Formulations for Potentials

We wish to define suitable time-varying potential consistent with Maxwell's equations to aid their solution. Recall from electrostatics, the potentials are defined according to

$$\boldsymbol{E} = -\nabla V \tag{1}$$

and

$$\boldsymbol{B} = \nabla \times \boldsymbol{A}. \tag{2}$$

Equation (2) seems okay, since

$$\nabla \cdot \boldsymbol{B} = \nabla \cdot \nabla \times \boldsymbol{A} \equiv 0 \tag{3}$$

However, Faraday's Law has a problem because on the one hand,

$$\nabla \times \boldsymbol{E} = \nabla \times (-\nabla V) \equiv 0, \tag{4}$$

yet we know that in fact

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \tag{5}$$

or equivalently

$$\nabla \times \left(\boldsymbol{E} + \frac{\partial \boldsymbol{A}}{\partial t} \right) = 0. \tag{6}$$

Since the term in parentheses is solenoidal (has no curl), it must be the gradient of some scalar function which we will call scalar electric potential Φ

$$\boldsymbol{E} + \frac{\partial \boldsymbol{A}}{\partial t} = -\nabla \Phi \quad \Rightarrow \quad \boldsymbol{E} = -\nabla \Phi - \frac{\partial \boldsymbol{A}}{\partial t}, \tag{7}$$

where we have retained the minus sign to be consistent with electrostatic relations. From Gauss' Law,

$$\nabla \cdot \boldsymbol{E} = \nabla \cdot \left(-\nabla \Phi - \frac{\partial \boldsymbol{A}}{\partial t} \right) = \frac{\rho_{ev}}{\epsilon_0}, \tag{8}$$

$$\nabla^2 \Phi + \frac{\partial}{\partial t}(\nabla \cdot \boldsymbol{A}) = -\frac{\rho_{ev}}{\epsilon_0} \tag{9}$$

From Ampère's Law,

$$\frac{1}{\mu_0} \nabla \times \nabla \times \boldsymbol{A} = \boldsymbol{J} + \epsilon_0 \left[\frac{\partial}{\partial t} \left(-\nabla \Phi - \frac{\partial \boldsymbol{A}}{\partial t} \right) \right] \tag{10}$$

$$-\nabla(\nabla \cdot \boldsymbol{A}) + \nabla^2 \boldsymbol{A} - \mu_0 \epsilon_0 \left(\nabla \frac{\partial \Phi}{\partial t} + \frac{\partial^2 \boldsymbol{A}}{\partial t^2} \right) = -\mu_0 \boldsymbol{J} \tag{11}$$

$$\nabla^2 \boldsymbol{A} - \frac{1}{c^2} \frac{\partial^2 \boldsymbol{A}}{\partial t^2} - \nabla \left(\nabla \cdot \boldsymbol{A} + \frac{1}{c^2} \frac{\partial \Phi}{\partial t} \right) = -\mu_0 \boldsymbol{J}. \tag{12}$$

We would really like to decouple equations (9) and (12). At this point, we need to recognize that the Helmholtz theorem states that a vector field is uniquely defined only when both its curl and divergence are specified. For example, say $A_y = A_z = 0$. Then, $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$ gives

$$B_x = 0 \tag{13}$$

$$B_y = \partial A_x / \partial z \tag{14}$$

$$B_z = -\partial A_x / \partial y \tag{15}$$

which provides no information on the possible variation of A_x with x. If we knew the divergence of $\boldsymbol{A}$, i.e.

$$\nabla \cdot \boldsymbol{A} = \frac{\partial A_x}{\partial x}, \tag{16}$$

our dilemma would be resolved.

The definition of $\boldsymbol{B}$ is also arbitrary in that the gradient of some scalar function could be added to $\boldsymbol{A}$ without changing $\boldsymbol{B}$; that is, a transformation of the form

$$\boldsymbol{A} \to \boldsymbol{A} + \nabla \Lambda, \tag{17}$$

does not change $\boldsymbol{B}$. Λ is a *gauge function*. $\boldsymbol{A}$ is unchanged because the curl of a gradient is zero. Similarly, $\boldsymbol{E}$ in (7) must be unchanged, requiring a corresponding transformation of Φ defined by

$$\Phi \to \Phi - \frac{\partial \Lambda}{\partial t} \tag{18}$$

yields

$$\boldsymbol{E} = \nabla \left(\Phi - \frac{\partial \Lambda}{\partial t} \right) - \frac{\partial}{\partial t}(\boldsymbol{A} + \nabla \Lambda) = -\nabla \Phi - \frac{\partial \boldsymbol{A}}{\partial t}. \tag{19}$$

(17) and (18) collectively define a *gauge transformation*. The choice of a certain Λ changes the specification of $\nabla \cdot \boldsymbol{A}$, but not the fields $\boldsymbol{E}$ and $\boldsymbol{B}$. This means these fields are *gauge invariant*. The apparent freedom in defining $\boldsymbol{A}$ and Φ means that we can choose them advantageously to suit the problem at hand. Since we wish to form a wave equation, the *Lorenz gauge*

$$\nabla \cdot \boldsymbol{A} + \frac{1}{c^2} \frac{\partial \Phi}{\partial t} \tag{20}$$

is particularly convenient, since then (12) becomes

$$\nabla^2 \boldsymbol{A} - \frac{1}{c^2} \frac{\partial^2 \boldsymbol{A}}{\partial t^2} = -\mu_0 \boldsymbol{J}, \tag{21}$$

which is the vector wave equation.

Note that in (21) and (9) we now must solve second-order PDEs for the potentials, but in exchange, these equations only depend on potentials and they can be differentiated to find the fields $\boldsymbol{E}$ and $\boldsymbol{B}$.

It is possible to use other potential functions to represent $\boldsymbol{E}$ and $\boldsymbol{H}$. For example, it is possible to define a Hertzian potential $\mathbf{\Pi}_e$ such that

$$\boldsymbol{A} = \mu_0\epsilon_0 \frac{\partial \mathbf{\Pi}_e}{\partial t} \tag{22}$$

$$\boldsymbol{E} = \nabla(\nabla \cdot \mathbf{\Pi}_e) - \mu_0\epsilon_0 \frac{\partial^2}{\partial t^2}\mathbf{\Pi}_e \tag{23}$$

$$\boldsymbol{H} = \mu_0\epsilon_0 \nabla \times \frac{\partial \mathbf{\Pi}_e}{\partial t} \tag{24}$$

which allows the fields $\boldsymbol{E}$ and $\boldsymbol{B}$ to be found upon solving the wave equation for $\mathbf{\Pi}_e$,

$$\nabla^2\mathbf{\Pi}_e - \mu_0\epsilon_0 \frac{\partial^2 \mathbf{\Pi}_e}{\partial t^2} = -\frac{1}{\epsilon_0}\boldsymbol{P}^i, \tag{25}$$

where $\boldsymbol{P}^i$ is an impressed polarization current that is independent of $\boldsymbol{E}$ and defined according to $\boldsymbol{D} = \epsilon_0\boldsymbol{E} + \boldsymbol{P} + \boldsymbol{P}^i$. There is also a dual Hertzian potential $\mathbf{\Pi}_m$ for magnetic current, which defines

$$\boldsymbol{F} = \mu_0\epsilon_0 \frac{\partial \mathbf{\Pi}_m}{\partial t} \tag{26}$$

$$\boldsymbol{D} = -\mu_0\epsilon_0 \nabla \times \frac{\partial \mathbf{\Pi}_h}{\partial t} \tag{27}$$

$$\boldsymbol{H} = \nabla(\nabla \cdot \mathbf{\Pi}_h) - \mu_0\epsilon_0 \frac{\partial^2}{\partial t^2}\mathbf{\Pi}_h \tag{28}$$

where $\boldsymbol{F}$ is the vector electric potential. The corresponding wave equation is

$$\nabla^2\mathbf{\Pi}_h - \mu_0\epsilon_0 \frac{\partial^2 \mathbf{\Pi}_h}{\partial t^2} = -\boldsymbol{M}^i, \tag{29}$$

where $\boldsymbol{M}^i$ is the impressed magnetization current.