Small Circular Loop

Recall,

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\int_C \boldsymbol{I}_e(\boldsymbol{r}')\frac{e^{-jkR}}{R}dl'. \tag{1}$$

Assume the radius of the loop is small with respect to the wavelength, so that the current along the loop can be treated as uniform: $I_\phi = I_0$. Then, in cylindrical coordinates,

$$\boldsymbol{I}_e(\rho', \phi', z')dl' = I_0 a d\phi' \hat{\boldsymbol{\phi}}'. \tag{2}$$

We now express $\boldsymbol{A}$ in cylindrical coordinates. Since $\boldsymbol{A}$ only has a ϕ-component,

$$\begin{aligned} A_\phi &= \frac{\mu_0}{4\pi}\int_0^{2\pi} I_0 a \hat{\boldsymbol{\phi}}' \cdot \hat{\boldsymbol{\phi}}\frac{e^{-jkR}}{R}d\phi' \\ &= \frac{\mu_0}{4\pi}\int_0^{2\pi} I_0 a\cos(\phi' - \phi)\frac{e^{-jkR}}{R}d\phi' \end{aligned} \tag{3}$$

There is no difference between ϕ-components in spherical and cylindrical coordinates, so we can use this term in spherical coordinates directly.

Next, we find

$$R = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2} \tag{4}$$

with

$$\begin{aligned} x &= r\sin\theta\cos\phi \quad & x' &= a\cos\phi' \\ y &= r\sin\theta\sin\phi \quad & y' &= a\sin\phi' \\ z &= r\cos\theta \quad & z' &= 0 \end{aligned} \tag{5}$$

then,

$$R = [r^2 - 2r\sin\theta\cos\phi a\cos\phi' - 2r\sin\theta\sin\phi a\sin\phi' + a^2]^{1/2} \tag{6a}$$

$$= [r^2 + a^2 - 2ar\sin\theta\cos(\phi - \phi')]^{1/2} \tag{6b}$$

The expression for A_ϕ becomes

$$A_\phi = \frac{\mu_0 a I_0}{4\pi}\int_0^{2\pi}\cos(\phi - \phi')\frac{\exp[-jk\sqrt{r^2 + a^2 - 2ar\sin\theta\cos(\phi - \phi')}]}{\sqrt{r^2 + a^2 - 2ar\sin\theta\cos(\phi - \phi')}}d\phi'. \tag{7}$$

Due to axial symmetry, A_ϕ does not depend on ϕ. Therefore, we evaluate it at any arbitrary observation angle; for convenience, let's take $\phi = 0$:

$$A_\phi = \frac{\mu_0 a I_0}{4\pi}\int_0^{2\pi}\cos\phi'\frac{\exp(-jk\sqrt{r^2 + a^2 - 2ar\sin\theta\cos\phi'})}{\sqrt{r^2 + a^2 - 2ar\sin\theta\cos\phi'}}d\phi'. \tag{8}$$

This integration is challenging. We approximate the integrand by expanding

$$f(a) = \frac{\exp(-jk\sqrt{r^2 + a^2 - 2ar\sin\theta\cos\phi'})}{\sqrt{r^2 + a^2 - 2ar\sin\theta\cos\phi'}}, \tag{9}$$

in a Maclaurin series (Taylor series about 0), which we first encountered when looking at phase errors surrounding the parallel-ray approximation:

$$f(x) = f(0) + f'(0)x + \frac{1}{2!}f''(0)x^2 + \cdots + \frac{1}{(n-1)!}f^{(n-1)}(0)x^{n-1} + \cdots, \tag{10}$$

which is valid if the loop is small ($a \to 0$). Then, keeping only the first two terms of the series,

$$f(0) = \frac{e^{-jkr}}{r} \tag{11a}$$
$$f'(0) = f'(0)a. \tag{11b}$$

To evaluate $f'(0)$, let

$$u(a) = \sqrt{r^2 + a^2 - 2ar\sin\theta\cos\phi'}. \tag{12}$$

Then,

$$f(u) = \frac{\exp(-jku)}{u} \tag{13}$$

and using the chain rule,

$$\frac{df}{da} = \frac{df}{du}\frac{du}{da} \tag{14a}$$
$$\frac{df}{du} = -jke^{-jku}u^{-1} - u^{-2}e^{-jku} \tag{14b}$$
$$\frac{du}{da} = \frac{\cancel{1}}{\cancel{2}}(r^2 + a^2 - 2ar\sin\theta\cos\phi')^{-1/2} \cdot (\cancel{2}a - \cancel{2}r\sin\theta\cos\phi'). \tag{14c}$$

Then, evaluating about $a = 0$ with $u(0) = r$,

$$f'(a)\big|_{a\to 0} = \left(\frac{-jke^{-jkr}}{r} - \frac{e^{-jkr}}{r^2}\right)\left(-\frac{\cancel{r}\sin\theta\cos\phi'}{\cancel{r}}\right), \tag{15}$$

and

$$\begin{aligned} f(a) &\approx \frac{e^{-jkr}}{r} + \left(\frac{jke^{-jkr}}{r} + \frac{e^{-jkr}}{r^2}\right)\sin\theta\cos\phi' a \\ &\approx e^{-jkr}\left[\frac{1}{r} + a\left(\frac{jk}{r} + \frac{1}{r^2}\right)\sin\theta\cos\phi'\right] \end{aligned} \tag{16}$$

Substituting (16) into (8),

$$\begin{aligned} A_\phi &\approx \frac{\mu_0 a I_0}{4\pi}\left\{\cancel{\int_0^{2\pi}\frac{e^{-jkr}}{r}}\cos\phi' d\phi' + a\left(\frac{jk}{r} + \frac{1}{r^2}\right)\sin\theta\underbrace{\int_0^{2\pi}\cos^2\phi'}_{\pi}\right\} \\ &\approx \frac{\mu_0 a^2 I_0}{4}e^{-jkr}\left(\frac{jk}{r} + \frac{1}{r^2}\right)\sin\theta \end{aligned} \tag{17}$$

We have now completed the radiation integral for $\boldsymbol{A}$. Using the usual process, we next find $\boldsymbol{H}$ using

$$\boldsymbol{H} = \frac{1}{\mu_0}\boldsymbol{\nabla} \times \boldsymbol{A} = \frac{1}{\mu_0}\left[\frac{\hat{\boldsymbol{r}}}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\phi \sin\theta) - \frac{\hat{\boldsymbol{\theta}}}{r}\frac{\partial}{\partial r}(rA_\phi)\right]. \tag{18}$$

The initial step for $\boldsymbol{H}$ is

$$\boldsymbol{H} = \frac{1}{\mu_0}\left[\frac{\hat{\boldsymbol{r}}}{r\sin\theta}\frac{\partial}{\partial theta}(A_\phi \sin\theta) - \frac{\hat{\boldsymbol{\theta}}}{r}\frac{\partial}{\partial r}(rA_\phi)\right], \tag{19}$$

leading to

$$\begin{aligned} H_r &= \frac{a^2 I_0}{4r\cancel{\sin\theta}}e^{-jkr}\left(\frac{jk}{r} + \frac{1}{r^2}\right)2\cancel{\sin\theta}\cos\theta \\ &= \frac{jka^2 I_0 \cos\theta}{2r^2}\left(1 + \frac{1}{jkr}\right)e^{-jkr} \end{aligned} \tag{20}$$

and

$$\begin{aligned} H_\theta &= -\frac{a^2 I_0}{4r}\sin\theta\frac{\partial}{\partial\theta}\left[e^{-jkr}\left(jk + \frac{1}{r}\right)\right] \\ &= -\frac{a^2 I_0}{4r}\sin\theta\left[k^2 e^{-jkr} - \frac{jke^{-jkr}}{r} - \frac{e^{-jkr}}{r^2}\right] \\ &= -\frac{(ka)^2 I_0}{4r}\sin\theta\left(1 + \frac{1}{jkr} - \frac{1}{(kr)^2}\right)e^{-jkr}. \end{aligned} \tag{21}$$

To find the electric field, we use

$$\begin{aligned} \boldsymbol{E} &= \frac{1}{j\omega\epsilon_0}\boldsymbol{\nabla} \times \boldsymbol{H} \\ &= \frac{\eta_0 (ka)^2 I_0}{4r}\sin\theta\left[1 + \frac{1}{jkr}\right]e^{-jkr}\hat{\boldsymbol{\phi}}. \end{aligned} \tag{22}$$