Asymptotic Description of Electromagnetic Pulse Propagation in Active Lorentzian Media

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1 Introduction

Pulse propagation through a dispersive, absorptive medium is a classic problem of interest in acoustics, ionospheric radio-wave propagation, and electromagnetic wave propagation in dielectric media, including optical fibers. Sommerfeld and Brillouin were among the early workers studying the wave propagation in linear, homogeneous, isotropic, causally dispersive media using asymptotic method of steepest descent [1]. Their analysis led to the discovery of two wave phenomena whose dynamical evolution preceded the evolution of the main signal. They referred to these as forerunners or precursors. In addition to Brillouin and Sommerfeld, Oughstun and coworkers have done an extensive work on precursors in passive media [2]. Surprisingly, there has been little work on the precursors in dispersive media other than passive Lorentzian. In light of recent experimental observation of superluminal velocities in active media [3], the study of the precursor fields in active media deserves a closer attention. In this paper, for the first time, we address the wave propagation in an active medium using the steepest descent method.

2 Active Lorentzian Medium

The asymptotic analysis requires that the behavior of the complex phase function appearing in the integral representation of the propagated field be known throughout the complex ω-plane. Therefore, a specific model of the frequency dispersion of the complex refractive index must be employed. The model that is used in this paper is a single-resonance active Lorentzian. It is a causal model of a dispersive dielectric, satisfying the Kramers- Kronig relations [4]. The complex index of refraction for this medium is

$$n(\omega) = \left[1 + \frac{\omega_p^2}{\omega^2 - \omega_0^2 + i2\delta\omega}\right]^{1/2}, \tag{1}$$

where, ω_0 is the resonance frequency, δ is the line-width of the resonance, and ω_p is the atomic plasma frequency. In typical situations, the inequalities $\delta < \omega_p < \omega_0$ are obeyed. Based on these inequalities an arbitrary set of parameters for the active Lorentzian medium is chosen as $\omega_0 = 4.0 \times 10^{15} Hz$, $\omega_p = 1.0 \times 10^{15} Hz$, and $\delta = 0.2 \times 10^{15} Hz$.

3 Asymptotic Evaluation of the Total Field Inside an Active Medium

The integral representation of an arbitrary electromagnetic wave propagated in the positive z-direction through a linear, homogeneous, isotropic, temporally dispersive

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medium occupying the half space $z \geq 0$ is given by [2]:

$$A(z,\theta) = \frac{1}{2\pi}\Re\Big\{i\int_{-\infty+ia}^{+\infty+ia} \tilde{f}(\omega)\exp[\frac{z}{c}\phi(\omega,\theta)]d\omega\Big\}, \tag{2}$$

where the phase function $\phi(\omega,\theta)$ defined as

$$\phi(\omega,\theta) = i\omega(n(\omega) - \theta), \tag{3}$$

is a complex function with $n(\omega)$ being the complex index of refraction. $A(z,\theta)$ represents any scalar component of the electric or magnetic field. Here, instead of using the time (t) as an independent variable, the dimensionless parameter $\theta = ct/z$, which is suitable for asymptotic approximations is used (c is the speed of light in free space). The function $\tilde{f}(\omega)$, which is defined as

$$\tilde{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{i\omega t}dt, \tag{4}$$

is the Fourier transform of the initial pulse $f(t) = A(0,t)$ at the input plane, $z = 0$. In this paper, we have assumed that the excitation $f(t)$ is a modulated step function with carrier frequency ω_c. In the integral expression of Eq.(2), the contour of integration C in the complex ω-plane is the straight line $\omega = \omega' + ia$, where $\omega' = \Re(\omega)$ ranges from negative to positive infinity and a is a fixed positive constant that is greater than the abscissa of absolute convergence for the function [5]. The first step in the asymptotic calculation of the propagated field is to determine the locations of the saddle points for $\phi(\omega,\theta)$. The second step is to find the value of $\phi(\omega,\theta) = X(\omega,\theta) + iY(\omega,\theta)$ at these points, and the regions of the complex ω-plane wherein $X(\omega,\theta)$ is less than the value of $X(\omega,\theta)$ at the dominant saddle point for a given value of θ. With all this information at hand, the final step is to deform the original integration path (C) to the steepest descent path and calculate the field at each value of θ for a predefined observation point. The phase function $\phi(\omega,\theta)$ is stationary at a saddle point; therefore, its derivative with respect to the complex frequency ω is zero at these points. Based on the equation for the phase function [Eq.(3)],

$$\phi'(\omega,\theta) = i(n(\omega) - \theta) + i\omega n'(\omega) = 0, \tag{5}$$

where the prime represents the first order derivative with respect to ω; therefore,

$$n(\omega) + \omega n'(\omega) - \theta = 0. \tag{6}$$

After some manipulations, Eq.(6) turns into an eight order polynomial that four of its roots are the saddle points of the phase function. They can be identified by plotting the phase function in the complex ω-plane. At $\theta = 1$, two of these saddle points are in the upper half ω-plane and the other two are in the lower half ω-plane. The saddle points that are in the upper half ω-plane are dominant. As θ increases from unity, one of the saddle points in the upper ω-plane moves downwards from infinity on the imaginary ω-axis and the other moves upwards on the imaginary ω-axis. At $\theta = \theta_1$, these two saddle points coalesce into a second order saddle point. As θ increases from θ_1, the two first order saddle points leave the imaginary ω-axis on opposite directions and move toward the branch lines of the phase function (as shown in Fig.1). We have numerically calculated the exact location of the saddle points in the complex ω-plane. Figure 2(a) shows the location of the two saddle points in the upper half ω-plane on the imaginary ω-axis when θ is close to unity and Fig.2(b) shows the two saddle points in the upper half ω-plane for $\theta > \theta_1$. The excitation $f(t)$ only has a single pole singularity at $\omega = \omega_c$. The original path of integration and the steepest descent path lie on the same side of the pole for $\theta < \theta_s$ and on opposite sides for $\theta > \theta_s$, where θ_s is the solution of the equation $Y(\omega_{sp}, \theta) = Y(\omega_c, \theta)$. The complex frequency ω_{sp} is the location of the saddle point that interacts with the pole singularity, and ω_c is the location of the pole. Therefore, the total field for $\theta < \theta_s$ is only the contribution of the saddle points and for $\theta > \theta_s$ it is the contribution of the saddle points and the pole both at the same time.

The asymptotic approximation of the total field at $1\mu m$ inside the active medium with the mentioned set of parameters is plotted in Fig.3(a) for $1 < \theta < 1.0038$ and Fig.3(b) for $1.0038 < \theta < 25$. As it can be seen from the inset of Fig.3(a), there is a high frequency oscillation for $1 < \theta < \theta_1 \simeq 1.0036$, that is due to the saddle points on the imaginary ω-axis. At $\theta = \theta_s \simeq 1.23$, the steepest descent path passes the pole at $\omega = \omega_c = 5 \times 10^{15}$ and the transient due to both the saddle points and the pole appears. As θ increases, the contribution of the saddle points fades and the steady-state part of the signal that is solely due to the pole becomes dominant. Although the analysis presented here has been done for a specific set of medium parameters, the general phase topography and the flow of the saddle points for an active Lorentzian medium is known and based on this information we can predict the transient behavior of the pulse. The result presented here were validated using the inverse Fourier transform technique. There is a good agreement between the results from the two techniques. Complete description of the asymptotic calculation and the comparison to the inverse Fourier transform will be discussed in the presentation.

4 Summary and Conclusion

The asymptotic description of the propagation of a step-modulated signal of fixed angular frequency ω_c into the half-space $z > 0$ that is occupied by a single-resonance active Lorentzian medium, which is a classical model of a inverted two-level atom, has been presented. The steepest descent method as an asymptotic technique provides the detailed time evolution of the pulse propagation in a linear, temporally dispersive active Lorentzian medium. The analysis and the numerical results show the relation of the location of the saddle points and the topography of the phase function in the complex ω-plane with the transient portion of the pulse that can be used to control the transient response of the medium.

References

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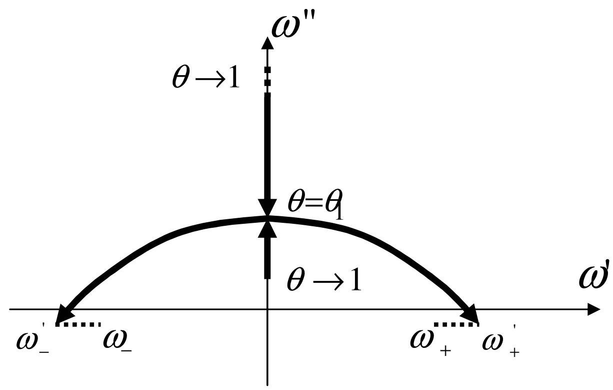

Figure 1: Locations of the upper half ω-plane saddle points as θ goes from one to infinity ($\omega'_{-}\omega_{-}$ and $\omega_{+}\omega'_{+}$ are the branch lines of the phase function)

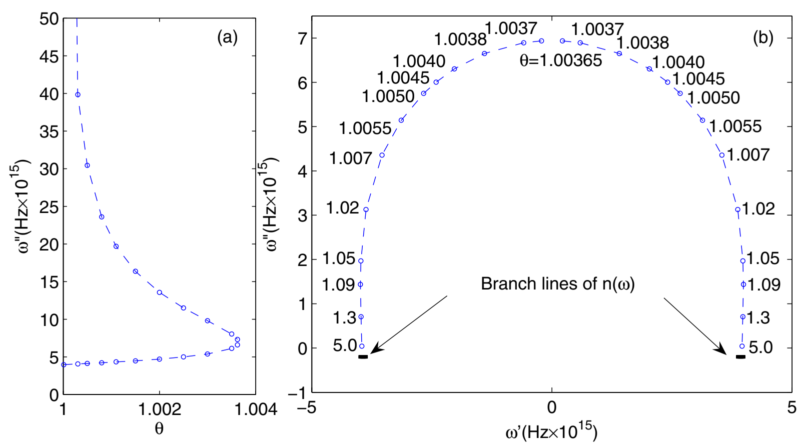

Figure 2: Locations of the upper half ω-plane saddle points as a function of θ (a) On the positive imaginary axis; (b) In the upper half ω-plane.

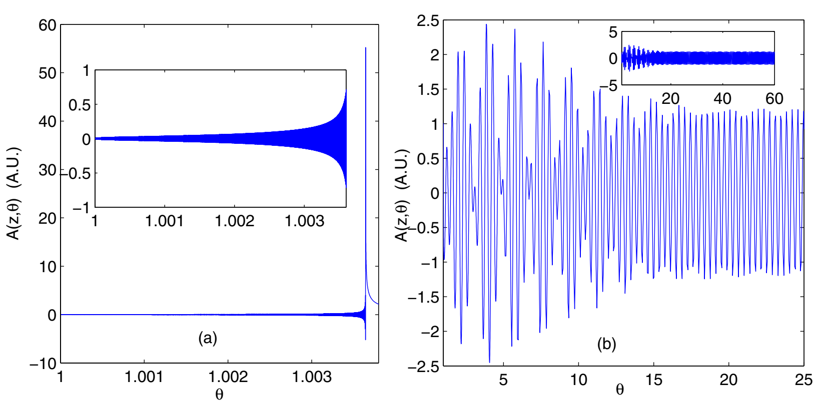

Figure 3: Total sampled field at $1\mu m$ inside the active medium, (a) $1 < \theta < 1.0038$; (b) $1.0038 < \theta < 25$