Circular Loop of Constant Current

We now analyze loop antennas that are larger in diameter than our infinitesimal loop example, but not so large that the current along the loops becomes nonuniform. That is, we will still assume that $I_\phi = I_0$ regardless of the source point angle ϕ'. Recall, from the small loop derivation, for an observation at any ϕ angle,

$$\begin{aligned} R &= [r^2 - 2r\sin\theta\cos\phi a\cos\phi' - 2r\sin\theta\sin\phi a\sin\phi' + a^2]^{1/2} \\ &= [r^2 + a^2 - 2ar\sin\theta\cos(\phi-\phi')]^{1/2} \\ &= [r^2 + a^2 - 2ar\sin\theta\cos\phi']^{1/2}. \end{aligned} \tag{1}$$

where the last line follows because we can conveniently set $\phi = 0$ since the loop is axially symmetric. If we are in the far-zone such that $r \gg a$, then

$$\begin{aligned} R &\approx \sqrt{r^2 - 2ar\sin\theta\cos\phi'} \\ &\approx r\sqrt{1 - \frac{2a}{r}\sin\theta\cos\phi'}. \end{aligned} \tag{2}$$

This is starting to look like the *parallel ray approximation* for the loop antenna. In fact, we can cast it into an even more familiar form by approximating the square root and defining an angle ψ_0 so that

$$R \approx r - a\sin\theta\cos\phi' \equiv r - a\cos\psi_0 \tag{3}$$

where ψ_0 is is defined such that $\cos\psi_0 = \hat{\boldsymbol{\rho}}' \cdot \hat{\boldsymbol{r}}$ in the $\phi = 0$ plane. This form of the approximation for R directly resembles the approximation we made for linear wire antennas along the z-axis.

Next, we determine the vector magnetic potential produced by the loop. As in the small loop case,

$$\begin{aligned} A_\phi &= \frac{\mu_0 I_0 a}{4\pi}\int_0^{2\pi} \cos\phi' \frac{e^{-jkR}}{R} d\phi' \\ &\approx \frac{\mu_0 I_0 a}{4\pi}\int_0^{2\pi} \cos\phi' \frac{\exp(-jkr)\exp(jka\sin\theta\cos\phi')}{r} d\phi' \\ &\approx \frac{\mu_0 I_0 a}{4\pi}\frac{e^{-jkr}}{r}\int_0^{2\pi} \cos\phi' \exp(jka\sin\theta\cos\phi') d\phi'. \end{aligned} \tag{4}$$

We separate the integral so that

$$A_\phi \approx \frac{\mu_0 I_0 a}{4\pi}\frac{e^{-jkr}}{r}\left[\int_0^{\pi} \cos\phi' \exp(jka\sin\theta\cos\phi') d\phi' + \int_\pi^{2\pi} \cos\phi' \exp(jka\sin\theta\cos\phi') d\phi'\right]. \tag{5}$$

Making the substitution $\phi' = \phi'' + \pi$ in the second integral,

$$A_\phi \approx \frac{\mu_0 I_0 a}{4\pi}\frac{e^{-jkr}}{r}\left[\int_0^{\pi} \cos\phi' \exp(jka\sin\theta\cos\phi') d\phi' - \int_0^{\pi} \cos\phi'' \exp(-jka\sin\theta\cos\phi'') d\phi''\right]. \tag{6}$$

Consulting Appendix V in the Balanis book, and in particular definition (V-36),

$$J_n(x) = \frac{j^{-n}}{\pi}\int_0^{\pi} \cos(n\phi) e^{jxcos\phi} d\phi \tag{7}$$

defines a *Bessel function of the first kind of order* n. Hence our expression for A_ϕ becomes

$$A_\phi \approx \frac{\mu_0 I_0 a}{4}\frac{e^{-jkr}}{r}\left[jJ_1(ka\sin\theta) - jJ_1(-ka\sin\theta)\right]. \tag{8}$$

Since $J_n(-z) = -J_n(z)$,

$$A_\phi \approx \frac{j\mu_0 I_0 a}{2}\frac{e^{-jkr}}{r}J_1(ka\sin\theta). \tag{9}$$

We can now find the far-zone expressions for $\boldsymbol{E}$ and $\boldsymbol{H}$. Bear in mind that our expression for $\boldsymbol{A}$ is already in the far-field. From our work in class, we know that in the far-zone, it is easy to relate $\boldsymbol{A}$ and $\boldsymbol{E}$ through

$$\boldsymbol{E} = -j\omega\boldsymbol{A}, \tag{10}$$

therefore

$$\boldsymbol{E} = \hat{\boldsymbol{\phi}}\frac{a\omega\mu_0 I_0}{2}\frac{e^{-jkr}}{r}J_1(ka\sin\theta) = \hat{\boldsymbol{\phi}}\frac{ak\eta_0 I_0}{2}\frac{e^{-jkr}}{r}J_1(ka\sin\theta) \tag{11}$$

and since $H_\theta = -E_\phi/\eta_0$,

$$\boldsymbol{H} = -\hat{\boldsymbol{\theta}}\frac{akI_0}{2}\frac{e^{-jkr}}{r}J_1(ka\sin\theta). \tag{12}$$