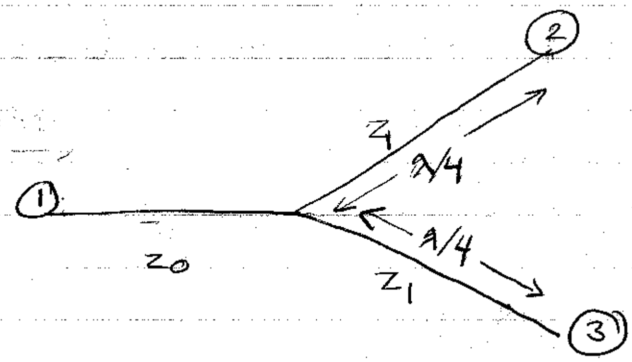
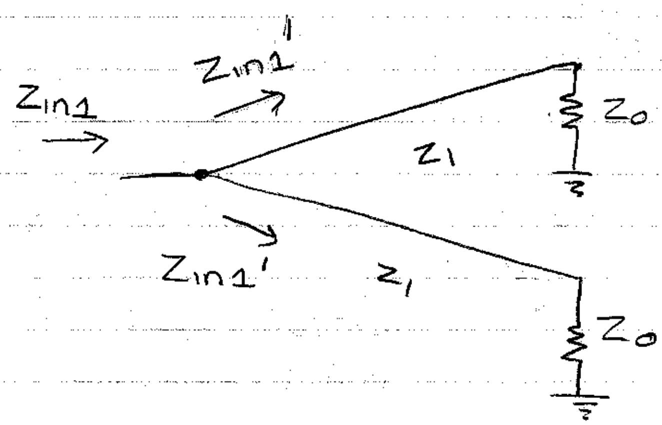
Question #1 Problem Set #7



Finding S_{11}



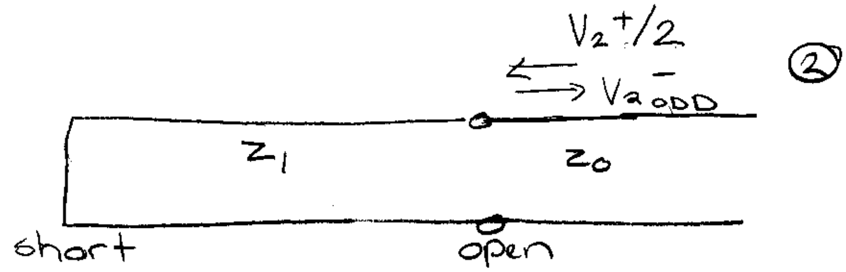
$$Z_{in} = Z_{in1}' \parallel Z_{in1}' = \frac{Z_{in1}'}{2} = \frac{Z_1^2}{Z_0}\left(\frac{1}{2}\right)$$

$$S_{11} = \frac{Z_{in} - Z_0}{Z_{in} + Z_0} = \frac{\frac{Z_1^2}{2Z_0} - Z_0}{\frac{Z_1^2}{2Z_0} + Z_0} = \frac{Z_1^2 - 2Z_0^2}{Z_1^2 + 2Z_0^2}$$

Finding S_{22} ($S_{22} = S_{33}$ due to symmetry)

Use even/odd analysis

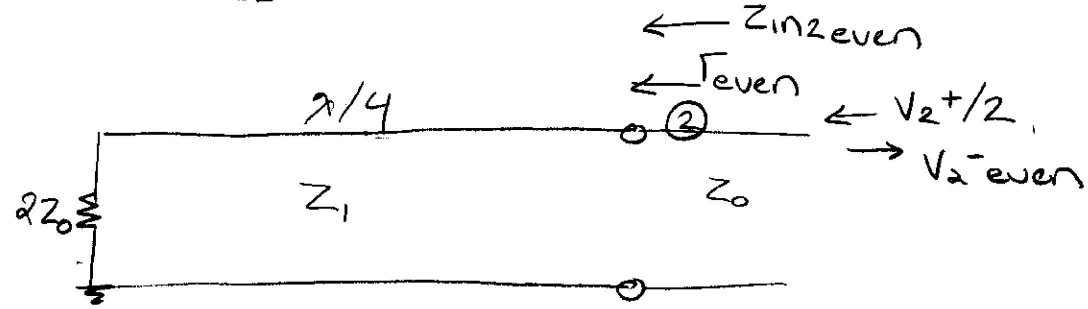
odd



since you see an open circuit looking into port 2

$$V_{2\,ODD}^- = V_2^+/2$$

even



$$\Gamma_{Even} = \frac{Z_{in2even} - Z_0}{Z_{in2even} + Z_0} \qquad Z_{in2even} = \frac{Z_1^2}{2Z_0}$$

$$= \frac{\frac{Z_1^2}{2Z_0} - Z_0}{\frac{Z_1^2}{2Z_0} + Z_0}$$

$$= \frac{Z_1^2 - 2Z_0^2}{Z_1^2 + 2Z_0^2}$$

$$\therefore \quad V_2^-even = \left(\frac{Z_1^2 - 2Z_0^2}{Z_1^2 + 2Z_0^2}\right) V_2^+/2$$

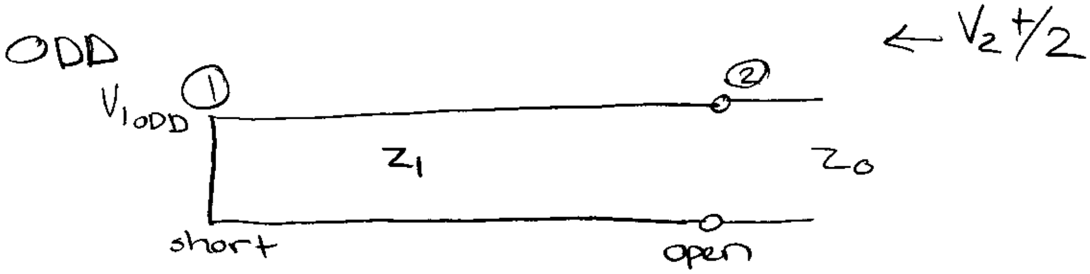
superpose both solutions

$$S_{22} = \frac{V_{2\,ODD}^- + V_{2\,even}^-}{V_2^+/2 + V_2^+/2}$$

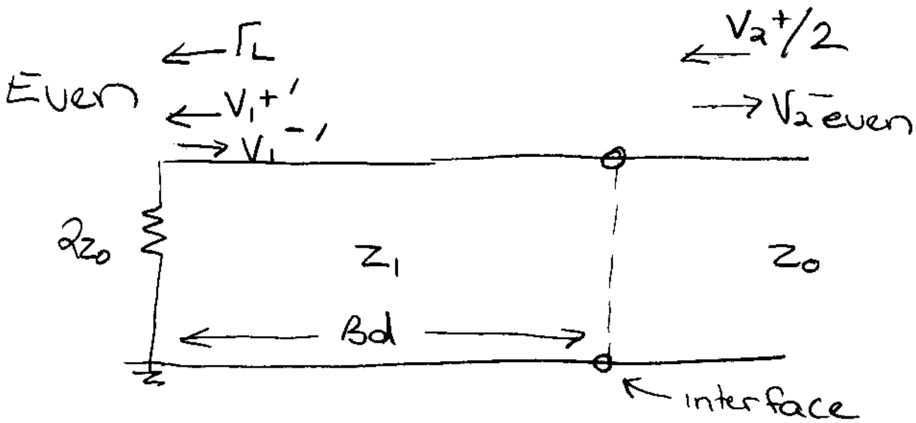
$$= \frac{V_2^+ + \left(\frac{Z_1^2 - 2Z_0^2}{Z_1^2 + 2Z_0^2}\right) V_2^+}{2V_2^+} = \frac{Z_1^2}{Z_1^2 + 2Z_0^2}$$

Finding S_{12} ($S_{12} = S_{21}$ due to reciprocity)

Use even/odd analysis



Since you see a short at port 1
$V_{1ODD} = 0V$ (voltage at port 1)



note: $V_2^-{}_{even} = \Gamma_{even} V_2^+/2$

Boundary condition at interface

$$V_1^{+\prime} e^{j\beta d} + V_1^- e^{-j\beta d} = \frac{V_2^+}{2} + V_2^-{}_{even}$$

$\therefore \beta d = \frac{2\pi}{\lambda}\frac{\lambda}{4} = \frac{\pi}{2}$ $\therefore$ $jV_1^{+\prime} - jV_1^- = \frac{V_2^+}{2} + V_2^-{}_{even}$

$$jV_1^{+\prime}(1 - \Gamma_L) = \frac{V_2^+}{2}(1 + \Gamma_{even})$$

$$V_1^{+\prime} = \frac{\frac{V_2^+}{2}(1 + \Gamma_{even})}{j(1 - \Gamma_L)}$$

$$S_{12} = \frac{V_1^{+'} + V_1^{-'} + 0}{\frac{V_2^+}{2} + \frac{V_2^+}{2}}$$

$$= \frac{V_1^{+'} + V_1^{-'}}{V_2^+}$$

$$= \frac{V_1^{+'}}{V_2^+}(1+\Gamma_L)$$

$$S_{12} = \frac{(1+\Gamma_L)(1+\Gamma_{even})}{2j(1-\Gamma_L)}$$

recall $\Gamma_{even} = \frac{z_1^2 - 2z_0^2}{z_1^2 + 2z_0^2}$ $\qquad \Gamma_L = \frac{2z_0 - z_1}{2z_0 + z_1}$

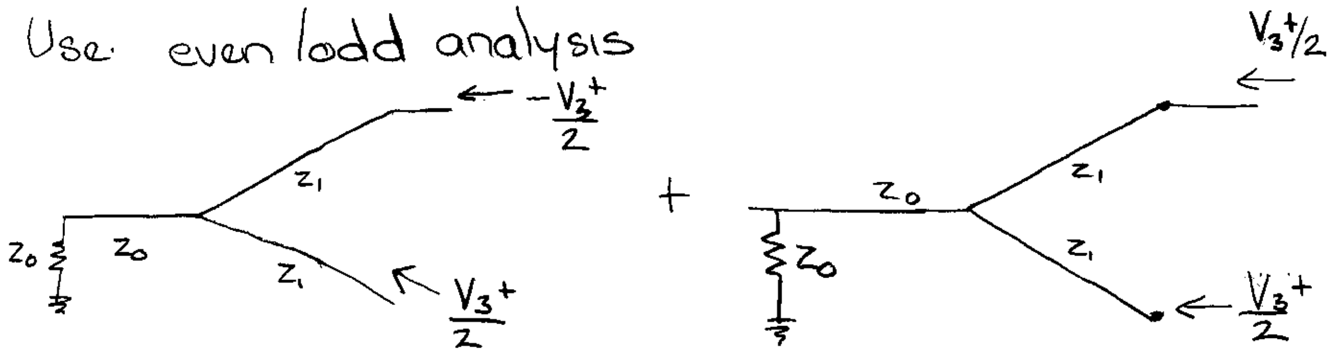
$$\therefore S_{12} = \frac{\left(\frac{\overset{2}{\cancel{4}} z_0}{\cancel{2z_0+z_1}}\right)\left(\frac{2z_1^2}{z_1^2+2z_0^2}\right)}{2j\left(\frac{\cancel{2}z_1}{\cancel{2z_0+z_1}}\right)}$$

$$= \frac{2z_1^2 z_0 / z_1}{(z_1^2 + 2z_0^2)j}$$

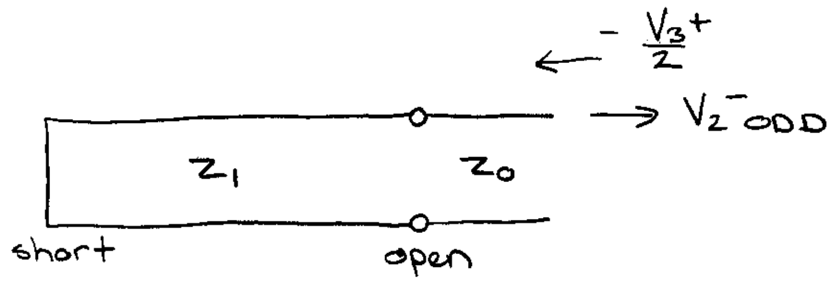
$$S_{12} = \frac{-2z_1 z_0 j}{z_1^2 + 2z_0^2}$$

Finding S_{23} ($S_{23} = S_{32}$ due to reciprocity, symmetry)

Use even/odd analysis

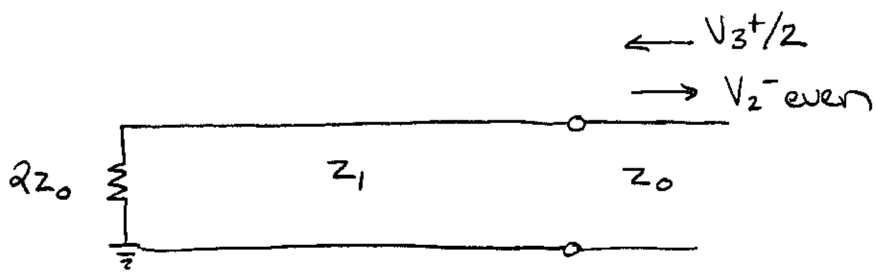


ODD



$$V_2^-{}_{\text{ODD}} = -\frac{V_3^+}{2}$$ since you see an open at port 2 (reflection coefficient = 1)

EVEN

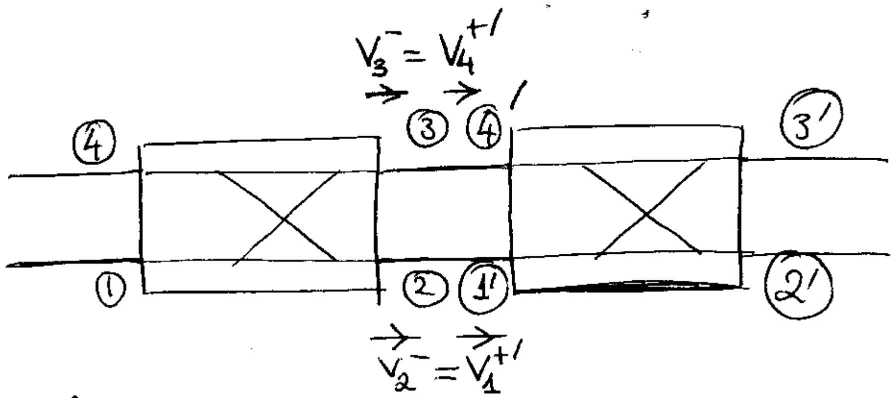


$$V_2^-{}_{\text{even}} = \left(\frac{Z_1^2 - 2Z_0^2}{Z_1^2 + 2Z_0^2}\right)\left(\frac{V_3^+}{2}\right)$$ same as V_2^- even in S_{22} calculation

$$S_{23} = \frac{V_2^-{}_{\text{ODD}} + V_2^-{}_{\text{EVEN}}}{\frac{V_3^+}{2} + \frac{V_3^+}{2}} = \frac{1}{2}\left(\frac{-4Z_0^2}{Z_1^2 + 2Z_0^2}\right)$$

$$S_{23} = \frac{-2Z_0^2}{Z_1^2 + 2Z_0^2}$$

7.3 :



The coupling $C = -20 \log \beta = 8.34\,dB \Rightarrow \beta = |S_{13}| = 0.383$

$\alpha = \sqrt{1-\beta^2} = 0.924$

In the first coupler, $V_1^+ = 1\angle 0°$:

Coupled-voltage : $V_3^- = j\beta V_1^+ = 0.383\angle 90°$

Through-voltage : $V_2^- = \alpha V_1^+ = 0.924\angle 0°$

V_2^-, V_3^- become the input to the second coupler :

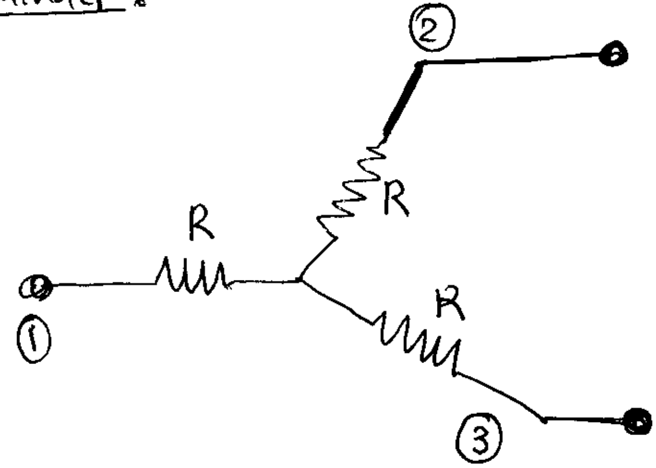
$$V_3^{-'} = j\beta V_1^{+'} + \alpha V_4^{+'} = j\beta V_2^- + \alpha V_3^- = 0.707\angle 90°$$

Also $V_2^{-'} = \alpha V_1^{+'} + j\beta V_4^{+'} = \alpha V_2^- + j\beta V_3^- = 0.707\angle 0°$

In other words, the output suggests that the cascaded $C = 8.34\,dB$ couplers function like a single 3dB hybrid.

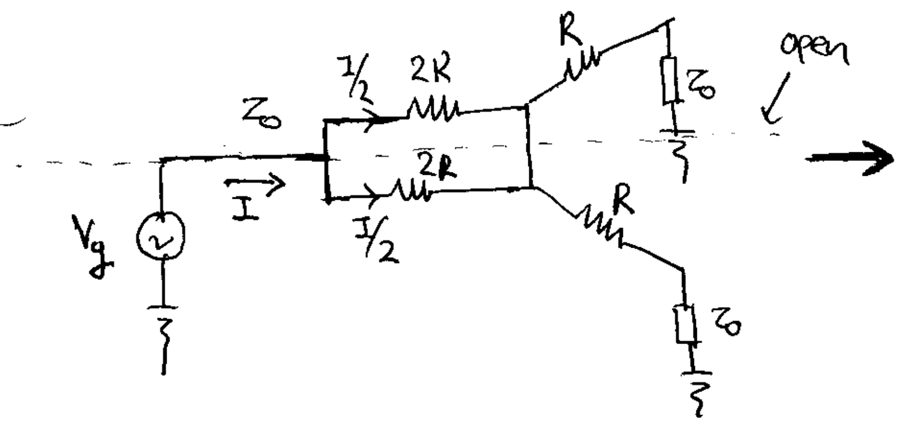
Problem 7.7:

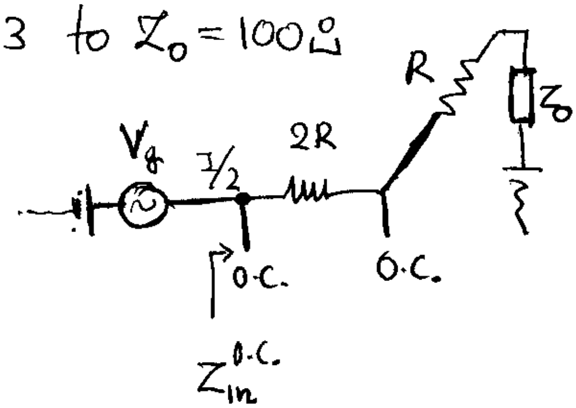
resistive divider:



Derivation of the S-matrix:

Excite port ① and match ports 2 and 3 to $Z_0 = 100\Omega$





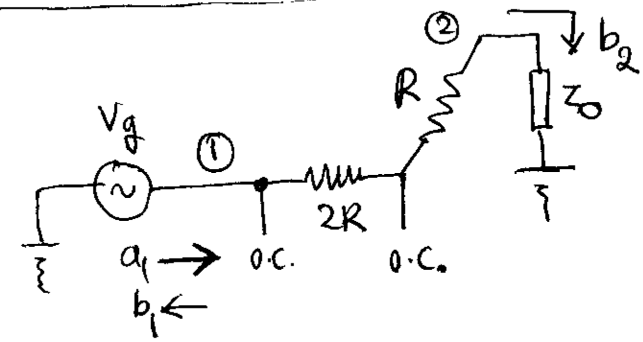
$$Z_{in} = \frac{V_g}{I} = \frac{1}{2}\,\frac{V_g}{I/2} = \frac{1}{2}\,Z_{in}^{o.c.} = \frac{1}{2}\left(2R+R+Z_0\right) = \frac{1}{2}\left(3R+Z_0\right)$$

For a matched port ①, $Z_{in} = Z_0$, i.e. $3R+Z_0 = 2Z_0 \Rightarrow$

$$\boxed{R = \frac{Z_0}{3} = 33\Omega}$$

Hence with $R = 33.3\Omega$, $\boxed{S_{11} = 0}$

Derivation of S_{21} and S_{31} :



$$V_1 = \sqrt{Z_0}\,(a_1 + \cancel{b_1}^{\,0}) = \sqrt{Z_0}\,a_1 = V_1^+$$
$$V_2 = \sqrt{Z_0}\,(\cancel{a_2}^{\,0} + b_2) = \sqrt{Z_0}\,b_2 = V_2^-$$

$$S_{21} = \frac{V_2^-}{V_1^+} = \frac{\sqrt{Z_0}\,b_2}{\sqrt{Z_0}\,a_1} = \frac{V_2}{V_1}$$

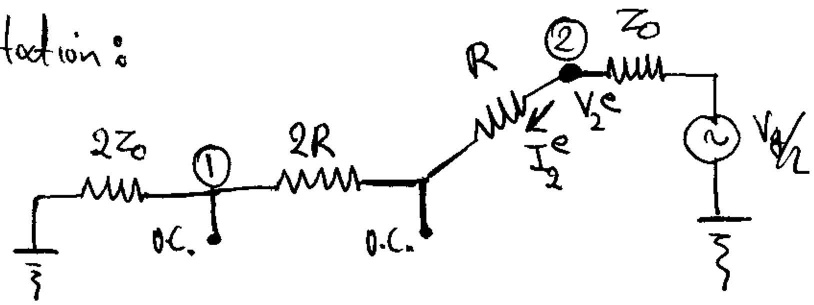
From the voltage-divider : $V_2 = V_1 \dfrac{Z_0}{3R+Z_0} = V_1 \dfrac{Z_0}{2Z_0} = \tfrac{1}{2} V_1$

I.e. $S_{21} = \tfrac{1}{2}$ and from symmetry $S_{31} = \tfrac{1}{2}$ as well.

From reciprocity, $S_{12} = S_{21} = \tfrac{1}{2}$
$S_{13} = S_{31} = \tfrac{1}{2}$

Deriation of S_{22}/S_{33}:

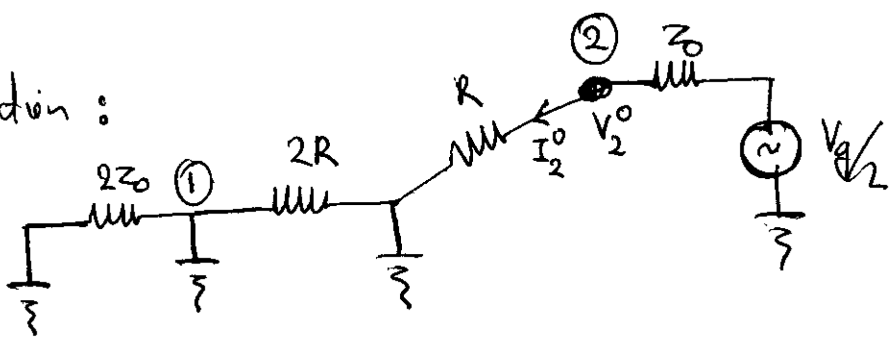
Even - Excitation:



$$V_2^e = \frac{V_g/2}{3R+3Z_0}(3R+2Z_0)$$
i.e. $V_2^e = \dfrac{3}{4}\dfrac{V_g}{2}$

$$I_2^e = \frac{V_g/2}{3R+3Z_0} = \frac{1}{4Z_0}\frac{V_g}{2}$$

Odd- Excitation :



$$V_2^o = \frac{V_g/2}{R+Z_0} R = \frac{1}{4}\frac{V_g}{2}$$
$$I_2^o = \frac{V_g/2}{R+Z_0} = \frac{3}{4Z_0}\frac{V_g}{2}$$

i.e. $Z_{in}^{②} = \dfrac{V_2^e + V_2^o}{I_2^e + I_2^o} = \dfrac{\frac{3}{4}+\frac{1}{4}}{\frac{1}{4}+\frac{3}{4}} Z_0 = Z_0$ // . Hence $S_{22} = 0$.
By symmetry $S_{33} = 0$.

Derivation of S_{32} (Isolation) : $S_{32} = \frac{V_3^-}{V_2^+} = \frac{V_3}{V_2}$

Even Excitation : Same as port 3 I.e $V_3^e = V_2^e = \frac{3}{4}\frac{V_g}{2} =$

Odd Excitation : $V_3^o = -V_2^o = -\frac{1}{4}\frac{V_g}{2}$

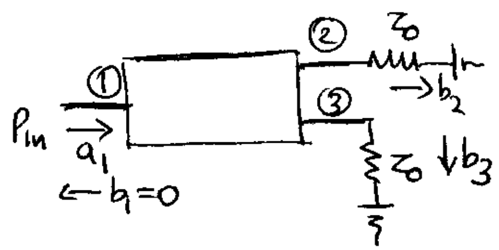
Therefore $S_{32} = \frac{V_3^e + V_3^o}{V_2^e + V_2^o} = \frac{V_2^e - V_2^o}{V_2^e + V_2^o} = \frac{\frac{3}{4} - \frac{1}{4}}{\frac{3}{4} + \frac{1}{4}} = 1/2$

I.e. there is no perfect isolation.

From reciprocity $S_{23} = 0$.

Finally

$$[S] = \frac{1}{2}\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$



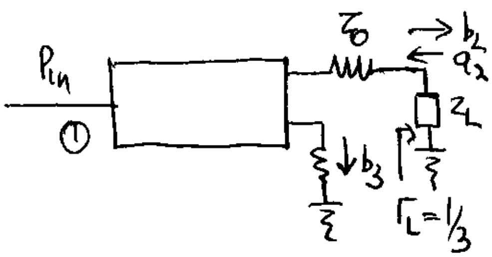
$P_3 = \frac{1}{2}|b_3|^2$

But $b_3 = S_{31}a_1 + S_{32}\cancel{a_2}^{0} + \cancel{S_{33}}^{0}a_3$

I.e $b_3 = S_{31}a_1$

Hence $P_3 = \frac{1}{2}|S_{31}|^2|a_1|^2 = |S_{31}|^2 P_{in}$

For port ② matched $P_3 = \frac{1}{4}P_{in}$



This time $b_3 = S_{31}a_1 + S_{32}a_2$

But $\frac{a_2}{b_2} = \Gamma_L = \frac{1}{3}$, I.e $b_3 = S_{31}a_1 + S_{32}\Gamma_L b_2$

Also $b_2 = S_{21}a_1 + \cancel{S_{22}}^{0}a_2 + S_{23}\cancel{a_3}^{0}$

Hence $b_3 = S_{31}a_1 + S_{32}\Gamma_L b_2 = S_{31}a_1 + S_{32}S_{21}\Gamma_L a_1$

and $b_3 = \left(\frac{1}{2} + \frac{1}{4}\frac{1}{3}\right)a_1 = 0.583a_1$

I.e $P_3' = \frac{1}{2}|a_1|^2(0.583)^2 = (0.583)^2 P_{in}$. Hence $10\log\frac{P_3}{P_3'} = 10\log\frac{0.34}{0.25} = 1.3\,dB$

Problem 7-32 :

The idea here is to exploit the symmetry that exists along the line Z_0. The input port (#1) is then treated as follows:

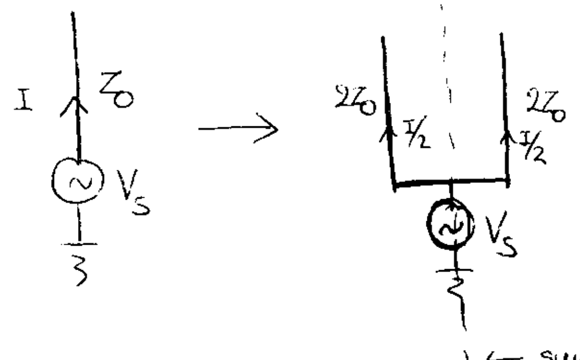


Fig. 1

Along the symmetry axis, I can place open-circuits (magnetic wall):

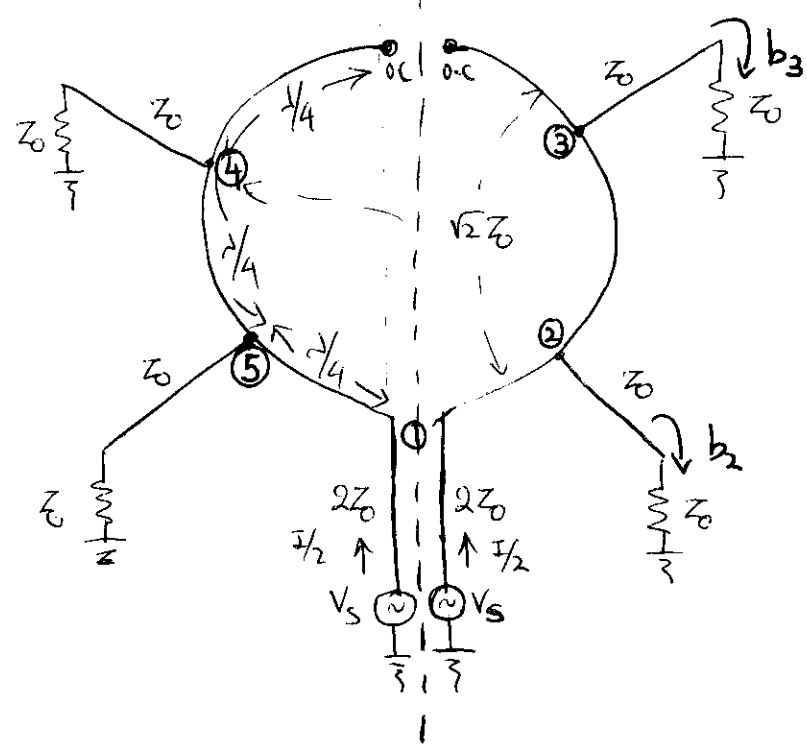


Fig. 2

Considering the right half:

1) The O.C. becomes a short at port-3. Therefore, $b_3 = 0$ and $S_{31} = 0$. By symmetry $S_{41} = 0$ as well.

2) The short at port ③ becomes an open at port ②. Therefore, for each half we have:

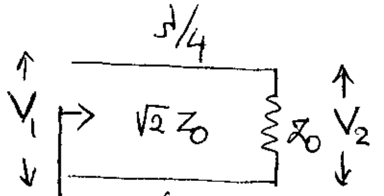
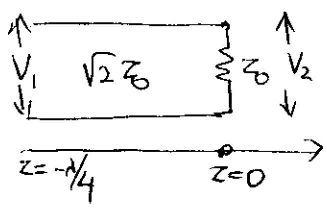


Fig. 3

$$Z_{in}^{h} = \frac{(\sqrt{2} Z_0)^2}{Z_0} = 2Z_0$$

Hence the input-impedance at port ① is $2Z_0 // 2Z_0 = Z_0$, i.e. port ① is matched and $b_1 = 0$ $(S_{11} = 0)$.

3)



$$V = V_+ \left(e^{-j\beta z} + \Gamma_L e^{j\beta z} \right)$$

$$\Rightarrow \left. \begin{aligned} V_2 &= V(z=0) = V_+ (1 + \Gamma_L) \\ V_1 &= V(z=-\lambda/4) = jV_+ (1 - \Gamma_L) \end{aligned} \right\} \Rightarrow \frac{V_2}{V_1} = \frac{1}{j} \frac{1+\Gamma_L}{1-\Gamma_L}$$

$$= \frac{1}{j} \frac{Z_0}{Z_0\sqrt{2}} = \frac{-j}{\sqrt{2}}$$

Therefore, $S_{21} = \frac{V_2}{V_1} = -j/\sqrt{2}$

Finally:

$$V_1^- = 0$$
$$V_2^- = V_5^- = S_{21} V^+ = -j/\sqrt{2}$$
$$V_3^- = V_4^- = 0$$

Note that $|S_{11}|^2 + |S_{21}|^2 + |S_{31}|^2 + |S_{41}|^2 + |S_{51}|^2 = 0 + 1/2 + 0 + 0 + 1/2 = 1$ (lossless condition). Power is equally split between ports ② and ⑤