BACKWARDS WAVES IN ANISOTROPIC LEFT-HANDED MEDIA

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ABSTRACT

The behavior of backward waves is considered from a purely wave propagation point of view. Beginning with the condition that the phase and group velocity vectors are anti-parallel a form for the index of refraction is derived which results in backwards wave behavior at all frequencies. The dispersion relation for this index is found and it is shown that, for the case considered, this index is necessarily negative. The requirement for backwards wave behavior –that the phase and group velocity are "perfectly" anti-parallel – is then relaxed and the situation is considered where these two velocity vectors have one or two anti-parallel components (the angle between them is between 90° and 270°). It is shown that this implies propagation through an anisotropic medium and that backwards wave behavior can still occur if at least one of the axes exhibits a negative index. This phenomenon, however, is not restricted to the propagation along the negative index axes.

KEY WORDS

Backwards waves, negative index, left-handed media.

1. Introduction

Recently, there has been much work dedicated to the study of media that exhibit backwards wave propagation [1-4]. Although more recent studies have focused on this phenomenon in left-handed media (LHM) this phenomenon is certainly not restricted to this type of media and has been investigated in various right-handed media (RHM) such as sheets of plasma [5]. Other RHM that have been shown to exhibit backwards wave behavior include traveling wave tubes and microwave amplifiers. Hence this phenomenon, defined as an anti-parallel Poynting vector $\vec{S}$ and wavevector $\vec{k}$, is not restricted to LHM although it is implied for propagation through a negative index passband. When studying backward wave propagation it is sometimes more convenient to use the group velocity as a substitute for the Poynting vector. This substitution, however, is only valid in the passband where the two vectors are parallel. Furthermore, when formulating the problem in terms of the group velocity, it also more convenient to replace the wavevector with the phase velocity.

In this paper we study the backwards wave phenomenon from a purely three dimensional wave propagation point of view. In section 2, beginning with the assumption that the group velocity and phase velocity are anti-parallel an expression for the index of refraction is derived which yields backwards waves at all frequencies, and it is shown to be identical to the presence of negative index of refraction. The condition that these two vectors are perfectly anti-parallel is then relaxed and the case is considered where only some of their components are anti-parallel. Section 3 discusses anisotropic media where the angles between the group velocity and phase velocity are calculated for a uniaxial RHM and LHM under some conditions. In section 4 we give our final thoughts and conclusions.

2. Backwards Waves

i) "Perfect" Backwards Waves

Let us examine the backward wave phenomenon from a three dimensional wave propagation point of view. Consider a plane wave propagating in an arbitrary medium with phase index $n(k)$. The question to be asked is the following: What functional form for the index will lead to the backward wave propagation? To answer this let us consider the expressions for the phase and group velocity in such a medium. The phase and group velocities of the propagating plane wave are given by

$$v_p = \frac{c_o}{n(k)}\hat{k}, \tag{1}$$

$$v_g = \nabla_k \frac{c_o k}{n(k)}, \tag{2}$$

where c_o is the vacuum speed of light and $\hat{k}$ is the unit vector in the direction of propagation. To enforce the condition that these two velocities are anti-parallel we set the group velocity equal to the negative of the phase velocity

$$\nabla_k \frac{c_o k}{n(k)} = -d \frac{c_o}{n(k)} \hat{k}, \qquad (3)$$

where d is an arbitrary positive constant which accounts for any difference in the magnitudes of the two velocities. The solution to this equation represents the index of a medium which supports backward wave behavior at all frequencies. Setting $d = 1$ for simplicity, the solution is

$$n(k) = bk^2 , \qquad (4)$$

where b is an arbitrary constant with units m^2. Since $k^2 > 0$ a negative index of refraction can only result for $b < 0$. The dispersion relation can be calculated from (4)

$$\omega(k) = \frac{c_o k}{n(k)} = \frac{c_o}{bk}, \qquad (5)$$

Assuming that only positive frequencies are physical, equation (5) results in the dispersion diagram shown in Fig. 1. In branch-I of Fig. 1 the wavevector, k, is negative implying a negative phase velocity. On the other hand, the local derivative at any point on Branch-I is positive signifying that the group velocity is positive everywhere. In Branch-II, on the other hand, the opposite is true and the phase velocity is positive whereas the group velocity is negative. Since the signs of the phase and group velocities are opposite everywhere in Fig. 1 the index derived from equation (3) does indeed yield backwards wave behavior at all frequencies.

In further discussing Fig. 1, assume that the propagating waves are generated by a source at $x = 0$ and that the waves' energy propagates in the $+x$ direction (i.e. away from the source). Under this assumption the solution of Branch-I implies that the phase velocity propagates towards the source (in the $-x$ direction) while the group velocity propagates away from the source (in the $+x$ direction). On the other hand, Branch-II implies a phase velocity which propagates away from the source and a group velocity which propagates towards the source. Since, in the passband, the group velocity is equal to the energy velocity the situation in Branch-II contradicts our assertion that the energy of the wave propagates away from the source. Hence, this case is not physical and is not a valid solution. The situation in Branch-I, where the group velocity is positive and the phase velocity is negative, is the only correct solution. This branch results from choosing $b < 0$ in equation (4) which is the condition for obtaining a negative index of refraction. The above results agree with those derived for a distributed system of series capacitances and shunt inductances (i.e. the dual of the simple transmission line model) [6]. To our knowledge, this is the first attempt to derive these results from the full wave (three-dimensional wave) propagation point of view.

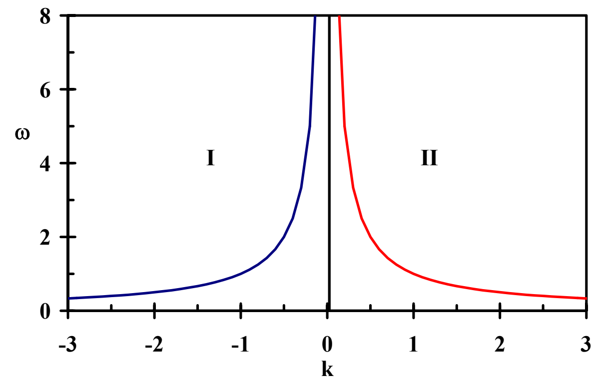

FIG. 1. Dispersion relation for index given in equation (4).

ii) "Imperfect" Backwards Waves

In the case considered above the phase velocity and group velocity are perfectly anti-parallel (that is, the angle between them is 180°). A more rigorous analysis would require solving a differential equation similar to that in (3) but for waves in which the angle between the phase and group velocities has values in the range $90 < \theta < 270$. For such waves, the phase and group velocities are not perfectly anti-parallel, but have anti-parallel components, such that they can still be considered as backward waves. The equation to be solved has the form

$$\frac{\hat{k}_x}{n(k)} + \frac{\hat{k}_y}{n(k)} + \frac{\hat{k}_z}{n(k)} - \frac{k\nabla_k n(k)}{n(k)^2} = d_x \frac{\hat{k}_x}{n(k)} + d_y \frac{\hat{k}_y}{n(k)} + d_z \frac{\hat{k}_z}{n(k)}, \qquad (6)$$

where the d_i (i=x, y, z) are arbitrary constants which account for the differences in magnitude and direction between the components of the group and phase velocities. Collecting terms with the same $\hat{k}_i$, equation (6) can be separated into three differential equations of the form

$$\frac{(1-d_i)}{n(k)}\hat{k}_i = \frac{k}{n(k)^2}\frac{\partial n(k)}{\partial k_i}\hat{k}_i, \text{[1]} \qquad (7)$$

where the i represent x, y, or z. The solutions to these three equations have the form

[1] The complexity of this problem could be reduced by choosing a coordinate system in which one of the primary axes is perpendicular to the plane containing the two velocity vectors, effectively reducing the problem to two dimensions.

$$n(k) = b_i (k^2)^{\frac{1-d_i}{2}}, \qquad (8)$$

Note that if d_i = -1 for all three solutions the problem reduces to the case of perfectly anti-parallel backward waves considered above. If, on the other hand, the d_i are not all identical the index of refraction has different forms along the three axes. This describes the case of an anisotropic medium. As will be shown in the next section, to achieve backward wave behavior we must choose $b_i < 0$ for at least one of the solutions.

3. Anisotropic Media

For simplicity let us consider a medium where the permeability is isotropic and the permittivity is anisotropic and given by

$$\overline{\overline{\varepsilon}} = \begin{bmatrix} \varepsilon_x & 0 & 0 \\ 0 & \varepsilon_y & 0 \\ 0 & 0 & \varepsilon_z \end{bmatrix}. \qquad (9)$$

Fig. 2 shows the *k*-space diagram for an electric field polarized in the k_x - k_y plane and parameters $\varepsilon_x = -2$, $\varepsilon_y = 1$, $\varepsilon_z = 1$, and $\mu = 1$. When the electric field is polarized in the k_y direction the index is positive and the wave propagates. For polarizations in the k_x direction the index is imaginary and there is no propagation. If we take the negatives of all the parameters used in generating the *k*-surface (i.e. $\varepsilon_x = 2$, $\varepsilon_y = -1$, $\varepsilon_z = -1$, and $\mu = -1$) the resulting *k*-space diagram will be identical to that in Fig. 2. However, in this case a k_y polarized wave sees a negative index instead of a positive one. The k_x polarization is again cutoff just as in the previous case due to the imaginary value of the index.

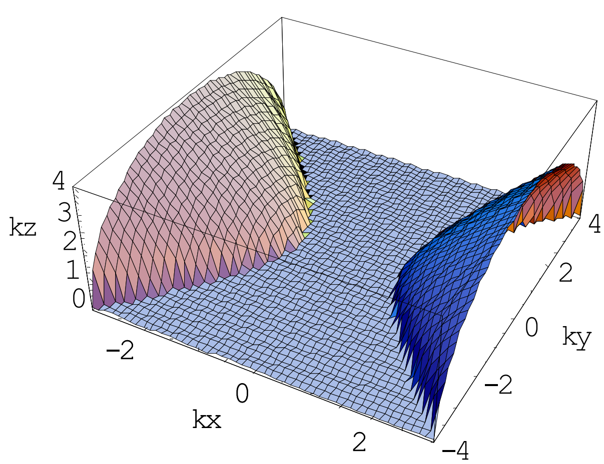

FIG. 2. *k*–Space diagrams for two anisotropic media. In both cases the media are uniaxial with $\varepsilon_y = \varepsilon_z$. In the RHM case the parameters are $\varepsilon_x = -2$, $\varepsilon_y = 1$, $\varepsilon_z = 1$, and $\mu = 1$. The parameters in the LHM case are $\varepsilon_x = 2$, $\varepsilon_y = -1$, $\varepsilon_z = -1$, and $\mu = -1$. Because the parameters in the LHM case are simply the negatives of those in the RHM case their *k*-surfaces are identical.

In each case, a cut was taken in the k_x - k_z plane and the angle between the group and phase velocity was calculated as a function of k_x (Fig. 3). For the RHM case the angle between the group and phase velocity begins at 0° when the wave begins to propagate (k_x = 1) and approaches 90° asymptotically in the high k_x limit (solid line). In the LHM case, on the other hand, the angle is 180° at the onset of propagation (k_x = 1) and decreases asymptotically towards 90° in the high k_x limit (dashed line). The results in Fig. 3 imply that the dot product between the phase velocity and group velocity vectors is always positive for the RHM, and negative for the LHM. In the LHM case, further analysis of the phase and group velocity vectors reveals that their components have the same sign in the k_z direction and that the backwards wave behavior is a result of their k_x components, which have opposite signs. Therefore, to achieve backwards wave behavior not all of the components of the phase and group velocity vectors require opposite signs.

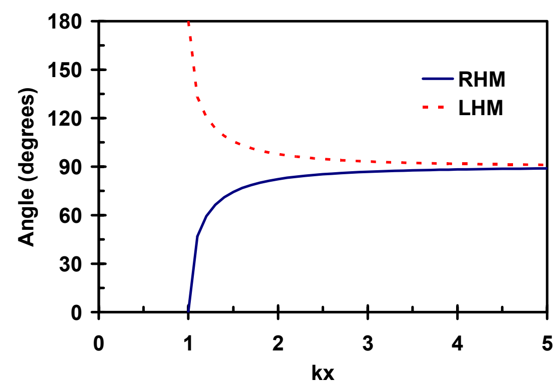

FIG. 3. Calculated angle between the phase and group velocity vectors for the RHM and LHM cases in FIG. 2.

4. Conclusions

The conditions required for backward wave propagation were investigated. This was done from a purely three dimensional wave propagation point of view starting with the assumption that the phase and group velocity were either perfectly anti-parallel or that some of their components were anti-parallel. In both cases the form of the index of refraction necessary to yield backwards wave behavior at all frequencies was derived. In the first case the index is isotropic and must necessarily be negative to yield this behavior. In the second case the medium is anisotropic and at least one of the axes is required to exhibit a negative index. The angles between the phase velocities and group velocities were calculated for RHM and LHM uniaxial media and it was shown that in the RHM case the angle varied from 0° to 90° degrees while in the LHM it varied from 180° to 90° degrees. Hence, in the RHM the dot product between the vectors is positive whereas in the LHM it is negative.

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