# Array Directivity

## 1 Broadside Array

For the case where $d \ll \lambda$, the normalized array factor is

$$(AF)_n \approx \frac{\sin\left(\frac{N}{2}kd\cos\theta\right)}{\frac{N}{2}kd\cos\theta} \tag{1}$$

and the corresponding radiation intensity for an array of isotropic elements is

$$U(\theta) = [(AF)_n]^2. \tag{2}$$

The average radiation intensity is

$$U_0 = \frac{P_{\text{rad}}}{4\pi} = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\left[\frac{\sin\left(\frac{N}{2}kd\cos\theta\right)}{\frac{N}{2}kd\cos\theta}\right]^2 \sin\theta d\theta. \tag{3}$$

Employing the substitution

$$Z = \frac{N}{2}kd\cos\theta \tag{4a}$$

$$dZ = -\frac{N}{2}kd\sin\theta d\theta \tag{4b}$$

we can write

$$U_0 = \frac{1}{2}\cdot\frac{-2}{Nkd}\int_{\frac{Nkd}{2}}^{-\frac{Nkd}{2}}\left[\frac{\sin Z}{Z}\right]^2 dZ \approx \frac{1}{Nkd}\int_{-\infty}^{\infty}\left[\frac{\sin Z}{Z}\right]^2 dZ \tag{5}$$

if $Nkd$ is large (the array is long). In this case

$$\int_{-\infty}^{\infty}\left[\frac{\sin Z}{Z}\right]^2 dZ = \pi \tag{6}$$

and

$$U_0 \approx \frac{\pi}{Nkd}, \tag{7}$$

$$D_0 = \frac{U_{\text{max}}}{U_0} = U_0^{-1} = \frac{Nkd}{\pi} = 2N\frac{d}{\lambda}. \tag{8}$$

## 2 Ordinary Endfire Array

Here we consider the case where $\theta_0 = 0^\circ$ or $\alpha = -kd$. The AF can then be written

$$(AF)_n \approx \frac{\sin\left[\frac{N}{2}kd(\cos\theta - 1)\right]}{\frac{N}{2}kd(\cos\theta - 1)} \tag{9}$$

with corresponding average radiation intensity

$$U_0 = \frac{1}{2}\int_0^\pi \left[\frac{\sin\left[\frac{N}{2}kd(\cos\theta - 1)\right]}{\frac{N}{2}kd(\cos\theta-1)}\right]^2 \sin\theta d\theta. \tag{10}$$

To evaluate this, we use a similar substitution as for the broadside case,

$$Z = \frac{N}{2}kd(\cos\theta - 1) \tag{11a}$$

$$dZ = -\frac{N}{2}kd\sin\theta d\theta \tag{11b}$$

so that

$$U_0 = -\frac{1}{Nkd}\int_0^{-Nkd}\left[\frac{\sin Z}{Z}\right]^2 dZ \approx \frac{1}{Nkd}\int_0^\infty \left[\frac{\sin Z}{Z}\right]^2 dZ \tag{12}$$

if $Nkd$ is large. $U_0$ is now half the value as it was for the broadside case, so that

$$D_0 \approx 4N\frac{d}{\lambda}. \tag{13}$$

# 3 Hansen-Woodyard Endfire Array

The Handsen-Woodyard array was derived to have an average radiation intensity given by

$$U_0 = \frac{1}{2}\left\{\frac{q(k-p)}{\sin[q(k-p)]}\right\}^2 \int_0^\pi \left\{\frac{\sin[q(k\cos\theta - p)]}{q(k\cos\theta - p)}\right\}^2 \sin\theta d\theta. \tag{14}$$

This quantity is minimized when $q(k-p) = -1.46)$. Under this condition,

$$U_0 = \frac{0.871}{Nkd}, \tag{15}$$

and

$$D_0 = \frac{Nkd}{0.871} = \frac{1}{0.554}\left(\frac{2Nkd}{\pi}\right) = 1.805\left[4N\left(\frac{d}{\lambda}\right)\right] \tag{16}$$

which is 1.8 times the directivity of an ordinary endfire array.