

Backward wave propagation in left-handed media with isotropic and anisotropic permittivity tensors

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The backward-wave phenomenon in an isotropic medium is investigated from a purely wave propagation point of view. The functional form for the index of refraction necessary to produce such behavior is derived using the condition that the phase and group-velocity vectors are antiparallel. It is shown that, in the case considered, the backward-wave propagation can be attained only in a medium where the index is negative. A more general case is then considered where the angle between the phase velocity and group velocity is allowed to vary between 90° and 270° . It is shown that such behavior requires propagation through an anisotropic medium where at least one of the axes has a negative index and the general form of the index along each of the three principal axes is derived. The condition that the group velocity must be positive in the transmission passband is then used to obtain the required minimum dispersion for a medium with negative index for both the isotropic and the anisotropic cases. © 2006 Optical Society of America

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1. INTRODUCTION

Recently, media that exhibit a negative index of refraction have been the topic of extensive work.¹⁻⁴ This is due to the development of the first negative index medium by Smith *et al.*⁵ and the subsequent explosion of publications on the subject that followed. An important characteristic of negative index media is that the wave vector \mathbf{k} , the electric field vector \mathbf{E} , and the magnetic field vector \mathbf{H} form a left-handed triplet, a relationship that has led them to be dubbed left-handed media (LHM). The aforementioned relation among the three vectors implies that the Poynting vector (\mathbf{S}) and the wave vector (\mathbf{k}) of a wave propagating in a LHM are necessarily antiparallel, which is the defining characteristic of the so-called backward wave. Therefore, LHM and consequently media with negative permittivity and permeability (negative index) are in essence backward wave media; but what about the reverse? Does an antiparallel relationship between the Poynting vector and the wave vector imply a unique negative index of refraction? And if so, then what would be the functional form of this index?

In this paper, the backward-wave phenomenon will be studied from a purely three-dimensional wave propagation point of view. To do so, it is convenient to restrict our attention to propagation through a transmission passband (i.e., away from regions of anomalous dispersion). In such a passband, the group velocity and Poynting vector point in the same direction and the problem can be formulated in terms of the group velocity instead of the Poynting vector. Furthermore, it is also convenient to substitute the propagation vector with the phase velocity. It should be noted that these substitutions are appropriate in the context of the present work, since we are only con-

cerned with the directions of the vectors and not necessarily their magnitudes.

Our analysis of backward-wave propagation and its relation to the index of refraction will be presented as follows. In Section 2 the index of refraction necessary to produce backward wave behavior is derived using the condition that the group and phase velocities are perfectly antiparallel (the angle between them is 180°). It is shown that this index is necessarily negative and isotropic. The condition that the phase and group-velocity vectors are perfectly antiparallel is then relaxed and the case is considered where the two vectors have at least one antiparallel component. In this case the medium is necessarily anisotropic requiring a negative index along at least one principal axis. Moreover, constraints on the functional form of the index of refraction are obtained using the condition that the group velocity must be positive in the transmission passband. Anisotropic uniaxial right-handed media (RHM) and LHM are further examined in Section 3 where the angles between the phase and group velocities under certain conditions are calculated. Section 4 contains our final thoughts and conclusions.

2. BACKWARD WAVES

A. Perfect Backward Waves

To examine the backward-wave phenomenon we start with a purely three dimensional wave propagation point of view, considering an arbitrary medium with phase index $n(k)$. The challenge is to find a functional form for the phase index that results in the propagating wave exhibiting backward behavior. To find this functional form, we begin with a known property of backward waves. That is,

the phase and group velocities of such waves must be antiparallel, where the phase and group velocities are given by

$$\mathbf{v}_p = \frac{c_0}{n(k)} \hat{k}, \quad (1)$$

$$\mathbf{v}_g = \nabla_k \frac{c_0 k}{n(k)}. \quad (2)$$

Here, c_0 is the speed of light in vacuum and \hat{k} is the unit vector in the direction of propagation. In an arbitrary medium where the relationship between the group velocity and phase velocity is not known, the two vectors can be related by

$$(v_{gx}, v_{gy}, v_{gz}) = (d_x v_{px}, d_y v_{py}, d_z v_{pz}), \quad (3)$$

where the d_i ($i=x, y, z$) are arbitrary constants that account for the differences in magnitude and direction between the components of the phase and group velocities. Note that if any of the components of the phase velocity are zero, Eq. (3) may not be valid. However, a new coordinate system can always be defined in which the phase velocity has no zero components. For the case of a perfect backward wave medium, the angle between the phase and group velocities must be 180° . This condition has two consequences for Eq. (3). First, we must choose $d_x = d_y = d_z = d$ so that the vectors are collinear. Second, d must be negative to ensure that the vectors are perfectly antiparallel. To enforce the fact that d must be negative, it will be written in the form $-|d|$. Hence, Eq. (3) becomes

$$\mathbf{v}_g = -|d| \mathbf{v}_p. \quad (4)$$

Using Eqs. (1) and (2) for the group and phase velocities in Eq. (4) gives

$$\nabla_k \frac{c_0 k}{n(k)} = -|d| \frac{c_0}{n(k)} \hat{k}. \quad (5)$$

The parameter d , as written in Eq. (5), is now an arbitrary constant that accounts for any difference in the magnitudes of the two velocity vectors. It is interesting to note that the phase index $n(k)$ that satisfies Eq. (5) will result in backward wave propagation for all frequencies. However, it should be noted that this result is purely mathematical in the sense that no physical medium currently exists that satisfies Eq. (5) for all frequencies. Setting $d=1$ for simplicity, the solution to the above vector differential equation is given by

$$n(k) = bk^2, \quad (6)$$

where b is an arbitrary constant with units of square meters. The dispersion relation corresponding to this index is

$$\omega(k) = \frac{c_0 k}{n(k)} = \frac{c_0}{bk}. \quad (7)$$

Figure 1 shows the dispersion diagram calculated from Eq. (7). The wave vector k in branch I of Fig. 1 is negative, implying a negative phase velocity. On the other hand, the local derivative at any point on branch I is positive,

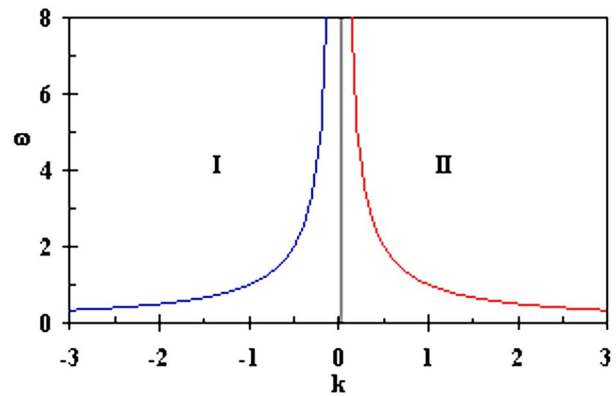


Fig. 1. (Color online) Dispersion relation for the index given in Eq. (7).

signifying that the group velocity is positive everywhere. In branch II the opposite is true. That is, at all points the phase velocity is positive and the group velocity is negative. Hence, as expected, the index derived from Eq. (4) yields backward wave behavior at all frequencies since the phase and group velocities are antiparallel everywhere.

It is important at this point to determine which branch in Fig. 1 presents a valid solution. Assume that the propagating waves are generated by a fixed source located at $\mathbf{r}=0$ and that the energy of the waves propagates away from the generator in the $+\hat{r}$ direction. For this configuration, the solution represented by branch I implies a group velocity that propagates away from the source ($+\hat{r}$ direction) and a phase velocity that propagates toward the source ($-\hat{r}$ direction). The solution represented by branch II, on the other hand, implies the opposite case with the group velocity propagating toward the source ($-\hat{r}$ direction) and the phase velocity propagating away from the source ($+\hat{r}$ direction). Since we have assumed propagation in the passband, where the group and energy velocities are the same, the solution in branch II directly violates our assumption that the energy of the wave propagates away from the source. Therefore, the situation in branch II is unphysical and does not present a valid solution. This can also be seen as follows. Branch II represents the case of a negative group velocity. It has previously been shown that in passive media the group velocity can be negative only in the stop band.^{2,6,7} Hence, branch II is again unphysical since we are assuming propagation through the passband of a passive medium. It should be noted that branch I can be obtained only by choosing $b < 0$ in Eq. (7). However, choosing $b < 0$ is also the condition used to obtain a negative index of refraction in Eq. (6). Therefore, we must conclude that for the case considered above, a negative index of refraction is a necessary condition for backward wave propagation. We may take note of the fact that the dispersion relation obtained above is in agreement with those derived for the dual of a simple transmission line model.⁸ This model, which consists of a series capacitor and a shunt inductor (instead of a customary series inductor and shunt capacitor), was the starting point for several of the first negative index media developed to date.^{9,10} To our knowledge, this is the first attempt to derive these results from the full wave propagation (3D) point of view.

B. Imperfect Backward Waves

In Subsection 2.A we considered a case where the phase and group-velocity vectors were perfectly antiparallel (the angle between them was 180°). To perform a more general analysis, the angle between the phase and group velocities should be allowed to have any value in the range of 90° < θ < 270°. In other words, we should allow the dot product of the unit vectors of the phase and group velocities to have any negative value. We will refer to these as imperfect backward waves. For such a wave, although the phase and group velocities are not antiparallel, they will have antiparallel components. Hence, the point of reference is once again Eq. (3), but now the d_i 's can no longer all be identical. In terms of the components of the group and phase velocities, from Eq. (3) we can write

$$v_{gi} = d_i v_{pi}, \quad (8)$$

where $i = x, y, z$. If we use the expressions of Eqs. (1) and (2) for the group velocity and phase velocity, Eq. (8) can be written as

$$c_0 \left(\frac{1}{n(k)} - \frac{k}{n(k)^2} \frac{\partial n(k)}{\partial k_i} \right) \frac{k_i}{k} = d_i \frac{c_0}{n(k)} \frac{k_i}{k}. \quad (9)$$

With a little algebra Eq. (9) can be rewritten in the form

$$\frac{(1 - d_i)}{n(k)} = \frac{k}{n(k)^2} \frac{\partial n(k)}{\partial k_i}. \quad (10)$$

The solution to the above differential equation is given by

$$n(k) = \alpha_i (k^2)^{(1-d_i)/2}, \quad (11)$$

where α_i is an arbitrary constant. Equation (11) shows that for this imperfect backward wave the index can have different forms along each of the three principal axes, which is the case for an anisotropic medium. This will be further examined in Section 3 where it will be shown that, to obtain imperfect backward wave propagation, the index along at least one of the axes must be negative. That is, we must have $\alpha_i < 0$ for at least one of the solutions in Eq. (11). Note that the simple case of an isotropic medium can be recovered from Eq. (11) by setting $d_i = d = -1$.

C. Minimum Required Dispersion in the Case of Negative Index Metamaterials

In the cases presented above a phase index was obtained that yielded backward wave behavior at all frequencies. Further insight into the possible functional forms and constraints on the phase index can be gained by rewriting the group velocity according to

$$\mathbf{v}_g = \left[\frac{c_0}{n(k)} - \frac{c_0 k}{n(k)^2} \frac{\partial n(k)}{\partial k} \right] \hat{\mathbf{k}}. \quad (12)$$

Since, as discussed previously, group velocity must be positive in the passband, Eq. (12) provides us with a constraint on the functional form of the index. In other words, any solution for the index that yields propagating backward waves must also give a positive result for the group velocity when substituted in Eq. (12). As an example let us assume the index can be written in the form

$$n(k) = \gamma k^p, \quad (13)$$

where γ and p are arbitrary constants. Substituting Eq. (13) into Eq. (12), the conditions for a positive group velocity become

$$p > 1 \text{ for } \gamma < 0, \quad (14a)$$

$$p < 1 \text{ for } \gamma > 0. \quad (14b)$$

Note that the phase index in Eq. (6) has the form given by Eq. (13), which satisfies the condition of inequality (14a), since $p=2$ and $\gamma=b < 0$. In other words, substituting Eq. (6) into Eq. (12) yields

$$v_g = -\frac{c_0}{bk^2} > 0. \quad (15)$$

The index given by Eq. (6) is isotropic. In the anisotropic case the constraints of inequalities (14a) and (14b) must be applied separately to each direction in which there is a passband. Using Eq. (11) in conjunction with inequalities (14), we now have a set of conditions that provide the framework for generating an index of refraction, either isotropic or anisotropic, which will produce the backward wave phenomenon. Finally, the conditions given in Eq. (13) and inequalities (14) also demonstrate an important characteristic of negative index media: In the negative index case ($\gamma < 0$), the medium is necessarily dispersive ($p > 1$).

3. ANISOTROPIC MEDIA

In Section 2 it was shown that a necessary condition for a perfect backward wave is an isotropic index of refraction. Furthermore, an analysis of the dispersion characteristics determined that the index was necessarily negative. In the case of the imperfect backward wave it was shown that the index could not be identical along all three principal axes. In other words, the medium must be anisotropic. However, no constraints on the sign of the index were established. In this section a general analysis will be performed to determine the requirements on the sign of the index along the principal axes. For simplicity let us consider a uniaxial medium where the permittivity and permeability are given by

$$\bar{\epsilon} = \epsilon_0 \begin{bmatrix} \epsilon_s & 0 & 0 \\ 0 & \epsilon_s & 0 \\ 0 & 0 & \epsilon_z \end{bmatrix}, \quad (16)$$

$$\mu = \mu_0 \mu_r, \quad (17)$$

where ϵ_0 and μ_0 are the free-space permittivity and permeability, respectively. From Eqs. (16) and (17) the optical axis of the uniaxial medium is directed along the z axis. In this case, a so-called ordinary wave will correspond to any wave propagating such that its \mathbf{D} vector is perpendicular to the optical axis (i.e., \mathbf{D} polarized in the x - y plane). In uniaxial media, ordinary wave propagation has an isotropic character. Since this type of behavior was examined in Section 2, the ordinary wave will not be considered here. If, on the other hand, the \mathbf{D} vector is polarized

in the plane containing both the optical axis and the propagation direction, then the wave is considered extraordinary. The k -space diagram for the extraordinary wave propagating through a medium with material parameters $\epsilon_s=1$, $\epsilon_z=-2$, and $\mu_r=1$ is shown in Fig. 2. It should be noted that since in the following analysis we are mainly concerned with the shape of the k surfaces, and since different values of ω only scale these surfaces, a convenient value of ω can be chosen. For simplicity, the plots in this section are generated using $\omega=1$.

For an extraordinary wave propagating along one of the three primary axes, Fig. 2 can be used to further elucidate the situation. For propagation in the x direction the extraordinary wave is polarized along the z axis. For this polarization, the index is imaginary so that there is no propagation in the x direction. Similarly, the index for an extraordinary wave propagating in the y direction will also be imaginary, cutting off propagation. On the other hand, an extraordinary wave with a propagation vector along the z axis (along the optical axis) will be polarized in the x - y plane, where the index is real and positive (RHM), and it will propagate. This is the point at which the k surfaces for the ordinary and extraordinary waves

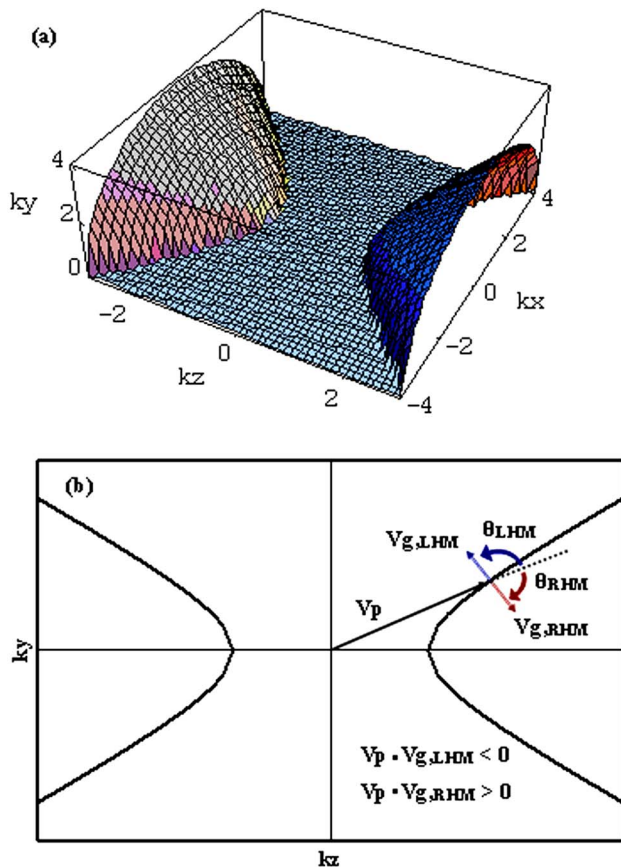


Fig. 2. (Color online) (a) 3D and (b) two-dimensional (2D) k -space diagrams for RHM and LHM two-sheeted hyperboloids. In both cases the media are uniaxial with $\epsilon_x = \epsilon_y = \epsilon_s$. In the RHM case the parameters are $\epsilon_s=1$, $\epsilon_z=-2$, and $\mu_r=1$. The parameters in the LHM case are $\epsilon_s=-1$, $\epsilon_z=2$, and $\mu_r=-1$. Because the parameters in the LHM case are simply the negatives of those in the RHM case, the k surfaces are identical. The angle between the phase velocity and group velocity for the RHM and LHM cases are shown on the 2D plots.

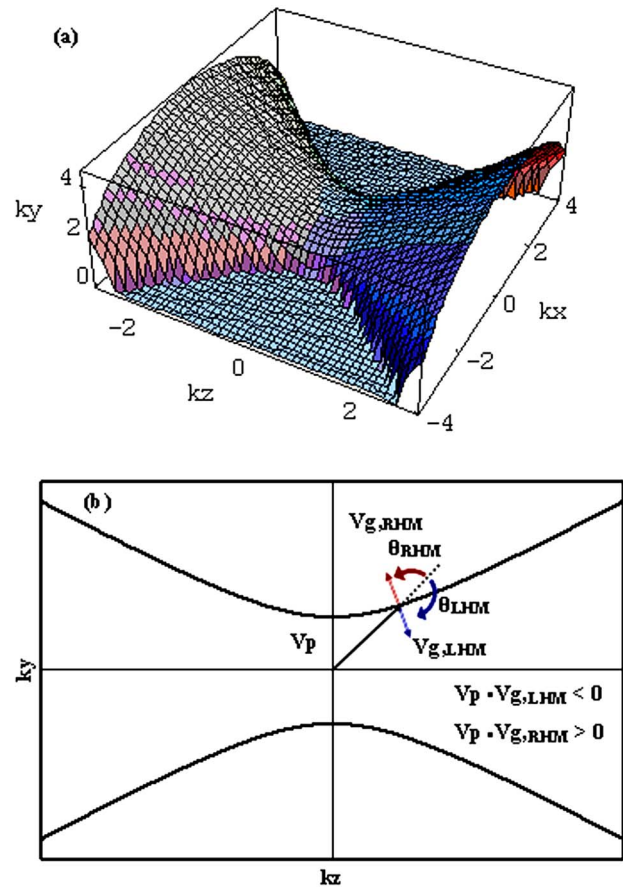


Fig. 3. (Color online) (a) 3D and (b) two-dimensional (2D) k -space diagrams for RHM and LHM one-sheeted hyperboloids. In both cases the media are uniaxial with $\epsilon_x = \epsilon_y = \epsilon_s$. In the RHM case the parameters are $\epsilon_s=-1$, $\epsilon_z=2$, and $\mu_r=1$. The parameters in the LHM case are $\epsilon_s=1$, $\epsilon_z=-2$, and $\mu_r=-1$. Because the parameters in the LHM case are simply the negatives of those in the RHM case, the k surfaces are identical. The angle between the phase velocity and group velocity for the RHM and LHM cases are shown on the 2D plots.

intersect so that the two types of waves cannot be distinguished. If we now negate all the parameters used to generate the k surface (i.e., $\epsilon_s=-1$, $\epsilon_z=2$, and $\mu_r=-1$), the resulting k -space diagram will be identical to that in Fig. 2. However, in this case waves propagating in the z direction will see a negative index (LHM) instead of a positive one whereas waves propagating in the x or y directions will again be cut off. The k surface in Fig. 2 is an example of a two-sheeted hyperboloid that results when the extraordinary wave in a uniaxial medium only propagates in the direction of the optical axis.

Figure 3 shows an example of a one-sheeted hyperboloid. This type of k surface results in a medium where only a wave with \mathbf{D} polarized in the direction of the optical axis can propagate. This implies that there is no propagating ordinary wave in this medium since, by definition, the \mathbf{D} vector for the ordinary wave must be polarized perpendicular to the optical axis. The k surface of Fig. 3 corresponds to an extraordinary wave for a medium with parameters $\epsilon_s=-1$, $\epsilon_z=2$, and $\mu_r=1$ in the RHM case, or $\epsilon_s=1$, $\epsilon_z=-2$, and $\mu_r=-1$ in the LHM case. In both cases, for propagation in the x and y directions, the ex-

traordinary wave is polarized along z where it sees a positive index in the RHM case or a negative index in the LHM case. For z -directed waves the extraordinary wave is polarized in the x - y plane. These waves are attenuated due to the fact that the permittivity and permeability have opposite signs. Table 1 summarizes the propagation behavior for the one- and two-sheeted hyperboloids.

As shown above, the k surface shapes do not change when the signs of the parameters ϵ_s , ϵ_z , and μ_r are reversed. In other words, the RHM surfaces are indistinguishable from the LHM surfaces for both the one- and the two-sheeted hyperboloids considered. Hence, if we are faced with the task of determining the sign of the index of refraction, the information provided by a visual inspection of these equifrequency surfaces alone is insufficient. To determine the sign of the index we must return to an analysis similar to that performed in Section 2, that is, we must examine the relationship between the phase and group-velocity vectors.

For both k surfaces a cut was taken in the k_y - k_z plane and the angle between the phase and group-velocity vectors was calculated as a function of k_z . Figure 4(a) shows the angles resulting from the two-sheeted surface of Fig. 2, and the angles calculated for the one-sheeted surface of Fig. 3 are shown in Fig. 4(b). For the RHM cases (solid curves) of both the one- and two-sheeted hyperboloids the angle between the phase velocity and group velocity begins at 0° at the onset of propagation ($k_z=1$ for the two-sheeted hyperboloid and $k_z=0$ for the one-sheeted hyperboloid) and approaches 90° asymptotical from below as k_z becomes large. On the other hand, in the LHM cases (dashed curves) the angle begins at 180° at the onset of propagation and approaches 90° asymptotical from above in the large k_z limit. Therefore, for both uniaxial media considered, the dot product between the unit vectors of the group velocity and phase velocity is always positive in the RHM cases and negative in the LHM cases. Hence, the LHM versions of the one- and two-sheeted hyperboloids both exhibit imperfect backward wave propagation. In addition, an analysis of the components of the phase and group velocities in the LHM cases for both structures shows that their k_y components are directed in the same direction and their backward wave behavior is therefore a result of oppositely directed k_z components. This emphasizes that imperfect backward wave behavior does not require all of the components of the phase and group velocities to be antiparallel.

In the above analysis the relationship between the phase velocity and group velocity in anisotropic LHM and RHM was discussed using equifrequency surfaces. In other words, the permittivity and permeability values were taken at a fixed frequency and were constant. A

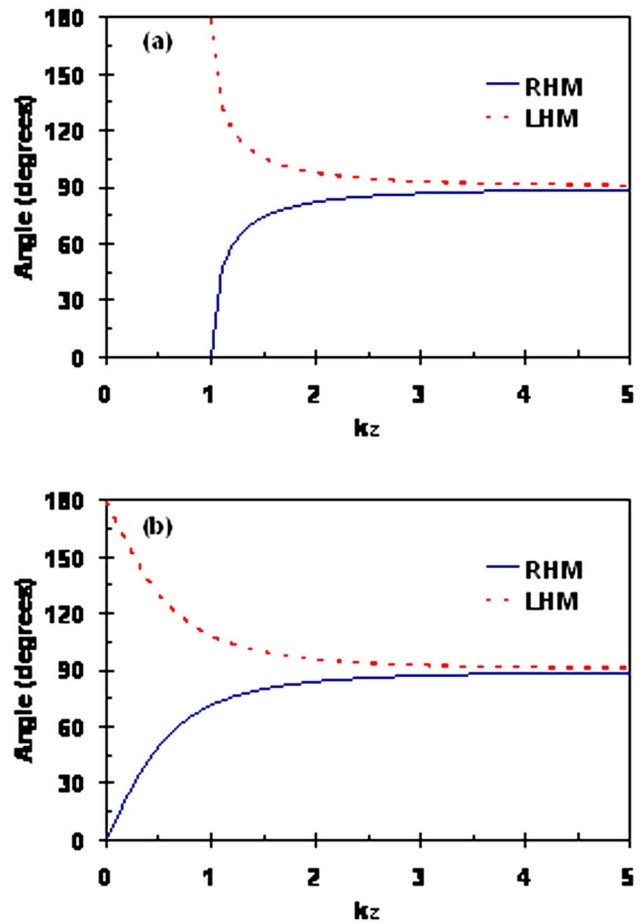


Fig. 4. (Color online) Calculated angle between the phase and group-velocity vectors for the RHM and LHM hyperboloidal k surfaces considered. (a) Two-sheeted hyperboloid (Fig. 2). (b) One-sheeted hyperboloid (Fig. 3).

question can then be asked in regard to the influence of temporal dispersion on the above analysis. To answer this we must keep the following in mind. First, the geometry of the k surface (an ellipsoid, a one-sheeted hyperboloid, a two-sheeted hyperboloid, or a null surface for which no propagation is allowed) is determined by the signs of the parameters ϵ and μ . Second, the dimensions of the k surfaces (i.e., the vertices of the ellipsoid or hyperboloid) are determined by the magnitudes of ϵ and μ . Therefore, if the parameters ϵ and μ are changing with frequency, instead of a single k surface there will be a family of k surfaces, each taken at a different frequency. In the frequency regions where the parameters do not change signs, each of these k surfaces will have the same geometry (i.e., ellipsoid, hyperboloid, no propagation) so that the sign of the dot product between the group velocity and phase velocity, and hence the general relationship between them (left or right handed), will not change sign. When one of the parameters changes signs, the geometry changes (for example, from a one-sheeted hyperboloid to a two-sheeted hyperboloid) and then a new relationship between the group and phase velocities emerges. This relationship is maintained until one of the parameters changes signs once again. Therefore, for the purposes of our analysis here, it is sufficient to analyze the general

Table 1. Index Seen by Different Polarizations in the Two- and One-Sheeted Hyperboloids

k Surface	Two-Sheeted		One-Sheeted	
Polarization	x - y plane	z (optical axis)	x - y plane	z (optical axis)
Index	± 1	$j\sqrt{2}$	j	± 2

behavior of the four possible geometries (a single ellipsoid, a one-sheeted hyperboloid, a two-sheeted hyperboloid, and a null surface) in the absence of dispersion, since the effects of the dispersion are only important to the extent that they can result in a change in the signs of the permittivity and permeability. The general case of an ellipsoid was considered in Section 2 and the one- and two-sheeted hyperboloids were examined in this section.

4. CONCLUSION

The problem of backward wave propagation was studied from a purely three-dimensional propagation point of view and two different types of backward waves were defined: perfect and imperfect. In a perfect backward wave the phase and group-velocity vectors are antiparallel (i.e., the angle between the vectors is 180°). In the second type, the imperfect backward wave, the dot product between the phase and group-velocity vectors is negative. Starting with the above relationships between the phase and group-velocity vectors, the forms of the indices necessary to produce these types of backward waves were obtained. In both cases, it was shown that backward wave behavior could only be the result of propagation through a negative index medium. In the case of the perfect backward wave, the index was necessarily negative and isotropic while for the imperfect backward wave the medium was anisotropic possessing a negative index along at least one of the three principal axes. Constraints on the functional form of the indices were then obtained using the condition that the group velocity must be positive in a transmission passband. Although at the moment the construction of left-handed media at optical frequencies remains a difficult task, such media could find use as beam-steering de-

vices and perfect optical lenses. In addition, for the anisotropic case, the polarization dependence of the media could be taken advantage of to make filters, polarization-dependent lenses, and beam-steering components.

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