Green's Functions in Radiation Problems

In solving general antenna radiation problems using potentials, the crux of the problem reduces to solving the inhomogeneous vector wave equation, given by

$$(\boldsymbol{\nabla}^2 + k^2)\boldsymbol{A}(\boldsymbol{r}) = -\mu \boldsymbol{J}(\boldsymbol{r}). \tag{1}$$

Let us solve the scalar equation

$$(\boldsymbol{\nabla}^2 + k^2)G(\boldsymbol{r}, \boldsymbol{r}') = -\delta(\boldsymbol{r} - \boldsymbol{r}'), \tag{2}$$

where $G(\boldsymbol{r}, \boldsymbol{r'})$ is the Green's function of this differential equation. The general vector solution[1] can then be expressed as

$$\boldsymbol{A}(\boldsymbol{r}) = \mu \iiint_v G(\boldsymbol{r}, \boldsymbol{r'})\boldsymbol{J}(\boldsymbol{r}')dv'. \tag{3}$$

We seek forms of G for specific radiation problems. To begin, let us consider the integral form of the Green's functions.

1 Green's Functions in Integral Form

To begin, let's construct the Green's function to a simple one-dimensional scalar problem, defined by

$$\frac{d^2\phi}{dx^2} + k^2\phi = f(x). \tag{4}$$

subject to the radiation condition,

$$\phi(x \to \infty) = \phi(x \to -\infty) = 0. \tag{5}$$

The Green's function for this case will satisfy

$$\frac{d^2G(x, x')}{dx^2} + k_0^2 G(x, x') = -\delta(x - x'). \tag{6}$$

Considering the homogenous version of this equation, it appears the source is radiating into an unbounded medium. Therefore, the most straightforward choice of solutions (eigenfunctions) are traveling waves given by

$$\psi(x, x') = \begin{cases} C(x')e^{-jkx} & x > x' \\ C(x')e^{+jkx} & x < x' \end{cases}, \tag{7}$$

since traveling-wave solutions satisfy the radiation condition. This is most obvious if one considers the case of lossy media, in which case the radition condition is met: the first solution satisfies the radiation condition at $x = \infty$ while the second satisfies it at $x = -\infty$. Let's focus on waves travelling in the $+x$ direction, so that

$$\psi(x, x') = C(x')e^{-jkx} \tag{8}$$

and $C(x)$ represents the amplitude of the traveling (plane) wave.

[1] This is a consequence of the fact that the free-space Green's function does not depend on the direction of $\boldsymbol{J}$.

Figure 1: 1D Green's function problem

Next, let us represent the Green's function via the Fourier transformation

$$G(x,x') = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{G}(k,x')e^{-jkx}dk, \tag{9}$$

whose Fourier transform pair is

$$\tilde{G}(k,x') = \int_{-\infty}^{\infty} G(x,x')e^{+jkx}dx. \tag{10}$$

In this Fourier analysis, $G(x,x')$ can be seen as a continuous spectrum of plane waves of the form of (8) whose amplitudes are given by $\tilde{G}(k,x')$. Let us compute the Fourier transform of the $\delta(x-x')$ term as

$$\tilde{\delta}(k,x') = \int_{-\infty}^{\infty} \delta(x-x')e^{+jkx}dx = e^{+jkx'}. \tag{11}$$

Taking the inverse Fourier transform,

$$\delta(x-x') = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left(e^{+jkx'}\right) e^{-jkx}dk. \tag{12}$$

Substituting (9) and (12) into (6), we obtain

$$\begin{aligned}
\frac{d}{dx^2}\frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{G}(k,x')e^{-jkx}dk + k_0^2\frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{G}(k,x')e^{-jkx}dk &= -\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{1}{2\pi}e^{+jkx'}\right)e^{-jkx}dk \\
\frac{1}{2\pi}\int_{-\infty}^{\infty}(-k^2+k_0^2)\tilde{G}(k,x')e^{-jkx}dk &= -\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{1}{2\pi}e^{+jkx'}\right)e^{-jkx}dk \\
\int_{-\infty}^{\infty}\left[(k_0^2-k^2)\tilde{G}(k,x') + e^{+jkx'}\right]e^{-jkx}dk &= 0
\end{aligned} \tag{13}$$

This equation is satisfied if

$$\tilde{G}(k,x') = -\frac{e^{+jkx'}}{k_0^2-k^2} = \frac{e^{+jkx'}}{k^2-k_0^2}. \tag{14}$$

(9) then becomes

$$\begin{aligned}
G(x,x') &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{+jkx'}}{k^2-k_0^2}e^{-jkx}dk \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-jk(x-x')}}{k^2-k_0^2}dk
\end{aligned} \tag{15}$$

To carry out the integral, we first observe that there are two poles in the integrand. Since the medium is slightly lossy, $k_0 = k_r + jk_i,\ k_i > 0$, this leads to the pole locations shown. We can adjust the contour as shown in Figure 2(a) to work around the poles and apply residue calculus to evaluate the integrals. The contour Γ_2 is located on a circular arc at an infinite radius from the origin. Note the figure is for $x > x'$, in order to yield physically meaningful solutions that satisfy the radiation condition.

$$\int_{-\infty}^{\infty} \frac{e^{-jk(x-x')}}{k^2 - k_0^2} dk = 2\pi j \sum (\text{residues of enclosed poles}) \tag{16}$$

$$G(x, x') = j \sum (\text{residues of enclosed poles}) - \frac{1}{2\pi} \int_{\Gamma_2} \frac{e^{-jk(x-x')}}{k^2 - k_0^2} dk \tag{17}$$

Since the contribution to the integral along Γ_2 is zero,

$$G(x, x') = j \sum (\text{residues of enclosed poles}). \tag{18}$$

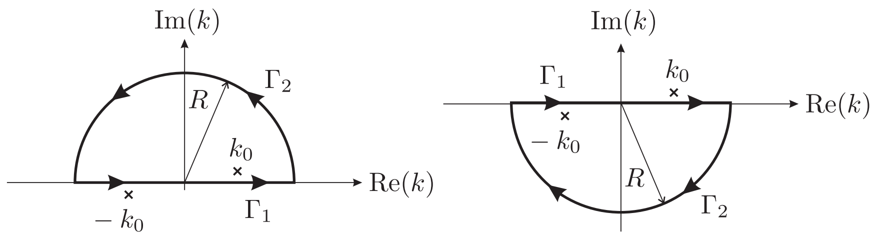

Figure 2: Integration contours in the complex k-plane

For $k = k_0$, the integrand represents a plane wave traveling in the $+x$ direction, while for $k = -k_0$, the integrand represents a plane wave traveling in the $-x$ direction. Since we are concerned with the $+x$ traveling wave, we only want the former to contribute. For this reason, the asymmetric contour path shown in Figure 2(a) is chosen to exclude the undesired pole. To compute the residues, recall that

$$\text{Res}(f, c) = \frac{1}{2\pi j} \int_{\gamma} f(z) dz = \lim_{z \to c} (z - c) f(z) \tag{19}$$

for simple poles, where γ is a contour that encircles the pole in a counterclockwise fashion. For $x > x'$,

$$\begin{aligned} G(x, x') &= j[\text{residue}(k = +k_0)] = j[\lim_{k \to +k_0} \frac{(k - k_0) e^{-jk(x-x')}}{(k - k_0)(k + k_0)}] \\ &= j \frac{e^{-jk_0(x-x')}}{2k_0}. \end{aligned} \tag{20}$$

Note that for $x < x'$, we use the contour in Figure 2(b) and obtain a very similar solution except the sign in the exponent differs,

$$G(x, x') = j \frac{e^{+jk_0(x-x')}}{2k_0}, \tag{21}$$

so the most general solution can be expressed for any sign of $x - x'$ as

$$G(x, x') = j \frac{e^{-jk_0|x-x'|}}{2k_0}. \tag{22}$$

2 Free Space Green's Function

In many antenna radiation problems, the antenna acts in free space leading to a very simple Green's function. Here way derive the Green's function by extending past the simple 1D case just considered. We will work in spherical coordinates for this problem. The three-dimensional free-space Green's function $G(\boldsymbol{r}|\boldsymbol{r}')$ satisfies

$$\boldsymbol{\nabla}^2 G(\boldsymbol{r},\boldsymbol{r}') + k_0^2 G(\boldsymbol{r},\boldsymbol{r}') = -\delta(\boldsymbol{r}-\boldsymbol{r}'). \tag{23}$$

Taking the three-dimensional Fourier Transform of both sides,

$$\iiint_{-\infty}^{\infty} (\boldsymbol{\nabla}^2 + k_0^2) G(\boldsymbol{r},\boldsymbol{r}') e^{j(k_x x + k_y y + k_z z)} dxdydz = -\iiint_{-\infty}^{\infty} \delta(\boldsymbol{r}-\boldsymbol{r}') e^{j(k_x x + k_y y + k_z z)} dxdydz. \tag{24}$$

Since $\delta(\boldsymbol{r}-\boldsymbol{r}') = \delta(x-x')\delta(y-y')\delta(z-z')$, the right hand side becomes

$$-\iiint_{-\infty}^{\infty} \delta(\boldsymbol{r}-\boldsymbol{r}') e^{j(k_x x + k_y y + k_z z)} dxdydz = -e^{j\boldsymbol{k}\cdot\boldsymbol{r}'}. \tag{25}$$

Meanwhile, in an analogous fashion to the 1D case, the left-hand side becomes

$$\iiint_{-\infty}^{\infty} (\boldsymbol{\nabla}^2 + k_0^2) G(\boldsymbol{r},\boldsymbol{r}') e^{j\boldsymbol{k}\cdot\boldsymbol{r}} dxdydz = (-k^2 + k_0^2) g_0(\boldsymbol{k},\boldsymbol{r}') \tag{26}$$

where $g_0(\boldsymbol{k},\boldsymbol{r}')$ is the three-dimensional Fourier Transform of $G(\boldsymbol{r},\boldsymbol{r}')$. Therefore, combining the left and right sides,

$$(-k^2 + k_0^2) g_0(\boldsymbol{k},\boldsymbol{r}') = -e^{j\boldsymbol{k}\cdot\boldsymbol{r}'} \Rightarrow g_0(\boldsymbol{k},\boldsymbol{r}') = -\frac{e^{j\boldsymbol{k}\cdot\boldsymbol{r}'}}{k_0^2 - k^2} \tag{27}$$

Note the similarity to (14). Taking the inverse Fourier Transform,

$$\begin{aligned} G(\boldsymbol{r},\boldsymbol{r}') &= -\frac{1}{(2\pi)^3} \iiint_{-\infty}^{\infty} \frac{e^{j\boldsymbol{k}\cdot\boldsymbol{r}'}}{k_0^2 - k^2} e^{-j\boldsymbol{k}\cdot\boldsymbol{r}} dk_x dk_y dk_z \\ &= \frac{1}{(2\pi)^3} \iiint_{-\infty}^{\infty} \frac{e^{-j\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{r}')}}{k^2 - k_0^2} dk_x dk_y dk_z = G(\boldsymbol{r}-\boldsymbol{r}') \end{aligned} \tag{28}$$

We now transform to spherical coordinates and carry out the integration over the sphere. We observe that the result of the integration does not depend on the orientation of the sphere, which is established by the difference vector $\boldsymbol{r}-\boldsymbol{r}'$. Therefore, this integral is obviously easier if we define $\boldsymbol{R} = \boldsymbol{r}-\boldsymbol{r}'$ so that

$$\begin{aligned} G(\boldsymbol{r}-\boldsymbol{r}') = G(\boldsymbol{R}) &= \frac{1}{(2\pi)^3} \int_0^{2\pi} \int_0^{\pi/2} \int_{-\infty}^{\infty} \frac{e^{-j\boldsymbol{k}\cdot\boldsymbol{R}}}{k^2 - k_0^2} k^2 \sin\theta dk d\theta d\phi \\ &= \frac{1}{(2\pi)^3} \int_0^{2\pi} \int_0^{\pi/2} \int_{-\infty}^{\infty} \frac{e^{-jkR\cos\theta}}{k^2 - k_0^2} k^2 \sin\theta dk d\theta d\phi \end{aligned} \tag{29}$$

Note that rather than integrating k from 0 to ∞ and θ from 0 to π, we have instead integrated k from $-\infty$ to ∞ and θ from 0 to $\pi/2$. This way, we can proceed with the integration over the poles in an identical fashion as the 1D case considered previously, with full control over the k contour. There is no azimuthal variation, so

$$G(\boldsymbol{R}) = \frac{2\pi}{(2\pi)^3} \int_0^{\pi/2} \int_{-\infty}^{\infty} \frac{e^{-jkR\cos\theta}}{k^2 - k_0^2} k^2 \sin\theta dk d\theta \tag{30}$$

Integration over θ is also straightforward,

$$\begin{aligned} G(\boldsymbol{R}) &= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\frac{1}{jkR}\left.\frac{e^{-jkR\cos\theta}}{k^2-k_0^2}\right|_0^{\pi/2}k^2dk \\ &= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\frac{1}{jR}\frac{1-e^{-jkR}}{k^2-k_0^2}kdk \\ &= \frac{1}{jR(2\pi)^2}\int_{-\infty}^{\infty}\frac{k}{k^2-k_0^2}dk-\frac{1}{jR(2\pi)^2}\int_{-\infty}^{\infty}\frac{ke^{-jkR}}{k^2-k_0^2}dk \\ &= -\frac{1}{jR(2\pi)^2}\int_{-\infty}^{\infty}\frac{ke^{-jkR}}{k^2-k_0^2}dk \end{aligned} \tag{31}$$

The last step follows because of the odd symmetry of the first integral. Also, for integration purposes, it is slightly more convenient if we can express the integrand such that as we integrate from $-\infty$ to ∞, the argument to the exponential in the integrand also increases positively. This changes the sign of the resulting integral:

$$G(\boldsymbol{R}) = -\frac{1}{jR(2\pi)^2}\int_{-\infty}^{\infty}\frac{ke^{-jkR}}{k^2-k_0^2}dk = \frac{1}{jR(2\pi)^2}\int_{-\infty}^{\infty}\frac{ke^{jkR}}{k^2-k_0^2}dk \tag{32}$$

For the final integration over k, we use the similar contour as discussed in the 1D case and residue theory. If I_1 denotes the integrand, for the pole at $k=-k_0$ included in the contour to yield a radially outward propagating solution,

$$\text{Res}(I_1,k_0) = \lim_{k\to -k_0}\frac{(k+k_0)ke^{jkR}}{(k-k_0)(k+k_0)} = \frac{e^{-jk_0R}}{2} \tag{33}$$

Therefore,

$$G(\boldsymbol{R}) = \frac{1}{jR(2\pi)^2}\cdot 2\pi j\text{Res}(I_1,k_0) = \frac{e^{-jk_0R}}{4\pi R} \tag{34}$$

or more generally,

$$G(\boldsymbol{r},\boldsymbol{r}') = \frac{e^{-jk_0|\boldsymbol{r}-\boldsymbol{r}'|}}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|}. \tag{35}$$

Returning to (23), the solution for $\boldsymbol{A}$ can be expressed as

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu}{4\pi}\iiint_v \boldsymbol{J}(\boldsymbol{r}')\frac{e^{-jk_0|\boldsymbol{r}-\boldsymbol{r}'|}}{|\boldsymbol{r}-\boldsymbol{r}'|}dv'. \tag{36}$$

or more compactly as

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu}{4\pi}\iiint_v \boldsymbol{J}(\boldsymbol{r}')\frac{e^{-jk_0R}}{R}dv'. \tag{37}$$

Extending the solution process to the vector electric potential,

$$\boldsymbol{F}(\boldsymbol{r}) = \frac{\epsilon}{4\pi}\iiint_V \boldsymbol{M}(\boldsymbol{r}')\frac{e^{-jk_0R}}{R}dv'. \tag{38}$$

Finally, there are cases where the source distribution is not represented as a surface density (m^{-2}), but rather as either a linear density (m^{-1}), or directly as an electric of magnetic current. For linear densities,

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu}{4\pi}\iint_S \boldsymbol{J}_s(\boldsymbol{r}')\frac{e^{-jkR}}{R}ds', \tag{39}$$

$$\boldsymbol{F}(\boldsymbol{r}) = \frac{\epsilon}{4\pi} \iint_S \boldsymbol{M}_s(\boldsymbol{r'}) \frac{e^{-jkR}}{R} ds'. \tag{40}$$

For electric and magnetic currents,

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu}{4\pi} \int_C \boldsymbol{I}_e(\boldsymbol{r'}) \frac{e^{-jkR}}{R} dl', \tag{41}$$

$$\boldsymbol{F}(\boldsymbol{r}) = \frac{\epsilon}{4\pi} \int_C \boldsymbol{I}_m(\boldsymbol{r'}) \frac{e^{-jkR}}{R} dl'. \tag{42}$$