MoM Using RWG Basis Functions

In this note we aim to derive expressions for the impedance matrix and voltage vector elements for a conducting structure in free space. The conducting structures are represented using RWG basis functions, defined as [1]

$$\boldsymbol{f}_n(\boldsymbol{r}') = \begin{cases} \frac{l_n}{2A_n^+}\boldsymbol{\rho}_n^+ & \boldsymbol{r}' \text{ in } T_n^+ \\ \frac{l_n}{2A_n^-}\boldsymbol{\rho}_n^- & \boldsymbol{r}' \text{ in } T_n^- \\ 0 & \text{elsewhere} \end{cases}. \tag{1}$$

The surface divergence of these basis functions are

$$\boldsymbol{\nabla}_S \cdot \boldsymbol{f}_n(\boldsymbol{r}') = \begin{cases} \frac{l_n}{A_n^+} & \boldsymbol{r}' \text{ in } T_n^+ \\ -\frac{l_n}{A_n^-} & \boldsymbol{r}' \text{ in } T_n^- \\ 0 & \text{elsewhere} \end{cases}. \tag{2}$$

1 Evaluation of Impedance Matrix

The electric field integral equation (EFIE) states that over the surface of the conductor S, the tangential component of the incident electric field ($\boldsymbol{E}^{inc}$) and the tangential component scattered electric field ($\boldsymbol{E}^s(\boldsymbol{r})$) satisfy

$$\boldsymbol{E}_{tan}^{inc}(\boldsymbol{r}) + \boldsymbol{E}_{tan}^{s}(\boldsymbol{r}) = 0, \quad \boldsymbol{r} \text{ on } S \tag{3}$$

assuming there are no losses in the conductor. The scattered electric field can be found as

$$\boldsymbol{E}^s(\boldsymbol{r}) = -j\omega\boldsymbol{A}(\boldsymbol{r}) - j\frac{1}{\omega\mu_0\epsilon_0}\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{A}(\boldsymbol{r})) = -j\omega\boldsymbol{A}(\boldsymbol{r}) - \boldsymbol{\nabla}\Phi(\boldsymbol{r}). \tag{4}$$

For linear current densities $\boldsymbol{J}_s$ and surface charge densities ρ_{es}, the vector magnetic potential $\boldsymbol{A}(\boldsymbol{r})$ and scalar electric potential $\Phi(\boldsymbol{r})$ can be found, respectively, as

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\iint_S \boldsymbol{J}_s(\boldsymbol{r'})\frac{e^{-jk_0R}}{R}ds', \tag{5}$$

and

$$\Phi(\boldsymbol{r}) = \iint_S \frac{\rho_{es}(\boldsymbol{r'})}{4\pi\epsilon_0}\frac{e^{-jk_0R}}{R}dv' \tag{6}$$

where $R = |\boldsymbol{r} - \boldsymbol{r}'|$.

The EFIE is tested using expansion functions which are the same as the basis functions (1) (Galerkin's method) according to

$$\langle\boldsymbol{E}^{inc}, \boldsymbol{f}_m\rangle = j\omega\langle\boldsymbol{A}, \boldsymbol{f}_m\rangle + \langle\nabla\Phi, \boldsymbol{f}_m\rangle \tag{7}$$

where

$$\langle \boldsymbol{f}, \boldsymbol{g} \rangle = \int_S \boldsymbol{f} \cdot \boldsymbol{g} \, ds. \tag{8}$$

Employing the equation of continuity, which on surfaces states

$$\boldsymbol{\nabla}_S \cdot \boldsymbol{J}_s = -j\omega\rho_{es}, \tag{9}$$

the scalar potential can be evaluated as

$$\Phi(\boldsymbol{r}) = \iint_S \frac{\rho_{es}(\boldsymbol{r'})}{4\pi\epsilon_0} \frac{e^{-jk_0R}}{R} ds' = -\frac{1}{j\omega 4\pi\epsilon_0} \iint_S \boldsymbol{\nabla}_S \cdot \boldsymbol{J}_s(\boldsymbol{r'}) \frac{e^{-jk_0R}}{R} ds' \tag{10}$$

In the moment method, the current density on the patch is represented as a summation of basis functions $\boldsymbol{J}_n$ with unknown amplitudes I_n. Therefore, we can write

$$\boldsymbol{J}(\boldsymbol{r'}) = \sum_n I_n \boldsymbol{f}_n(\boldsymbol{r'}) \tag{11}$$

Carrying out the testing procedure at each interior edge in the mesh yields

$$\langle \boldsymbol{E}^{inc}, \boldsymbol{f}_m \rangle = j\omega \langle \boldsymbol{A}, \boldsymbol{f}_m \rangle + \langle \nabla\Phi, \boldsymbol{f}_m \rangle. \tag{12}$$

The testing function $\boldsymbol{f}_m$ is defined over two regions. Therefore, the reaction integral is divided over two regions,

$$\left\langle \left\{ \begin{array}{c} \boldsymbol{E}^i \\ \boldsymbol{A} \end{array} \right\}, \boldsymbol{f}_m \right\rangle = l_m \left[\frac{1}{2A_m^+} \iint_{T_m^+} \left\{ \begin{array}{c} \boldsymbol{E}^i \\ \boldsymbol{A} \end{array} \right\} \cdot \boldsymbol{\rho}_m^+ ds + \frac{1}{2A_m^-} \iint_{T_m^-} \left\{ \begin{array}{c} \boldsymbol{E}^i \\ \boldsymbol{A} \end{array} \right\} \cdot \boldsymbol{\rho}_m^- \right] ds, \tag{13}$$

$$\langle \nabla\Phi, \boldsymbol{f}_m \rangle = -\iint_S \Phi \boldsymbol{\nabla}_S \cdot \boldsymbol{f}_m ds = -l_m \left(\frac{1}{A_m^+} \iint_{T_m^+} \Phi ds - \frac{1}{A_m^-} \iint_{T_m^-} \Phi ds \right). \tag{14}$$

The first equality in (14) results from a surface vector calculus identity and the properties of the basis function (see equation (A3.47) in [2]).

The reaction integrals above can be approximated by evaluating field quantities at the centre of each triangle $\boldsymbol{r}_m^{c\pm}$, so that

$$\left\langle \left\{ \begin{array}{c} \boldsymbol{E}^{inc} \\ \boldsymbol{A} \end{array} \right\}, \boldsymbol{f}_m \right\rangle \approx \frac{l_m}{2} \left[\left\{ \begin{array}{c} \boldsymbol{E}^{inc}(\boldsymbol{r}_m^{c+}) \\ \boldsymbol{A}(\boldsymbol{r}_m^{c+}) \end{array} \right\} \cdot \boldsymbol{\rho}_m^{c+} + \left\{ \begin{array}{c} \boldsymbol{E}^{inc}(\boldsymbol{r}_m^{c-}) \\ \boldsymbol{A}(\boldsymbol{r}_m^{c-}) \end{array} \right\} \cdot \boldsymbol{\rho}_m^{c-} \right]. \tag{15}$$

Similarly, for the scalar potentials,

$$\langle \boldsymbol{\nabla}\Phi(\boldsymbol{r}), \boldsymbol{f}_m(\boldsymbol{r'}) \rangle \approx l_m [\Phi(\boldsymbol{r}_m^{c+}) - \Phi(\boldsymbol{r}_m^{c-})] \tag{16}$$

Equation (12) is enforced at every triangle edge $m = 1, 2, 3, \ldots$. The fields are evaluated at the centre of the respective triangles, $\boldsymbol{r}_m^{c-}$ and $\boldsymbol{r}_m^{c+}$ associated with edge m. The impedance matrix can then be defined as

$$Z_{mn} = l_m \left[j\omega \left(\boldsymbol{A}(\boldsymbol{r}_m^{c+}) \cdot \frac{\boldsymbol{\rho}_m^{c+}}{2} + \boldsymbol{A}(\boldsymbol{r}_m^{c-}) \cdot \frac{\boldsymbol{\rho}_m^{c-}}{2} \right) + \Phi(\boldsymbol{r}_m^{c-}) - \Phi(\boldsymbol{r}_m^{c+}) \right]. \tag{17}$$

where the index j indicates that the jth basis function is used to find the potential. (17) can be written more compactly as

$$Z_{mn} = l_m \left[j\omega \left(\boldsymbol{A}^+_{mn} \cdot \frac{\boldsymbol{\rho}^{c+}_m}{2} + \boldsymbol{A}^-_{mn} \cdot \frac{\boldsymbol{\rho}^{c-}_m}{2} \right) + \Phi^-_{mn} - \Phi^+_{mn} \right] \tag{18}$$

where a function $F^\pm$ indicates that it is evaluated at the field position $\boldsymbol{r}^{c\pm}_m$. The potentials in (18) are given by

$$\boldsymbol{A}^\pm_{mn} = \frac{\mu}{4\pi} \iint_S \boldsymbol{f}^j(\boldsymbol{r}') \frac{e^{-jkR^\pm_m}}{R^\pm_m} ds' \tag{19}$$

$$\Phi^\pm_{mn} = -\frac{1}{4\pi j\omega\epsilon} \iint_S \boldsymbol{\nabla}_S \cdot \boldsymbol{f}^j(\boldsymbol{r}') \frac{e^{-jkR^\pm_m}}{R^\pm_m} ds' \tag{20}$$

$$R^\pm_m = |\boldsymbol{r}^{c\pm}_m - \boldsymbol{r}'| \tag{21}$$

The basis and weighting functions $\boldsymbol{f}^j$ are defined over *two* regions, one in the (+) triangle and one in the (-) triangle. The testing function has been evaluated over two regions already in the reaction integral. The potentials must also be expanded over the basis function regions. Hence, evaluating a potential at a given field position at the centre of a (+) or (-) triangle receives contributions from two source triangles,

$$\boldsymbol{A}^\pm_{mn} = \frac{\mu}{4\pi} \iint_{T^+_n} \frac{l_n}{2A^+_n} \boldsymbol{\rho}^+_n \cdot \frac{e^{-jkR^\pm_m}}{R^\pm_m} ds' + \frac{\mu}{4\pi} \iint_{T^-_n} \frac{l_n}{2A^-_n} \boldsymbol{\rho}^-_n \cdot \frac{e^{-jkR^\pm_m}}{R^\pm_m} ds' \tag{22}$$

$$\Phi^\pm_{mn} = -\frac{1}{4\pi j\omega\epsilon} \iint_{T^+_n} \frac{l_n}{A^+_n} \frac{e^{-jkR^\pm_m}}{R^\pm_m} ds' + \frac{1}{4\pi j\omega\epsilon} \iint_{T^-_n} \frac{l_n}{A^-_n} \frac{e^{-jkR^\pm_m}}{R^\pm_m} ds' \tag{23}$$

Substituting into (18),

$$\begin{aligned} Z_{mn} = \; & l_m \left\{ j\omega \frac{\mu l_n}{8\pi} \left[\left(\iint_{T^+_n} \frac{1}{A^+_n} \boldsymbol{\rho}^+_n \frac{e^{-jkR^+_m}}{R^+_m} ds' + \iint_{T^-_n} \frac{1}{A^-_n} \boldsymbol{\rho}^-_n \frac{e^{-jkR^+_m}}{R^+_m} ds' \right) \cdot \frac{\boldsymbol{\rho}^{c+}_m}{2} + \right. \right. \\ & \left. \left(\iint_{T^+_n} \frac{1}{A^+_n} \boldsymbol{\rho}^+_n \frac{e^{-jkR^-_m}}{R^-_m} ds' + \iint_{T^-_n} \frac{1}{A^-_n} \boldsymbol{\rho}^-_n \frac{e^{-jkR^-_m}}{R^-_m} ds' \right) \cdot \frac{\boldsymbol{\rho}^{c-}_m}{2} \right] - \\ & \frac{l_n}{4\pi j\omega\epsilon} \left[\left(\iint_{T^+_n} \frac{1}{A^+_n} \frac{e^{-jkR^-_m}}{R^-_m} ds' - \iint_{T^-_n} \frac{1}{A^-_n} \frac{e^{-jkR^-_m}}{R^-_m} ds' \right) + \right. \\ & \left. \left. \left(\iint_{T^+_n} \frac{1}{A^+_n} \frac{e^{-jkR^+_m}}{R^+_m} ds' - \iint_{T^-_n} \frac{1}{A^-_n} \frac{e^{-jkR^+_m}}{R^+_m} ds' \right) \right] \right\} \end{aligned} \tag{24}$$

We now group integrations by region,

$$\begin{aligned} Z_{mn} &= l_m \left\{ j\omega \frac{\mu l_n}{16\pi} \left[\frac{1}{A_n^+} \iint_{T_n^+} \left(\boldsymbol{\rho}_n^+ \cdot \boldsymbol{\rho}_m^{c+} \frac{e^{-jkR_m^+}}{R_m^+} + \boldsymbol{\rho}_n^+ \cdot \boldsymbol{\rho}_m^{c-} \frac{e^{-jkR_m^-}}{R_m^-} \right) ds' \; + \right. \right. \\ & \left. \frac{1}{A_n^-} \iint_{T_n^-} \left(\boldsymbol{\rho}_n^- \cdot \boldsymbol{\rho}_m^{c+} \frac{e^{-jkR_m^+}}{R_m^+} + \boldsymbol{\rho}_n^- \cdot \boldsymbol{\rho}_m^{c-} \frac{e^{-jkR_m^-}}{R_m^-} \right) ds' \right] + \\ & \frac{l_n}{4\pi j\omega\epsilon} \left[\frac{1}{A_n^+} \iint_{T_n^+} \left(\frac{e^{-jkR_m^+}}{R_m^+} - \frac{e^{-jkR_m^-}}{R_m^-} \right) ds' \; - \right. \\ & \left. \left. \frac{1}{A_n^-} \iint_{T_n^-} \left(\frac{e^{-jkR_m^+}}{R_m^+} - \frac{e^{-jkR_m^-}}{R_m^-} \right) ds' \right] \right\} \end{aligned} \tag{25}$$

Surface integrals over a given triangle are approximated by summing over 9 sub-triangles composing the triangle of interest, derived using barycentric subdivision. That is,

$$\iint_{T_n} g(\boldsymbol{r}) ds \approx \frac{A_n}{9} \sum_{k=1}^{9} g(\boldsymbol{r}_k^c) \tag{26}$$

where $\boldsymbol{r}_k^c$ is the centre of the kth sub-triangle. The impedance matrix is then approximated as

$$\begin{aligned} Z_{mn} &\approx l_m \left\{ j\omega \frac{\mu l_n}{144\pi} \left[\sum_{k=1}^{9} \left(\boldsymbol{\rho}_{j,k}^{c+} \cdot \boldsymbol{\rho}_m^{c+} \frac{e^{-jkR_m^+}}{R_m^+} + \boldsymbol{\rho}_{j,k}^{c+} \cdot \boldsymbol{\rho}_m^{c-} \frac{e^{-jkR_m^-}}{R_m^-} \right) ds' \; + \right. \right. \\ & \left. \sum_{k=1}^{9} \left(\boldsymbol{\rho}_{j,k}^{c-} \cdot \boldsymbol{\rho}_m^{c+} \frac{e^{-jkR_m^+}}{R_m^+} + \boldsymbol{\rho}_{j,k}^{c-} \cdot \boldsymbol{\rho}_m^{c-} \frac{e^{-jkR_m^-}}{R_m^-} \right) ds' \right] + \\ & \frac{l_n}{36\pi j\omega\epsilon} \left[\sum_{k=1}^{9} \left(\frac{e^{-jkR_m^+}}{R_m^+} - \frac{e^{-jkR_m^-}}{R_m^-} \right) ds' \; - \right. \\ & \left. \left. \sum_{k=1}^{9} \left(\frac{e^{-jkR_m^+}}{R_m^+} - \frac{e^{-jkR_m^-}}{R_m^-} \right) ds' \right] \right\} \end{aligned} \tag{27}$$

where $\rho_{j,k}^{c\pm}$ is the centre of the kth sub-triangle in triangle j. We note that the constant outside the parentheses on the first line is the factor `FactorA` in the `rwg3.m` code [3]. The constant outside the parentheses on the third line is `FactorFi`, while the inner most term in parentheses is `gP - gM` in `impmet.m`.

2 Evaluation of Voltage Vector

The voltage vector is given by

$$V_m = \left\langle \boldsymbol{E}^{inc}(\boldsymbol{r}), \boldsymbol{f}_m(\boldsymbol{r}') \right\rangle = \iint_S \boldsymbol{f}_m(\boldsymbol{r}) \cdot \boldsymbol{E}^{inc}(\boldsymbol{r}) ds \tag{28}$$

we see that the voltage vector is the reaction of the weighting function $\boldsymbol{f}_m$ with the incident electric field. Since the basis function is defined over two triangles, we expect for each edge (each impedance matrix element) that there will be a contribution from both field triangles. It appears that rather than carry out a Barycentric summation for the voltage vectors, the following approximation is used [3]:

$$V_m \approx \boldsymbol{f}_m(\boldsymbol{r}) \cdot \boldsymbol{E}^{inc}(\boldsymbol{r}) ds. \tag{29}$$

It then follows from (1) that

$$V_m \approx l_m \left[\boldsymbol{E}^{inc}\left(\boldsymbol{r}_m^{c+}\right) \cdot \frac{\boldsymbol{\rho}_m^{c+}}{2} + \boldsymbol{E}^{inc}\left(\boldsymbol{r}_m^{c-}\right) \cdot \frac{\boldsymbol{\rho}_m^{c-}}{2} \right] \tag{30}$$

References

[1] S. Rao, D. Wilton, and A. Glisson, "Electromagnetic scattering by surfaces of arbitrary shape," *IEEE Transactions on Antennas and Propagation*, vol. 30, no. 3, pp. 409 – 418, May 1982.

[2] J. van Bladel, *Electromagnetic Fields*, 2nd ed. Wiley-IEEE Press, 2007.

[3] S. N. Makarov, *Antenna and EM Modeling with MATLAB*. New York: John Wiley and Sons, 2002.