frac{1}{4R_g} \quad \text{- - - (10)}$$

Finally (8), (10) give the required expression:

$$\boxed{P = P_A \frac{(1-|\Gamma_G|^2)(1-|\Gamma_L|^2)}{|1-\Gamma_L\Gamma_G e^{-2j\beta l}|^2}}, \quad P_A = \frac{1}{8}\frac{|V_g|^2}{R_g}$$

(b) When the load is matched to the line, $\Gamma_L = 0 \Rightarrow$

$$P = P_A \left(1 - |\Gamma_G|^2\right)$$

I.e, there is power reflected at the generator given by $P_A |\Gamma_G|^2$. This does not allow for maximum power transfer.

(c) When the generator is matched to the line, $\Gamma_G = 0 \Rightarrow$

$$P = P_A \left(1 - |\Gamma_L|^2\right)$$

This time there is power reflected at the load: $P_A |\Gamma_L|^2$.

(d) For a conjugate-matched line $Z_{in} = Z_g^*$.

In this case, $\Gamma(\ell) = \Gamma_L e^{-2j\beta\ell} = \dfrac{Z_{in} - Z_0}{Z_{in} + Z_0} = \dfrac{Z_g^* - Z_0}{Z_g^* + Z_0} = \Gamma_G^*$

Hence, $|\Gamma_L| = |\Gamma_G^*| = |\Gamma_G| = |\Gamma(\ell)|$

Also, $|1 - \Gamma_G \Gamma_L e^{-2j\beta\ell}| = |1 - \Gamma_G \Gamma(\ell)| = |1 - \Gamma_G \Gamma_G^*| = \left(1 - |\Gamma_G|^2\right)$

Hence $P = P_A \dfrac{\left(1 - |\Gamma_G|^2\right)\left(1 - |\Gamma_L|^2\right)}{|1 - \Gamma_L \Gamma_G e^{-2j\beta\ell}|^2} = P_A \dfrac{\cancel{\left(1 - |\Gamma_G|^2\right)\left(1 - |\Gamma_G|^2\right)}}{\cancel{\left(1 - |\Gamma_G|^2\right)^2}} = P_A$

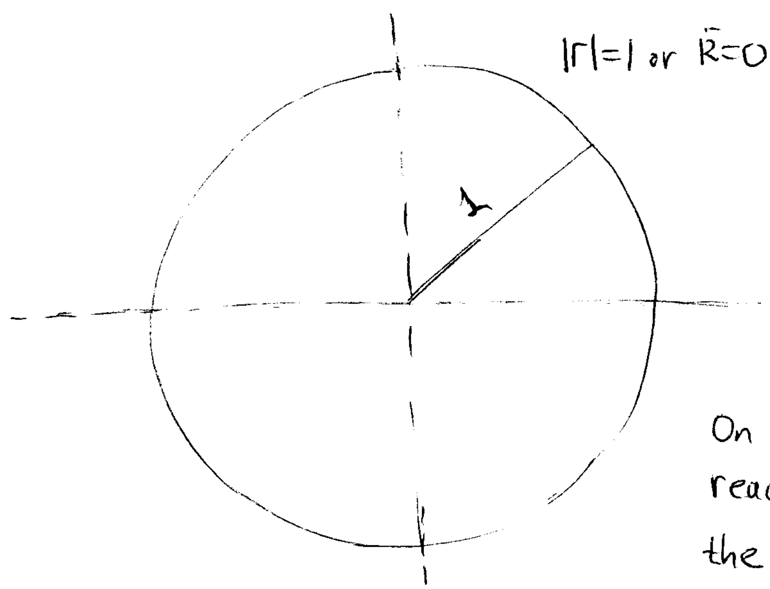
QED

Problem ②

$$|\Gamma| = \left| \frac{Z_L - Z_0}{Z_L + Z_0} \right| = \left| \frac{jX - Z_0}{jX + Z_0} \right| = \left| \frac{Z_0 - jX}{Z_0 + jX} \right|$$

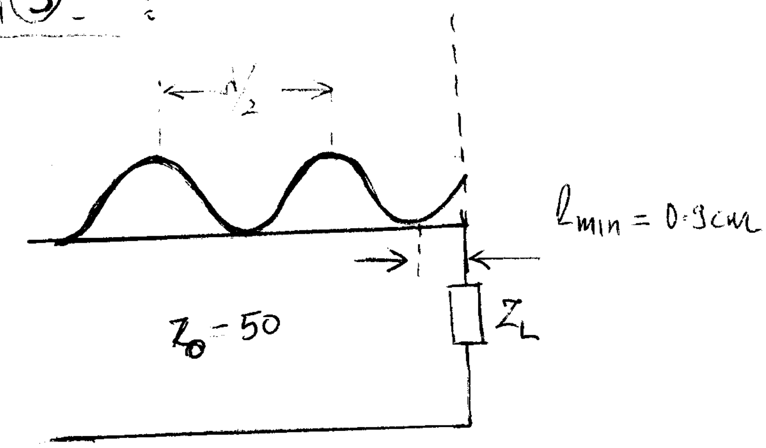
But with Z_0, X real the phasor $Z_c = Z_0 + jX$ has a conjugate: $Z_c^* = Z_0 - jX$. Hence if $Z_c = |Z_c| e^{j\theta_c}$, $Z_c^* = |Z_c| e^{-j\theta_c}$

and $$|\Gamma| = \frac{|Z_c|\,|e^{j\theta_c}|}{|Z_c|\,|e^{-j\theta_c}|} = 1$$ //



On the Smith Chart all reactive loads lie on the $|\Gamma|=1$ external circle.

- Problem (3) :



Distance between successive minima $= 2.1$ cm $= \lambda/2 \Rightarrow \boxed{\lambda = 4.2\text{cm}}$

$$SWR = \frac{1+|\Gamma_L|}{1-|\Gamma_L|} = 2.5 \Rightarrow |\Gamma_L| = \frac{SWR-1}{SWR+1} = \frac{1.5}{3.5} = 0.428 \Rightarrow \boxed{|\Gamma_L| = 0.428}$$

- Let $\Gamma_L = |\Gamma_L| e^{j\theta}$. Condition for first minimum:

$$|V| = |V_0^+| \, |1 + |\Gamma_L| e^{j(\theta - 2\beta l)}| \Rightarrow V_{min} = |V_0^+| (1 - |\Gamma_L|)$$

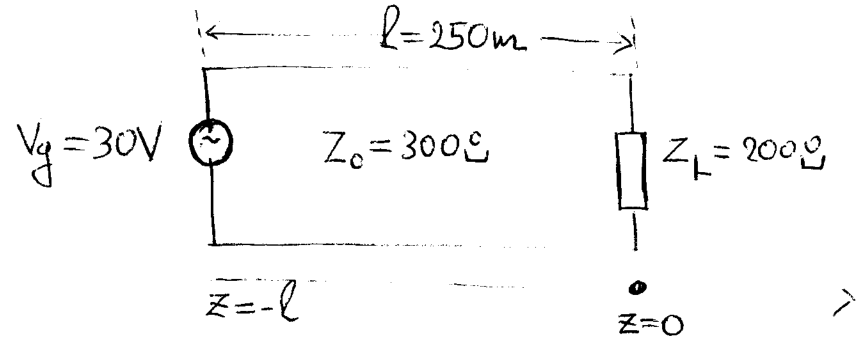
when $\theta - 2\beta l_{min} = -\pi \Rightarrow \theta = 2\beta l_{min} - \pi$

i.e. $\theta = 2 \frac{2\pi}{\lambda} l_{min} - \pi = 2 \frac{2\pi}{4.2} 0.9 - \pi = -0.4488$ rad ($\sim 26°$)

Hence $\boxed{\Gamma_L = 0.428 e^{-j0.4488}}$

and $Z_L = Z_0 \frac{1+\Gamma_L}{1-\Gamma_L} = 50 \frac{1+\Gamma_L}{1-\Gamma_L} = (99 - 45)j\ \Omega$

Problem 4



The reflection coefficient at the load is $\Gamma_L = \dfrac{Z_L - Z_0}{Z_L + Z_0} = \dfrac{200-300}{200+300} = -0.2$

Since the line is air-filled $V_\phi = c = 2.998 \times 10^8 \ m/s$.

Hence the propagation constant $\beta = \dfrac{\omega}{c} = \dfrac{2\pi \times 2.86 \times 10^6}{2.998 \times 10^8} = 0.06 \ rad/m$

The reflection coefficient at the input of the line ($z=-\ell$) is:

$$\Gamma(\ell) = \Gamma_L e^{-2\gamma\ell} = -0.2 e^{-2(4\times10^{-4} + j0.06)(250)} = 0.164 \angle -98.77^\circ$$

where $\gamma = \alpha + j\beta = 4 \times 10^{-4} + j0.06$

The line equations are

$$\begin{cases} V(-\ell) = V_o^+ e^{\gamma\ell}(1 + \Gamma(\ell)) \\ I(-\ell) = \dfrac{V_o^+}{Z_o} e^{\gamma\ell}(1 - \Gamma(\ell)) \end{cases}$$

But we know that at the input $V(-\ell) = V_g \Rightarrow$

$$V_g = V_o^+ e^{\gamma\ell}(1 + \Gamma(\ell))$$

Problem ④ (cont.):

From which,

$$|V_0^+| = \frac{|V_g|}{|1+\Gamma(\ell)|} e^{-\alpha \ell} = 27.46\,V$$

Power delivered to the input of the line:

$$P_{in} = \frac{1}{2} Re\left\{ V(-\ell)\, I^*(-\ell) \right\} = \frac{|V_0^+|^2}{2Z_0}\left(1 - |\Gamma(\ell)|^2\right) e^{2\alpha \ell} = 1.49W$$

Power delivered to the load:

$$P_L = \frac{1}{2}\left\{ V(0)\, I^*(0) \right\} = \frac{|V_0^+|^2}{2Z_0}\left(1 - |\Gamma_L|^2\right) = 1.21\,W$$

Power lost in Line:

$$P_{Loss} = P_{in} - P_L = 1.49 - 1.21 = 280\,mW$$

Power reflected at load:

$$P_{L,refl} = \frac{|V_0^+|^2}{2Z_0} |\Gamma_L|^2 = 50\ mW$$