b) $TM_1$ mode

From Eqs. (10.64) and (10.65):

\[
\begin{align*}
E_y^0(y) &= E_0 \cos \left( \frac{\pi y}{b} \right), \\
H_x^0(y) &= -\frac{E_0}{\eta_0 \sqrt{1-(c_0/b)^2}} \sin \left( \frac{\pi y}{b} \right),
\end{align*}
\]

\[f_c = \frac{1}{2b/\mu_0} = 5 \times 10^9 \text{ (Hz).}\]

\[P_{av} = \frac{w}{2} \int_0^b E_y^0(y) H_x^0(y) dy = \frac{w b E_0^2}{4 \eta_0 \sqrt{1-(c_0/b)^2}}.
\]

Max. \( \frac{P_{av}}{W} = \frac{b (3 \times 10^4)}{4 \eta_0 \sqrt{1-(c_0/b)^2}} = 2.07 \times 10^8 \text{ (W/m)} = 207 \text{ (MW/m)}.\]

c) $TE_1$ mode

From Eqs. (10.84) and (10.85):

\[
\begin{align*}
E_x^0(y) &= E_0 \sin \left( \frac{\pi y}{b} \right), \\
H_y^0(y) &= \frac{E_0}{\eta_0 \sqrt{1-(c_0/f)^2}} \sin \left( \frac{\pi y}{b} \right).
\end{align*}
\]

\[P_{av} = \frac{w}{2} \int_0^b E_x^0(y) H_y^0(y) dy = \frac{w b E_0^2}{4 \eta_0 \sqrt{1-(c_0/f)^2}}.
\]

Max. \( \frac{P_{av}}{W} = \frac{b (3 \times 10^4)}{4 \eta_0 \sqrt{1-(c_0/f)^2}} = 1.55 \times 10^8 \text{ (W/m)} = 155 \text{ (MW/m)}.\]

P.10-12 a) $TM_{21}$ mode b) $TE_{11}$ mode

**Electric field lines**

**Magnetic field lines**

P.10-13 Equations (10-134) through (10-137) for $TM_n$ mode:

\[
E_x^0(x,y) = \frac{2 \alpha u}{\hbar^2} \left( \frac{\pi}{a} \right) E_0 \cos \left( \frac{\pi x}{a} \right) \sin \left( \frac{\pi y}{b} \right).
\]

\[
E_y^0(x,y) = \frac{2 \alpha u}{\hbar^2} \left( \frac{\pi}{b} \right) E_0 \sin \left( \frac{\pi x}{a} \right) \cos \left( \frac{\pi y}{b} \right).
\]
\[ E_x^0(x,y) = E_0 \sin\left(\frac{n_x}{a}\right) \sin\left(\frac{n_y}{b}\right), \]
\[ H_x^0(x,y) = \frac{i\omega}{\varepsilon} \left(\frac{n_x}{b}\right) E_0 \sin\left(\frac{n_x}{a}\right) \cos\left(\frac{n_y}{b}\right), \]
\[ H_y^0(x,y) = -\frac{i\omega}{\mu} \left(\frac{n_y}{a}\right) E_0 \cos\left(\frac{n_x}{a}\right) \sin\left(\frac{n_y}{b}\right). \]

a) Surface current densities:
\[ \mathbf{J}_x(y=0) = \bar{\alpha}_n \times \mathbf{H} \bigg|_{y=0} = \bar{\alpha}_y [\bar{\alpha}_x H_x^0(x,y) + \bar{\alpha}_y H_y^0(x,y)] \]
\[ = -\bar{\alpha}_x H_x^0(x,y) = -\bar{\alpha}_x \frac{i\omega}{\varepsilon} E_0 \sin\left(\frac{n_x}{a}\right) e^{i\beta y} \]
\[ = \mathbf{J}_x(y=b). \]
\[ \mathbf{J}_x(x=0) = \bar{\alpha}_n \times \mathbf{H} \bigg|_{x=0} = \bar{\alpha}_x [\bar{\alpha}_x H_x^0(0,y) + \bar{\alpha}_y H_y^0(0,y)] \]
\[ = \bar{\alpha}_x H_x^0(0,y) = -\bar{\alpha}_x \frac{i\omega}{\mu} E_0 \cos\left(\frac{n_y}{b}\right) e^{-i\beta x} \]
\[ = \mathbf{J}_x(x=a). \]

P. 10-14 Rectangular waveguide: \( a = 7.21 \text{ cm}, \ b = 3.40 \text{ cm}. \)

Eq. (10-140): \( (\lambda_c)_{mn} = \frac{2}{\sqrt{(\frac{m}{a})^2 + (\frac{n}{b})^2}} \).

Modes with the shortest \( \lambda_c < 5 \text{ cm} \) are:

<table>
<thead>
<tr>
<th>Mode</th>
<th>TE_{10}</th>
<th>TE_{20}</th>
<th>TE_{01}</th>
<th>TE_{11}/TM_{11}</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \lambda ) (cm)</td>
<td>14.4</td>
<td>7.20</td>
<td>6.80</td>
<td>6.15</td>
</tr>
</tbody>
</table>

a) For \( \lambda = 10 \text{ cm} \), the only propagating mode is TE_{10}.

b) For \( \lambda = 5 \text{ cm} \), the propagating modes are: TE_{10}, TE_{20}, TE_{01}, TE_{11}, and TM_{11}.
\[ u_{en} = u \sqrt{1 - \left(\frac{f}{f_{0}}\right)^2} \]

For the TE\(_{10}\) mode, \( f_{0} = \frac{u}{2a} \).

\[ u_{en} = u \sqrt{1 - \left(\frac{u}{2a f_{0}}\right)^2} = \frac{f}{\sqrt{\mu \varepsilon \varepsilon_{0} f_{0}}} \sqrt{1 - \left(\frac{f}{f_{0}}\right)^2} \]

**P.10-15**

\[ f_{c} = \frac{f}{2\sqrt{\mu \varepsilon}} \sqrt{\frac{(m/a)^2 - (n/b)^2}{(m/a)^2}} = \frac{1}{2a \sqrt{\mu \varepsilon}} F(m,n) \]

<table>
<thead>
<tr>
<th>Modes</th>
<th>F(m,n)</th>
<th>Modes</th>
<th>F(m,n)</th>
</tr>
</thead>
<tbody>
<tr>
<td>TE(_{10})</td>
<td>1</td>
<td>TE(<em>{10}), TE(</em>{01})</td>
<td>1</td>
</tr>
<tr>
<td>TE(<em>{01}), TE(</em>{20})</td>
<td>2</td>
<td>TE(<em>{11}), TM(</em>{11})</td>
<td>(\sqrt{2})</td>
</tr>
<tr>
<td>TE(<em>{11}), TM(</em>{11})</td>
<td>(\sqrt{5})</td>
<td>TE(<em>{22}), TE(</em>{20})</td>
<td>2</td>
</tr>
<tr>
<td>TE(_{02})</td>
<td>4</td>
<td>TM(_{12})</td>
<td>(\sqrt{5})</td>
</tr>
<tr>
<td>TM(_{11})</td>
<td>(\sqrt{17})</td>
<td>TM(_{22})</td>
<td>2(\sqrt{2})</td>
</tr>
<tr>
<td>TM(_{12})</td>
<td>(\sqrt{20})</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**P.10-16**

Let \( a = k b \), \( 1 < k < 2 \). \( f_{c} = \frac{3 \times 10^8}{2a} \sqrt{m^2 + k^2 n^2} \).

\( f = 3 \times 10^8 \) (Hz), \( \lambda = c/f = 0.1 \) (m).

\( a = k b \), \( 1 < k < 2 \). \( f_{c} = \frac{3 \times 10^8}{2a} \sqrt{m^2 + k^2 n^2} \).

\( a = 6.5 \) (cm) and \( b = 3.5 \) (cm).

**P.10-17**

\( u_{p} = \frac{c}{\sqrt{1 - (\lambda/2a)^2}} = 4.70 \times 10^8 \) (m/s),

\[ \lambda_{g} = \frac{2\pi}{\sqrt{1 - (\lambda/2a)^2}} = 0.157 \) (m) = 15.7 \) (cm),

\( \beta = \frac{2\pi}{\lambda_{g}} = 40.1 \) (rad/m),

\( Z_{TE} = \frac{\eta_{0}}{\sqrt{1 - (\lambda/2a)^2}} = 590 \) (\(\Omega\)).
b) From Eqs. (10-159), (10-160), and (10-158):
\[ E_y^0 = E_0 \sin \left( \frac{\pi x}{a} \right) \]
\[ H_x^0 = -\frac{E_0}{\eta_0} \sqrt{1 - \left( \frac{f_0}{f} \right)^2} \sin \left( \frac{\pi x}{a} \right) \]
\[ H_z^0 = \frac{f_0}{\eta_0} \frac{E_0}{\eta_0} \cos \left( \frac{\pi x}{a} \right) \]
\[ P_{av} = \frac{1}{2} \int_0^a \int_0^b \left( -E_y^0 H_z^0 \right) dx \, dy = \frac{E_0^2 ab}{4 \eta_0} \sqrt{1 - \left( \frac{f_0}{f} \right)^2} \]

For \( P_{av} = 10^3 \) (W) at the load (antenna), assuming
under matched conditions:
\[ |E_y^0| = E_0 = 94.800 \, (\text{V/m}), \quad |H_x^0| = 127.4 \, (\text{A/m}), \quad |H_z^0| = 167.6 \, (\text{A/m}) \]
The waveguide is 1(m) long. The field intensities are higher at the sending end by a factor of \( e^{ax} \approx 1.138 \).
\[ \text{Max.} |E_y^0| = 10.788 \, (\text{V/m}), \quad \text{Max.} |H_x^0| = 213.3 \, (\text{A/m}), \quad \text{Max.} |H_z^0| = 190.7 \, (\text{A/m}). \]

\( \tilde{f} \)\( (x=0) = \tilde{a}_y x (\tilde{a}_x H_x^0 + \tilde{a}_z H_z^0) \)\( (y=0) = \tilde{a}_y \left( \frac{f_0}{f} \right) \frac{E_0}{\eta_0} \)
\[ |\tilde{f} (x=0)| = |H_x^0| = 167.6 \, (\text{A/m}) \]
\[ \tilde{f} (y=0) = \tilde{a}_y x (\tilde{a}_x H_x^0 + \tilde{a}_z H_z^0) \]\( (y=0) = -\tilde{a}_z H_x^0 (x,0) + \tilde{a}_x H_z^0 (x,0) \)
\[ |\tilde{f} (y=0)| = \left[ \left( H_x^0 \right)^2 + \left( H_z^0 \right)^2 \right]^{1/2} = \frac{E_0}{\eta_0} \left\{ \left[ \left( \frac{f_0}{f} \right)^2 + 1 - 2 \left( \frac{f_0}{f} \right)^2 \sin^2 \left( \frac{\pi x}{a} \right) \right] \right\}^{1/2} \]
which is maximum at \( x = a/2 \).
At the sending end:\[ \text{Max.} |\tilde{f}| = \frac{E_0}{\eta_0} \sqrt{1 - \left( \frac{f_0}{f} \right)^2} \times 1.138 = 213.3 \, (\text{A/m}) \]

d) Total amount of average power dissipated in 1(m) of waveguide:
\[ P_d = 1000 \, (e^{2\pi k} - 1) = 1000 \, (e^{0.02\pi} - 1) = 26.2 \, (\text{W}). \]

P.10-21. From problem P.10-20, we have
\[ P_{av} = \frac{E_0^2 ab}{4 \eta_0} \sqrt{1 - \left( \frac{f_0}{f} \right)^2} \times \sqrt{1 - \left( \frac{f_0}{f} \right)^2} = 0.745. \]
\[ \therefore \text{Max.} P_{av} = \frac{(3 \times 10^6)^4 \times (2.25 \times 10^6)}{4 \times 120 \pi} \times 0.745 = 10^4 \, (\text{W}) = 1 \, (\text{MW}). \]
Problem 8.39  A hollow rectangular waveguide is to be used to transmit signals at a carrier frequency of 6 GHz. Choose its dimensions so that the cutoff frequency of the dominant TE mode is lower than the carrier by 25% and that of the next mode is at least 25% higher than the carrier.

Solution:

For $m = 1$ and $n = 0$ (TE$_{10}$ mode) and $u_{p0} = c$ (hollow guide), Eq. (8.106) reduces to

$$f_{10} = \frac{c}{2a}.$$ 

Denote the carrier frequency as $f_0 = 6$ GHz. Setting

$$f_{10} = 0.75 f_0 = 0.75 \times 6 \text{ GHz} = 4.5 \text{ GHz},$$

we have

$$a = \frac{c}{2f_{10}} = \frac{3 \times 10^8}{2 \times 4.5 \times 10^9} = 3.33 \text{ cm}.$$ 

If $b$ is chosen such that $a > b > \frac{a}{2}$, the second mode will be TE$_{01}$, followed by TE$_{20}$ at $f_{20} = 9$ GHz. For TE$_{01}$,

$$f_{01} = \frac{c}{2b}.$$ 

Setting $f_{01} = 1.25 f_0 = 7.5$ GHz, we get

$$b = \frac{c}{2f_{01}} = \frac{3 \times 10^8}{2 \times 7.5 \times 10^9} = 2 \text{ cm}.$$
Problem 8.40  A TE wave propagating in a dielectric-filled waveguide of unknown permittivity has dimensions \( a = 5 \text{ cm} \) and \( b = 3 \text{ cm} \). If the \( x \)-component of its electric field is given by

\[
E_x = -36 \cos(40\pi x) \sin(100\pi y) \sin(2.4\pi \times 10^{10} t - 52.9\pi z), \quad (\text{V/m})
\]

determine:

(a) the mode number,
(b) \( \varepsilon_r \) of the material in the guide,
(c) the cutoff frequency, and
(d) the expression for \( H_y \).

Solution:

(a) Comparison of the given expression with Eq. (8.110a) reveals that

\[
\frac{m\pi}{a} = 40\pi, \quad \text{hence } m = 2
\]

\[
\frac{n\pi}{b} = 100\pi, \quad \text{hence } n = 3.
\]

Mode is TE\(_{23}\).

(b) From \( \sin(\omega t - \beta z) \), we deduce that

\[
\omega = 2.4\pi \times 10^{10} \text{ rad/s}, \quad \beta = 52.9\pi \text{ rad/m}.
\]

Using Eq. (8.105) to solve for \( \varepsilon_r \), we have

\[
\varepsilon_r = \frac{c^2}{\omega^2} \left[ \beta^2 + \left( \frac{m\pi}{a} \right)^2 + \left( \frac{n\pi}{b} \right)^2 \right] = 2.25.
\]

(c)

\[
u_{p0} = \frac{c}{\sqrt{\varepsilon_r}} = \frac{3 \times 10^8}{\sqrt{2.25}} = 2 \times 10^8 \text{ m/s}.
\]

\[
f_{23} = \frac{\nu_{p0}}{2} \sqrt{\left( \frac{2}{a} \right)^2 + \left( \frac{3}{b} \right)^2} = 10.77 \text{ GHz}.
\]
(d)

\[
Z_{TE} = \frac{E_x}{H_y} = \eta \sqrt{\frac{1}{1 - \left(\frac{f_23}{f}\right)^2}} \\
= \frac{377}{\sqrt{\varepsilon_r}} \sqrt{1 - \left(\frac{10.77}{12}\right)^2} = 569.9 \Omega.
\]

Hence,

\[
H_y = \frac{E_x}{Z_{TE}} \\
= -0.063 \cos(40\pi x) \sin(100\pi y) \sin(2.4\pi \times 10^{10}t - 52.9\pi z) \quad (A/m).
\]
Problem 8.41  A waveguide filled with a material whose \( \varepsilon_r = 2.25 \) has dimensions \( a = 2 \) cm and \( b = 1.4 \) cm. If the guide is to transmit 10.5-GHz signals, what possible modes can be used for the transmission?

Solution:
Application of Eq. (8.106) with \( u_{p_0} = c/\sqrt{\varepsilon_r} = 3 \times 10^8/\sqrt{2.25} = 2 \times 10^8 \) m/s, gives:

\[
\begin{align*}
    f_{10} & = 5 \text{ GHz (TE only)} \\
    f_{01} & = 7.14 \text{ GHz (TE only)} \\
    f_{11} & = 8.72 \text{ GHz (TE or TM)} \\
    f_{20} & = 10 \text{ GHz (TE only)} \\
    f_{21} & = 12.28 \text{ GHz (TE or TM)} \\
    f_{12} & = 15.1 \text{ GHz (TE or TM)}.
\end{align*}
\]

Hence, any one of the first four modes can be used to transmit 10.5-GHz signals.
Problem 8.43  A waveguide, with dimensions $a = 1$ cm and $b = 0.7$ cm, is to be used at 20 GHz. Determine the wave impedance for the dominant mode when
(a) the guide is empty, and
(b) the guide is filled with polyethylene (whose $\varepsilon_r = 2.25$).

Solution:  
For the TE$_{10}$ mode, 
\[ f_{10} = \frac{u_{02}}{2a} = \frac{c}{2a\sqrt{\varepsilon_r}}. \]

When empty, 
\[ f_{10} = \frac{3 \times 10^8}{2 \times 10^{-2}} = 15 \text{ GHz}. \]

When filled with polyethylene, $f_{10} = 10$ GHz.

According to Eq. (8.111), 
\[ Z_{TE} = \frac{\eta}{\sqrt{1 - (f_{10}/f)^2}} = \frac{\eta_0}{\sqrt{\varepsilon_r} \sqrt{1 - (f_{10}/f)^2}}. \]

When empty, 
\[ Z_{TE} = \frac{377}{\sqrt{1 - (15/20)^2}} = 570 \Omega. \]

When filled, 
\[ Z_{TE} = \frac{377}{\sqrt{2.25} \sqrt{1 - (10/20)^2}} = 290 \Omega. \]