Lorentz Reciprocity Theorem and Antennas

The reciprocity theorem is important in establishing the transmitting and receiving properties of antennas. Here, we derive the reciprocity theorem and apply it to a two-antenna system.

Assume we have to sets of sources, $\boldsymbol{J}_1, \boldsymbol{M}_1$ and $\boldsymbol{J}_2, \boldsymbol{M}_2$, inside a linear, isotropic, and homogeneous medium. These sources produce field pairs $\boldsymbol{E}_1, \boldsymbol{H}_1$ and $\boldsymbol{E}_2, \boldsymbol{H}_2$ respectively. These fields satisfy Maxwell's equations:

$$\nabla \times \boldsymbol{E}_1 = -\boldsymbol{M}_1 - j\omega\mu\boldsymbol{H}_1 \tag{1a}$$
$$\nabla \times \boldsymbol{H}_1 = \boldsymbol{J}_1 + j\omega\epsilon\boldsymbol{E}_1 \tag{1b}$$

$$\nabla \times \boldsymbol{E}_2 = -\boldsymbol{M}_2 - j\omega\mu\boldsymbol{H}_2 \tag{2a}$$
$$\nabla \times \boldsymbol{H}_2 = \boldsymbol{J}_2 + j\omega\epsilon\boldsymbol{E}_2. \tag{2b}$$

Let's take the dot product of (1a) and $\boldsymbol{H}_2$, as well as (2b) and $\boldsymbol{E}_1$:

$$\boldsymbol{H}_2 \cdot \nabla \times \boldsymbol{E}_1 = -\boldsymbol{H}_2 \cdot \boldsymbol{M}_1 - j\omega\mu\boldsymbol{H}_2 \cdot \boldsymbol{H}_1 \tag{3a}$$
$$\boldsymbol{E}_1 \cdot \nabla \times \boldsymbol{H}_2 = \boldsymbol{E}_1 \cdot \boldsymbol{J}_2 + j\omega\epsilon\boldsymbol{E}_1 \cdot \boldsymbol{E}_2 \tag{3b}$$

Subtracting these equations yields

$$\boldsymbol{E}_1 \cdot \nabla \times \boldsymbol{H}_2 - \boldsymbol{H}_2 \cdot \nabla \times \boldsymbol{E}_1 = \boldsymbol{E}_1 \cdot \boldsymbol{J}_2 + j\omega\epsilon\boldsymbol{E}_1 \cdot \boldsymbol{E}_2 + \boldsymbol{H}_2 \cdot \boldsymbol{M}_1 + j\omega\mu\boldsymbol{H}_2 \cdot \boldsymbol{H}_1. \tag{4}$$

Applying the vector identity

$$\nabla \cdot (\boldsymbol{A} \times \boldsymbol{B}) = \boldsymbol{B} \cdot (\nabla \times \boldsymbol{A}) - \boldsymbol{A} \cdot (\nabla \times \boldsymbol{B}) \tag{5}$$

gives

$$\nabla \cdot (\boldsymbol{H}_2 \times \boldsymbol{E}_1) = \boldsymbol{E}_1 \cdot \boldsymbol{J}_2 + j\omega\epsilon\boldsymbol{E}_1 \cdot \boldsymbol{E}_2 + \boldsymbol{H}_2 \cdot \boldsymbol{M}_1 + j\omega\mu\boldsymbol{H}_2 \cdot \boldsymbol{H}_1 = -\nabla \cdot (\boldsymbol{E}_1 \times \boldsymbol{H}_2). \tag{6}$$

Similarly, repeating the same process beginning with the dot product of (1b) and $\boldsymbol{E}_2$, and (2a) and $\boldsymbol{H}_1$,

$$\nabla \cdot (\boldsymbol{H}_1 \times \boldsymbol{E}_2) = \boldsymbol{E}_2 \cdot \boldsymbol{J}_1 + j\omega\epsilon\boldsymbol{E}_2 \cdot \boldsymbol{E}_1 + \boldsymbol{H}_1 \cdot \boldsymbol{M}_2 + j\omega\mu\boldsymbol{H}_1 \cdot \boldsymbol{H}_2 = -\nabla \cdot (\boldsymbol{E}_2 \times \boldsymbol{H}_1). \tag{7}$$

Subtracting these two equations,

$$-\nabla \cdot (\boldsymbol{E}_1 \times \boldsymbol{H}_2 - \boldsymbol{E}_2 \times \boldsymbol{H}_1) = \boldsymbol{E}_1 \cdot \boldsymbol{J}_2 + \boldsymbol{H}_2 \cdot \boldsymbol{M}_1 - \boldsymbol{E}_2 \cdot \boldsymbol{J}_1 - \boldsymbol{H}_1 \cdot \boldsymbol{M}_2. \tag{8}$$

This equation is the *Lorentz reciprocity theorem* in differential (point) form. The integral form is more useful for our purposes, so we take the volume integral of both sides,

$$-\iiint_V \nabla \cdot (\boldsymbol{E}_1 \times \boldsymbol{H}_2 - \boldsymbol{E}_2 \times \boldsymbol{H}_1) dv' = \iiint_V (\boldsymbol{E}_1 \cdot \boldsymbol{J}_2 + \boldsymbol{H}_2 \cdot \boldsymbol{M}_1 - \boldsymbol{E}_2 \cdot \boldsymbol{J}_1 - \boldsymbol{H}_1 \cdot \boldsymbol{M}_2) dv'. \tag{9}$$

Employing the divergence theorem, we arrive at the Lorentz reciprocity theorem in integral form,

$$-\oint_S (\boldsymbol{E}_1 \times \boldsymbol{H}_2 - \boldsymbol{E}_2 \times \boldsymbol{H}_1) \cdot d\boldsymbol{s}' = \iiint_V (\boldsymbol{E}_1 \cdot \boldsymbol{J}_2 + \boldsymbol{H}_2 \cdot \boldsymbol{M}_1 - \boldsymbol{E}_2 \cdot \boldsymbol{J}_1 - \boldsymbol{H}_1 \cdot \boldsymbol{M}_2) dv'. \tag{10}$$

For antenna problems, a more useful form of the reciprocity theorem is found by considering one set of sources contained inside a sphere of infinite radius, and the other set of sources at some point on the surface of the sphere. If we evaluate the fields produced by the sources at the centre of the sphere on the sphere surface, they represent fields associated with a plane wave since we are very far from the sources. This means that $\boldsymbol{E}$ and $\boldsymbol{H}$ are related through

$$\boldsymbol{H} = \frac{1}{\eta}\hat{\boldsymbol{n}} \times \boldsymbol{E}, \tag{11}$$

where $\hat{\boldsymbol{n}}$ is a unit vector normal to the sphere. Hence,

$$\begin{aligned}(\boldsymbol{E}_1 \times \boldsymbol{H}_2 - \boldsymbol{E}_2 \times \boldsymbol{H}_1) \cdot \hat{\boldsymbol{n}} &= (\hat{\boldsymbol{n}} \times \boldsymbol{E}_1) \cdot \boldsymbol{H}_2 - (\hat{\boldsymbol{n}} \times \boldsymbol{E}_2) \cdot \boldsymbol{H}_1 \\ &= \eta \boldsymbol{H}_1 \cdot \boldsymbol{H}_2 - \eta \boldsymbol{H}_2 \cdot \boldsymbol{H}_1 = 0. \end{aligned} \tag{12}$$

Therefore,

$$-\oint_S (\boldsymbol{E}_1 \times \boldsymbol{H}_2 - \boldsymbol{E}_2 \times \boldsymbol{H}_1) \cdot d\boldsymbol{s}' = 0 \tag{13}$$

and the reciprocity theorem becomes

$$\iiint_V (\boldsymbol{E}_1 \cdot \boldsymbol{J}_2 - \boldsymbol{H}_1 \cdot \boldsymbol{M}_2) dv' = \iiint_V (\boldsymbol{E}_2 \cdot \boldsymbol{J}_1 - \boldsymbol{H}_2 \cdot \boldsymbol{M}_1) dv'. \tag{14}$$

Note that the operation described by the equations above is often called the *reaction* between fields and sources, e.g.

$$\langle 1, 2 \rangle = \iiint_V (\boldsymbol{E}_1 \cdot \boldsymbol{J}_2 - \boldsymbol{H}_1 \cdot \boldsymbol{M}_2) dv' \tag{15}$$

represents the reaction (coupling) of the fields $(\boldsymbol{E}_1, \boldsymbol{H}_1)$ to the sources $(\boldsymbol{J}_2, \boldsymbol{M}_2)$. The reciprocity theorem then becomes

$$\langle 1, 2 \rangle = \langle 2, 1 \rangle. \tag{16}$$

This form of the theorem can be readily applied to antenna problems. To analyze the implications of reciprocity, we will consider the two antenna network as shown in Figure 1, where the antennas are assume to be far from each other so that this form of the reciprocity theorem holds.

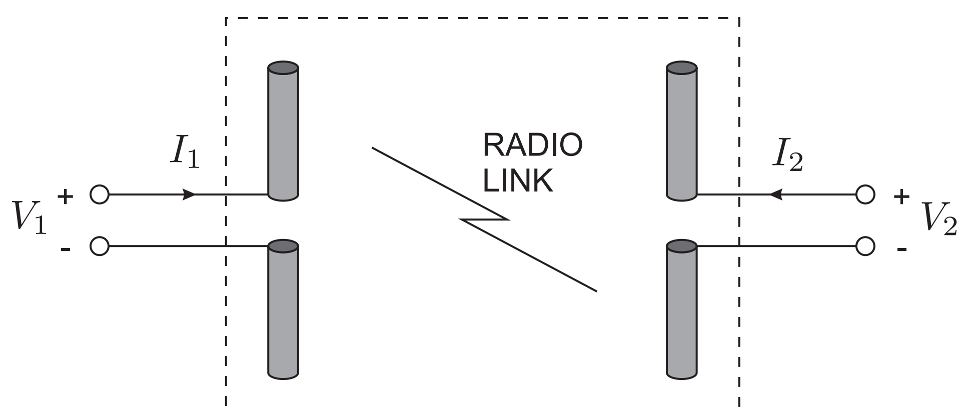

Figure 1: Two-port antena network

Recall from circuit theory than an unknown "black box" with two ports is fully characterized if we know the terminal voltage (V_1, V_2) and currents (I_1, I_2) of the device. These quantities are related through

$$V_1 = Z_{11}I_1 + Z_{12}I_2 \tag{17}$$
$$V_2 = Z_{21}I_1 + Z_{22}I_2, \tag{18}$$

or simply,

$$[V] = [Z][I], \tag{19}$$

where $[Z]$ is the *impedance matrix* of the two-port network. Z_{11} and Z_{22} are called *self-impedances* of the system while Z_{12} and Z_{21} are called *mutual impedances* of the network. Each of the impedances can be found through open-circuiting ports of the network, such that

$$Z_{mn} = \left.\frac{V_m}{I_n}\right|_{I_k=0 \text{ for } k\neq n}. \tag{20}$$

Now, let's return to (14), and consider a simplified situation involving only electric current sources for clarity. In this case, (14) becomes

$$\iiint_V (\boldsymbol{E}_1 \cdot \boldsymbol{J}_2) dv' = \iiint_V (\boldsymbol{E}_2 \cdot \boldsymbol{J}_1) dv'. \tag{21}$$

Let's assume the two antennas are ideal dipole antennas, driven by ideal current generators (with infinite source impedances).

Thinking of the structure of the antenna, $\boldsymbol{E}_2$, the field produced at antenna 1 by antenna 2 (source $\boldsymbol{J}_2$) only has a nonzero projection with $\boldsymbol{J}_1$ across the antenna terminals, since $\boldsymbol{J}_1$ is only nonzero along the conducting part of the antenna, and $\boldsymbol{E}_2$ is only nonzero across the gap because of PEC boundary conditions. If we assume that the current $\boldsymbol{J}_1$ can be represented as a linear current (existing only over a contour instead of a volume) and furthermore the current is uniform across the gap,

$$\iiint \boldsymbol{E_2} \cdot \boldsymbol{J}_1 = \int_C E_2 I_1 d\ell = I_1 \int_C E_2 d\ell = -V_1^{oc} I_1, \tag{22}$$

where V_1^{oc} is the open-circuited voltage produced at antenna 1 as a result of the incident field produced by antenna 2 ($\boldsymbol{E}_2$). Similarly, at antenna 2,

$$\iiint \boldsymbol{E_1} \cdot \boldsymbol{J}_2 = -V_2^{oc} I_2 \tag{23}$$

According to the reciprocity theorem (21), equations (22) and (23) must be equal:

$$V_1^{oc} I_1 = V_2^{oc} I_2 \tag{24}$$

or

$$\frac{V_1^{oc}}{I_2} = \frac{V_2^{oc}}{I_1}. \tag{25}$$

From the definition of impedance matrices,

$$Z_{21} = \left.\frac{V_2}{I_1}\right|_{I_2=0} = \frac{V_2^{oc}}{I_1} \tag{26}$$

and

$$Z_{12} = \left.\frac{V_1}{I_2}\right|_{I_1=0} = \frac{V_1^{oc}}{I_2}, \tag{27}$$

therefore,

$$Z_{12} = Z_{21}. \tag{28}$$

This is the fundamental definition of a reciprocal two-port circuit, because we see that if we excite port 1 with a current source of amplitude I, the open circuit-voltage at port 2 is $Z_{21}I$ while if we flip the current source to port 2, the open-circuit voltage at port 1 is $Z_{12}I = Z_{21}I$ which is the same result as with the current source at port 1.

The consequence of this on antennas is found as follows. Consider an experiment where we measure the transmit pattern of antenna (1), using a second receiving antenna (2) moving about a circle of fixed radius about the transmitting antenna while remaining co-polarized with the transmission. If we measure the open-circuit voltage at antenna 2, we know it is equal to

$$V_2^{oc}(\theta) = Z_{21}(\theta)I. \tag{29}$$

Now consider a second experiment where antenna (2) is used as the transmitter and antenna (1) as the receiver. Antenna (2) moves in an identical manner as the first experiment, while this time we measure the open-circuit voltage at the terminals of antenna 1, which is equal to

$$V_1^{oc}(\theta) = Z_{12}(\theta)I = Z_{21}(\theta)I. \tag{30}$$

Hence we see the "open-circuit voltage" pattern measured by both experiments is identical. Therefore, we conclude that **the transmit and receive patterns of an antenna are the same**.