Multiple Reflections on Transmission Lines

The solutions for the voltage and current along a lossless transmission line have been initially derived as

\[
V(z') = \frac{I_L}{2} (Z_L + Z_0) \left[ e^{j\beta z'} + \Gamma_L e^{-j\beta z'} \right]
\]

(1)

\[
I(z') = \frac{I_L}{2Z_0} (Z_L + Z_0) \left[ e^{j\beta z'} - \Gamma_L e^{-j\beta z'} \right]
\]

(2)

However, to actually evaluate these expressions requires that we connect the transmission line to a source (via a source impedance \(Z_g\)), and find the load current \(I_L\). This can be done, as proposed in the textbook, by determining the voltage and current at the input to the line,

\[
V_{in} = V(z' = \ell) = \frac{I_L}{2} (Z_L + Z_0)e^{j\beta \ell} \left[ 1 + \Gamma_L e^{-j2\beta \ell} \right]
\]

(3)

\[
I_{in} = I(z' = \ell) = \frac{I_L}{2Z_0} (Z_L + Z_0)e^{j\beta \ell} \left[ 1 - \Gamma_L e^{-j2\beta \ell} \right]
\]

(4)

and using the fact from KVL that

\[
V_{in} = V_g - I_{in} Z_g
\]

(5)

to solve for \(I_L\), or actually, \(\frac{I_L}{2} (Z_L + Z_0)e^{j\beta \ell}\).

You may wonder what physically happens when the generator is connected to the transmission line, since the source impedance \(Z_g\) may create an additional impedance discontinuity (if \(Z_g \neq Z_0\)), leading to reflection at the generator end in addition to those at the load end. Indeed, this is the case, and you end up getting multiple reflections between the generator and the load. In fact, we can use this to come up with an expression for \(V(z')\) and \(I(z')\) in a more intuitive way than using the expressions involving \(I_L\), which require that \(I_L\) be found first.

To begin, we recognize that that the voltage wave initially launched onto the line has an amplitude of

\[
V_0^+ = \frac{Z_0}{Z_0 + Z_g} V_g,
\]

(6)

which is what would result if the line were infinitely long \((Z_{in} = Z_0)\). The voltage travelling wave is then given by

\[
V_0(z) = V_0^+ e^{-j\beta z}.
\]

(7)

where \(z\) is the distance from the generator (and \(z'\) is the distance from the load, \(z' = \ell - z\).) When the wave reaches the load, it is reflected. Knowing the load reflection coefficient \(\Gamma_L\), we can determine an expression for the reflected wave as

\[
V_0'(z') = V_0^- e^{-j\beta \ell} e^{-j\beta z'} = \Gamma_L V_0^+ e^{-j\beta \ell} e^{-j\beta z'}.
\]

(8)

Notice that there are two exponential terms: one to account for the wave travelling one line-length \(\ell\), and another to account for the distance we have travelled from the load back to the generator, \(z'\).
When the wave reaches the generator, it is re-reflected back into the transmission line according to the reflection coefficient of the generator,

$$\Gamma_g = \frac{Z_g - Z_0}{Z_g + Z_0}. \quad (9)$$

This wave travels in the $+z$ direction and is given by

$$V_1(z) = \Gamma_g V_0^- = \Gamma_g \frac{V_0^+ e^{-j2\beta\ell} e^{-j\beta z}}{z_0^+}, \quad (10)$$

where we note that we have incremented the superscript to denote that this is the second pass the wave is making through the transmission line. The exponential terms reflect the fact that the wave has now made two complete passes through the transmission line, plus the distance from the generator, for a total distance travelled of $2\ell + z$. The second reflection from the load has

$$V_1'(z') = V_1^e^{-j\beta z} = \Gamma_L V_1^+ e^{-j\beta \ell} e^{-j\beta z'} = \Gamma_L^2 \Gamma_g V_0^+ e^{-j3\beta \ell} e^{-j\beta z'}. \quad (11)$$

This process repeats over and over. The total voltage on the line is given by the superposition of these waves,

$$V(z, z') = V_0(z) + V_0'(z') + V_1(z) + V_1'(z') + V_2(z) + V_2'(z') + \cdots$$

$$= V_0^e^{-j\beta z} + \Gamma_L V_0^e^{-j3\beta \ell} e^{-j\beta z'} + \Gamma_g \Gamma_L V_0^e^{-j3\beta \ell} e^{-j\beta z} + \Gamma_L^2 \Gamma_g V_0^e^{-j3\beta \ell} e^{-j\beta z'} + \cdots \quad (12)$$

Factoring out the $V_0^e^{-j\beta z}$ term,

$$V(z, z') = V_0^e^{-j\beta z} \left[1 + \Gamma_L e^{-j2\beta \ell} e^{-j\beta z'} + \Gamma_g \Gamma_L e^{-j2\beta \ell} e^{-j\beta z} + \Gamma_L^2 \Gamma_g e^{-j3\beta \ell} e^{-j\beta z'} + \cdots \right] \quad (13)$$

Since $\ell - z = z'$,

$$V(z, z') = V_0^e^{-j\beta z} \left[1 + \Gamma_L e^{-j2\beta z'} + \Gamma_g \Gamma_L e^{-j2\beta \ell} + \Gamma_L^2 \Gamma_g e^{-j2\beta \ell} e^{-j2\beta z'} + \Gamma_L^2 \Gamma_g e^{-j4\beta \ell} e^{-j2\beta \ell} e^{-j2\beta z'} + \cdots \right]$$

$$= V_0^e^{-j\beta z} \left[1 + \Gamma_g \Gamma_L e^{-j2\beta \ell} + \Gamma_L^2 \Gamma_g e^{-j4\beta \ell} + \cdots \right.$$

$$+ \Gamma_L e^{-j2\beta z'} + \Gamma_L^2 \Gamma_g e^{-j2\beta \ell} + \cdots$$

$$= V_0^e^{-j\beta z} \left[1 + \Gamma_g \Gamma_L e^{-j2\beta \ell} + \Gamma_L^2 \Gamma_g e^{-j4\beta \ell} + \cdots \right.$$

$$\Gamma_L e^{-j2\beta z'} \left(1 + \Gamma_L \Gamma_g e^{-j2\beta \ell} + \Gamma_L^2 \Gamma_g e^{-j4\beta \ell} + \cdots \right)$$

$$= V_0^e^{-j\beta z} \left(1 + \Gamma_L e^{-j2\beta z'} \right) \left[1 + \Gamma_g \Gamma_L e^{-j2\beta \ell} + \Gamma_L^2 \Gamma_g e^{-j4\beta \ell} + \cdots \right] \quad (14)$$

The last term is the form of a series

$$1 + x + x^2 + x^3 + x^4 + \cdots = \sum_{k=0}^{\infty} x^k$$

$$1 + x + x^2 + x^3 + x^4 + \cdots = \sum_{k=0}^{\infty} x^k \quad (15)$$
where \( x = \Gamma_g \Gamma_L e^{-j2\beta\ell} \). Since \( |\Gamma_g| \leq 1 \) and \( \Gamma_L \leq 1 \), this series is convergent and equal to

\[
\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}
\]  

so \( V(z') \) can be expressed as

\[
V(z, z') = \frac{Z_0 V_g}{Z_0 + Z_g} e^{-j\beta z} \frac{1 + \Gamma_L e^{-j2\beta z'}}{1 - \Gamma_g \Gamma_L e^{-j2\beta\ell}}.
\]  

(17)

Through an analogous process,

\[
I(z, z') = \frac{V_g}{Z_0 + Z_g} e^{-j\beta z} \frac{1 - \Gamma_L e^{-j2\beta z'}}{1 - \Gamma_g \Gamma_L e^{-j2\beta\ell}}.
\]  

(18)

Note that this entire multiple-reflection process is captured within the \( I_L \) term. Evaluating (1) at the generator \((z' = \ell)\),

\[
V(z') = \frac{I_L}{2} (Z_L + Z_0) e^{j\beta\ell} \left[ 1 + \Gamma_L e^{-j2\beta\ell} \right].
\]  

(19)

Comparing this to (17) evaluated at the generator \((z = 0, z' = \ell)\), we can see that

\[
\frac{I_L}{2} (Z_L + Z_0) e^{j\beta\ell} = \frac{Z_0 V_g}{Z_0 + Z_g} \frac{1}{1 - \Gamma_g \Gamma_L e^{-j2\beta\ell}}.
\]  

(20)

Since we associate the second term with the infinite series, the infinite multiple-reflection process can be seen to be embedded in the \( I_L \) term.