A Non-Uniform Mesh High-Order Finite-Difference Time-Domain Method based on Biorthogonal Interpolating Functions

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***Abstract*— This paper addresses, for the first time, the question of non-uniform meshing within the context of a recently formulated high-order finite-difference technique that employs Deslauriers-Dubuc biorthogonal interpolating basis functions for the expansion of electromagnetic field components in space. To that end, a domain overriding approach is developed and shown to present a very low level of reflections at interfaces between meshes of different densities, enhancing the accuracy and versatility of the Deslauriers-Dubuc scheme (which lends itself to the modeling of complex, inhomogeneous media, unlike earlier similar methods), while providing a generic approach to embedding subgrids within high-order finite-difference meshes.**

***Index Terms*— FDTD, higher-order finite-differences, subgridding.**

I. Introduction

THE introduction of the Multiresolution Time Domain (MRTD) technique in [1], indicated a systematic process to formulate time-domain numerical techniques for Maxwell's equations with highly linear dispersion properties, overcoming a well-known disadvantage of the Finite-Difference Time-Domain (FDTD). These properties were achieved by using smooth basis functions for the expansion of field components in space, as opposed to rectangular pulse functions, implicitly employed by FDTD [2]. On the other hand, the use of smooth basis functions can render the modeling of inhomogeneous media computationally expensive, due to the overlap of neighboring basis functions over a small, albeit finite interval. This challenge was very effectively addressed in [3], through the choice of a biorthogonal interpolating basis, formulated by Deslauriers and Dubuc, for the expansion of field components. A salient feature of these even symmetric basis functions is that their value is zero at all integers except for zero ($\phi(n) = \delta_{n,0}$, where ϕ is a Deslauriers-Dubuc scaling function and δ denotes Kronecker's delta). Consider, for simplicity, a one-dimensional case of wave propagation along the $z-$axis, and fields E_x and H_y. In the field expansions:

$$
\begin{aligned}
E_x(z,t) &= \sum_{n,i} {}_nE_i \phi\left(\frac{z}{\Delta z} - i\right) h\left(\frac{t}{\Delta t} - n\right) \\
H_y(z,t) &= \sum_{n,i} {}_{n'}H_{i'} \phi\left(\frac{z}{\Delta z} - i'\right) h\left(\frac{t}{\Delta t} - n'\right)
\end{aligned}
\tag{1}
$$

where $i' = i + 1/2$, $n' = m + 1/2$ and h is a rectangular pulse function, the coefficients ${}_nE_i$ (${}_{n'}H_{i'}$) represent the actual values of the electric (magnetic) field component E_x (H_y) at $z = i\Delta z$ ($z' = i'\Delta z$) and $t = n\Delta t$ ($t' = n'\Delta t$) and hence Deslauriers-Dubuc schemes are said to possess an *exact interpolating property*. In addition, their field update equations incorporate localized material properties, in just as straightforward manner as FDTD. For example, the electric field update equation is:

$$
\begin{aligned}
{}_{n+1}E_i^x &= \frac{1 - \sigma_i \Delta t / 2\epsilon_i}{1 + \sigma_i \Delta t / 2\epsilon_i} \, {}_nE_i^x \\
&- \frac{\Delta t / \epsilon_i \Delta z}{1 + \sigma_i \Delta t / 2\epsilon_i} \sum_{p=-L_s}^{L_s - 1} \alpha(p) \, {}_{n+\frac{1}{2}}H^y_{i+p+\frac{1}{2}}
\end{aligned}
\tag{2}
$$

where $\alpha(p)$ are the connection coefficients (given in [3]) and L_s the stencil of the scheme [1], [3], while ϵ_i and σ_i are the dielectric permittivity and conductivity of the simulated medium at $z = i\Delta z$. However, (2) also implies that the scheme cannot capture material variations at a scale smaller than the cell size (which the linear numerical dispersion of the method allows to be relatively large), in contrast to earlier schemes (for example, [1], [4]).

Mesh refinement, through the introduction of wavelets (as in [5]), can potentially overcome this problem. Yet, it would add significant complexity to the scheme, also eliminating the exact interpolating property that originally motivated its use. Alternatively, a non-uniform mesh can be employed, modifying the field expansion (1) into:

$$
E_x(z,t) = \sum_k \sum_{n,i \in \mathcal{R}_k} {}_nE_i^{\phi} \phi\left(\frac{z}{\Delta z_k} - i\right) h\left(\frac{t}{\Delta t} - n\right)
\tag{3}
$$

where different cell sizes Δz_k have been used in subgrids $\mathcal{R}_k$. Interestingly, despite the large volume of research on subgridding for the conventional FDTD method [6]–[8], detailed studies of their performance [9] and the recent demonstration of adaptively moving subgrids [10], mesh refinement either static or dynamic in the context of high-order MRTD techniques has remained unexplored. In the latter, the use of multi-point finite-differences significantly complicates the enforcement of boundary conditions between subgrids and renders this problem theoretically

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intriguing, in addition to its interest from an application point of view.

This paper is aimed at proposing a systematic methodology to address this question. The second order Deslauriers-Dubuc (DD_2) scheme of [3] (which employs *fourth* order spatial finite differences) is used to demonstrate the formulation of the proposed technique and to perform its numerical evaluation. A domain overriding approach, partially assisted by the exact interpolating property of the Deslauriers-Dubuc functions is developed and shown to present a robust solution to this problem.

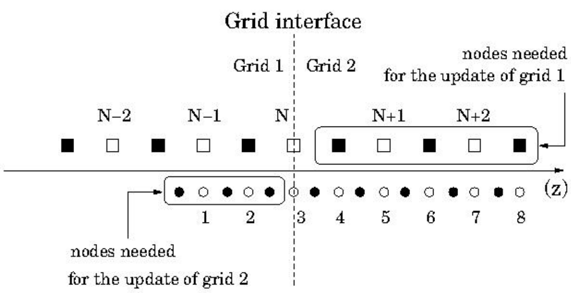


Fig. 1. Interface between two DD_2 meshes of cell sizes Δz (grid 1) and $\Delta z/2$ (grid 2); □ : electric field node in grid 1; ■ : magnetic field node in grid 1; ○ : electric field node in grid 2; • : magnetic field node in grid 2.

II. Implementation of Subgridding: A Domain Overriding Approach

A. Methodology

In this section, the proposed implementation of a 2:1 subgrid in a DD_2 scheme will be presented in detail for a one-dimensional problem (with electric field $E_x(z,t)$ and magnetic field $H_y(z,t)$), to clarify its most important aspects. Then, its generalization to two/three dimensions, as well as higher mesh refinement ratios, will be discussed. In the following, ϕ denotes the second order Deslauriers-Dubuc scaling function.

For the DD_2 scheme, the stencil $L_s = 3$. From (2), the update of an electric (magnetic) field component requires three magnetic (electric) field nodes in the backward and three in the forward direction. Fig. 1 depicts the case of two meshes matched at an electric field node at $z = N\Delta z$, along with the position of the nodes of each mesh inside the other that are needed for the update equations of boundary points, along the stencil of the scheme, to be carried out. Electric nodes in both grids have been numbered, with N being the total number of interior cells in the first grid. The magnetic nodes are half a cell of the corresponding grid away and numbered accordingly.

First, the computation of the electric fields of grid 1 within grid 2 is considered. By the exact interpolating property of the basis functions of the DD_2 scheme and the fact that these nodes are collocated with grid 2 nodes, ${}_kE^{I,x}_{N+l} = {}_kE^{II,x}_{3+2l}$, $l = 1, 2$, where the superscripts I, II indicate that the field coefficients belong to grid 1 or 2, respectively. Now, consider the magnetic field node of grid 1 at $z = (N+0.5)\,\Delta z$, ${}_{k+\frac{1}{2}}H^{I,y}_{N+\frac{1}{2}} \equiv H_y\left((N+0.5)\Delta z, (k+0.5)\Delta t\right)$. From the magnetic field expansion (1) in grid 2 (with cell size $\Delta z_2 = \Delta z/2$):

$$\begin{aligned} {}_{k+\frac{1}{2}}H^{I,y}_{N+\frac{1}{2}} &= \phi(1/2)\left({}_{k+\frac{1}{2}}H^{II,y}_{3+\frac{1}{2}} + {}_{k+\frac{1}{2}}H^{II,y}_{4+\frac{1}{2}}\right) \\ &+ \phi(3/2)\left({}_{k+\frac{1}{2}}H^{II,y}_{2+\frac{1}{2}} + {}_{5+\frac{1}{2}}H^{II,y}_{4+\frac{1}{2}}\right) \end{aligned} \tag{4}$$

where the weight terms $\phi(1/2)$ and $\phi(3/2)$ are present because the magnetic field nodes they correspond to are half and one and a half cell of grid 2 away, respectively, from the point where the field is calculated. Note that $\phi(2.5) = 0$ and therefore no more terms are involved in (4). Yet, one of the terms in (4), ${}_{k+\frac{1}{2}}H^{II,y}_{2+\frac{1}{2}}$ is among the nodes of grid 2 that are within grid 1, and therefore determined by the fields of grid 1. The relevant expression (deduced from the magnetic field expansion in grid 1) is:

$$\begin{aligned} &{}_{k+\frac{1}{2}}H^{II,y}_{2+\frac{1}{2}} = \phi(9/4)\,{}_{k+\frac{1}{2}}H^{I,y}_{N-3+\frac{1}{2}} + \\ &\phi(5/4)\,{}_{k+\frac{1}{2}}H^{I,y}_{N-2+\frac{1}{2}} + \phi(1/4)\,{}_{k+\frac{1}{2}}H^{I,y}_{N-1+\frac{1}{2}} + \\ &\phi(3/4)\,{}_{k+\frac{1}{2}}H^{I,y}_{N+\frac{1}{2}} + \phi(7/4)\,{}_{k+\frac{1}{2}}H^{I,y}_{N+1+\frac{1}{2}} \end{aligned} \tag{5}$$

The latter makes clear that ${}_{k+\frac{1}{2}}H^{II,y}_{2+\frac{1}{2}}$ also depends on ${}_{k+\frac{1}{2}}H^{I,y}_{N+\frac{1}{2}}$ (the reverse is also true by (4)) and therefore, the interface of Fig. 1 with FDTD-type interpolations of boundary nodes cannot work in this case. This observation

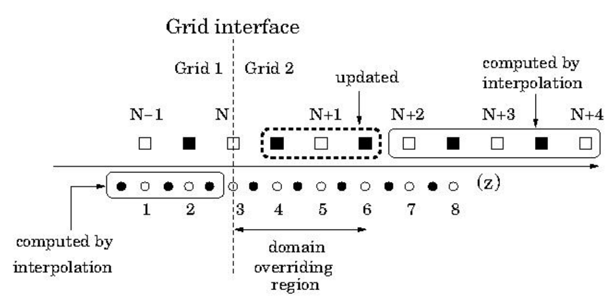


Fig. 2. The proposed method for connecting two DD_2 meshes of cell sizes Δz (grid 1) and $\Delta z/2$ (grid 2); the symbols are as in Fig. 1.

leads to the idea of employing *domain overriding*, which is based on the definition of nodes of grid 1, within the grid 2 region, that are *updated* via regular update equations, instead of interpolated. Fig. 2 describes the proposed method, that entails the definition of one additional electric and two magnetic nodes of grid 1 within the grid 2 region, indicating the area where field update equations are performed in both grids. Again, these updates are possible by the addition of L_s more nodes that are interpolated

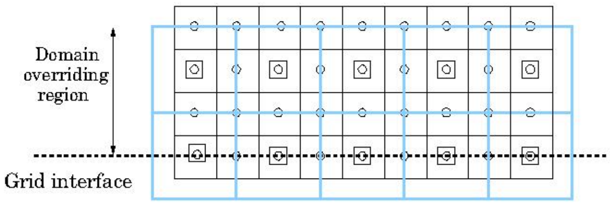


Fig. 3. Interface between and domain overriding region of two DD_2 meshes (mesh refinement ratio 2:1) in two-dimensions (TE case); $\square$: normal electric field node in grid 1; $\circ$: normal electric field node in grid 2.

from grid 2. For example:

$$\begin{aligned} {}_{k+\frac{1}{2}}H_s^{I,y} &= \frac{1-\sigma_s^*\Delta t/2\mu_s}{1+\sigma_s^*\Delta t/2\mu_s}\,{}_{k-\frac{1}{2}}H_s^{I,y} \\ &- \frac{\Delta t/\mu_s\Delta z}{1+\sigma_s^*\Delta t/2\mu_s}\sum_{p=-3}^{2}\alpha(p)\,{}_kE_{N+2+p}^{I,x} \end{aligned} \tag{6}$$

with $s = N+1+\frac{1}{2}$ and σ^* the magnetic conductivity. Also, ${}_kE_{N+2}^{I,x} = {}_kE_7^{II,x}$, ${}_kE_{N+3}^{I,x} = {}_kE_9^{II,x}$, ${}_kE_{N+4}^{I,x} = {}_kE_{11}^{II,x}$. Therefore, no extra operations are needed to compute these values. For the magnetic field nodes, interpolation equations similar to (4) are applied. The difference now is that due to the presence of the overriding region, they do not involve interpolated nodes of grid 1.

The width of the overriding region, which includes up to $H_{N+1+\frac{1}{2}}^{I,y}$ is defined to ensure that the interpolation of all electric and magnetic nodes of grid 2 within grid 1 will involve updated (and *not* interpolated) nodes of grid 1 and vice versa. It is easy to show that if this is guaranteed for the grid of the larger cell size, it automatically holds for the refined grid as well. In general, if a mesh refinement ratio q had been considered and a DD_p scheme had been employed, the width of the overriding region would be less or equal to:

$$(2p-1-1/2q)\Delta z \tag{7}$$

in each direction. Note that in the case of Fig. 2, the width is $1.25\Delta z < 2.75$, because $\phi(2.25) = \phi(2.75) = 0$. As a result, nodes beyond $H_{N+1+\frac{1}{2}}^{I,y}$ do not contribute to the interpolated fields of grid 2.

In the FDTD subgridding literature, the question of how the order of interpolations may affect stability and accuracy has been repeatedly considered [7], [9]. Note that in the case of the Deslauriers-Dubuc function based methods, the field expansions of the form of (1) are themselves high-order interpolation formulas and naturally provide for the enforcement of boundary conditions at subgrid interfaces.

B. Calculation of field interpolation coefficients

Applying (1) to compute field values at arbitrary points (such as in (5)) requires the computation of ϕ at several rational numbers. For dyadic mesh refinements, this computation can be accomplished at a pre-processing stage, by means of the dilation equation satisfied by ϕ:

$$\phi(x) = \sum_{k=0,\pm1,\pm3} h_k\phi(2x-k) \tag{8}$$

where h_k are the filter coefficients used for the construction of ϕ [3], [11]. Recursively applying this formula, the values of ϕ at arbitrary dyadic rationals can be efficiently deduced.

C. Generalization to higher dimensions

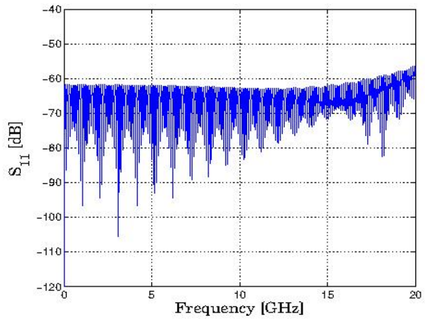
The extension of the proposed method to higher dimensions is straightforward. Fig. 3 presents a two-dimensional TE extension of the domain overriding technique (only normal electric field nodes in the overriding region are shown). Again, collocation of electric field nodes, albeit not necessary, has been pursued. Implementing this technique goes through the identification and storage of grid points involved in boundary updates and the computation of the interpolation coefficients. The cost of the interface is low even in two and three dimensions, since the operations are limited to boundary points.

III. Numerical Results

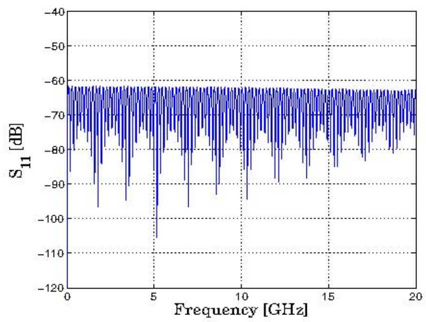
In this section, simulation results for a one-dimensional 2:1 DD_2 subgrid are shown. A Gaussian pulse of the form $\exp(-((t-t_0)/T_s)^2)$, with $T_s = 25$ ps, $t_0 = 3T_s$ excites a domain of 2000 cells of cell size Δ, interfaced to 2000 cells of cell size $\Delta/2$. The domain is terminated in 16-cell matched absorbers at both ends. A global time step, corresponding to one half of the Courant limit of the dense mesh [3], is used. Multiple cell sizes have been tested. Fig. 4 presents the reflection coefficient S_{11} at the interface between the two meshes for Δ= 0.125 and 0.075 cm. The behavior shown is typical of all numerical experiments that have been performed; reflections are below -60 dB. The practically reflection less propagation of the pulse is also shown in Fig. 5. Since these results have been derived with finite domains, reflections from the absorbers have also contributed to the already low observed level of reflections. It is noted though that the results do not include the angular dependence of the interface reflections, which is the most important question to be studied next. Finally, the late-time behavior of the DD_2 subgrid has been studied over millions of time-steps. No late-time instability has been identified. Fig. 6 shows the typical behavior of the electric field (when Δ= 0.125 cm), at 2×10^6 time steps.

IV. Conclusions

A domain overriding approach has been proposed for the implementation of subgridding in the Deslauriers-Dubuc based finite-difference time-domain schemes. Numerical experiments have demonstrated a low level of reflections at grid interfaces, as a result of the inherent capability of the Deslauriers-Dubuc technique to support high



(a) Δ = 0.125 cm



(b) Δ = 0.075 cm

Fig. 4. Reflection coefficient as a function of frequency at a 2:1 DD_2 grid interface, implemented with domain overriding, for two cell sizes.

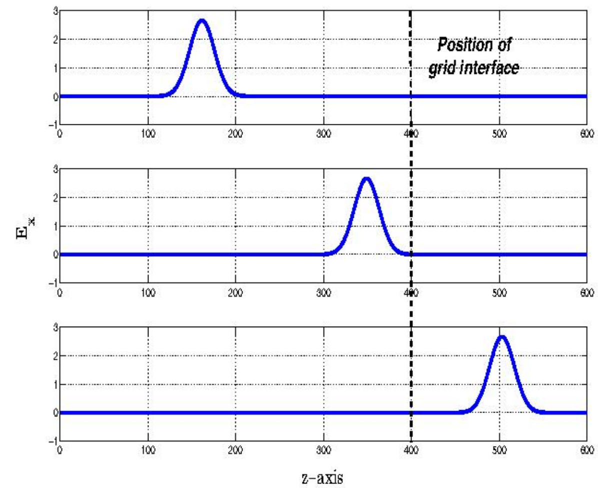


Fig. 5. Gaussian pulse propagation through the 2:1 DD_2 grid interface (Δ = 0.125 cm). Dashed line indicates the position of the interface.

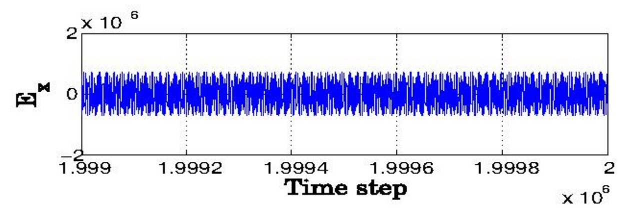


Fig. 6. Stable late time behavior of the 2:1 DD_2 grid interface.

order field interpolations that are employed to connect adjacent grids of different cell size by enforcing field boundary conditions. Hence, while the domain overriding approach is generic and potentially useful for the realization of subgrids in high-order time-domain methods such the FDTD(2,4) [12], it is particularly well-suited to the Deslauriers-Dubuc scheme. Current work focuses on convergence and accuracy studies of the proposed method (similar to [9]) and applications related to the modeling of surface resonances developing at boundaries between positive and negative index media, a phenomenon that requires highly accurate, yet localized, finite-difference computations.

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