

# Review of Maxwell's Equations

## 1 Field Quantities

Before reviewing Maxwell's equations let's recall some basic field quantities and their units.

Symbol	Quantity	Units
$\vec{D}(\vec{r}, t)$	Electric flux density	Coulombs/m <sup>2</sup> (C/m <sup>2</sup> )
$\rho_v(\vec{r}, t)$	Electric charge density	C/m <sup>3</sup>
$\Phi(\vec{r}, t)$	Magnetic flux in Webers	Webers (Wb)
$\vec{B}(\vec{r}, t)$	Magnetic flux density	Wb/m <sup>2</sup> or Teslas (T)
$\vec{E}(\vec{r}, t)$	Electric field	Volts/m (V/m)
$\vec{H}(\vec{r}, t)$	Magnetic field	Amperes/m (A/m)
$\vec{J}(\vec{r}, t)$	Conduction current density	A/m <sup>2</sup>

We note that each quantity is in general a function of position in space (here expressed in terms of a position vector  $\vec{r}$ , which will be dropped for compactness. Each quantity is a function of time  $t$  unless where are considering an electrostatic situation.

Constitutive relations, which relate some of the parameters are, for simple (linear, isotropic, reciprocal, non-dispersive, and homogenous) media:

$$\vec{D}(\vec{r}, t) = \epsilon \vec{E}(\vec{r}, t) \quad (1)$$

$$\vec{B}(\vec{r}, t) = \mu \vec{H}(\vec{r}, t) \quad (2)$$

$$\vec{J}(\vec{r}, t) = \sigma \vec{E}(\vec{r}, t), \quad (3)$$

where  $\epsilon$ ,  $\mu$ , and  $\sigma$  are the permittivity, permeability, and conductivity (respectively) of the medium where the equations are evaluated. For the purposes of this course, we will only consider the case where these quantities are scalars (i.e. not tensors, which are associated with anisotropic media). Furthermore, we will restrict ourselves to  $\mu = \mu_0$  (non-magnetic materials only), and  $\sigma \geq 0$  (passive media). In free space,  $\epsilon_0 = 8.854 \times 10^{-12}$  F/m,  $\mu_0 = 4\pi \times 10^{-7}$  H/m, and  $\sigma_0 = 0$  S/m.

It's easiest to recall Maxwell's equations by starting with some well-known relations we remember from electrostatics, and the extending these relations to electrodynamics, from which Maxwell's equations are borne.

## 2 Gauss' Law

Gauss' law in integral form is

$$\oiint_S \vec{E}(\vec{r}, t) \cdot d\vec{s} = \frac{Q(t)}{\epsilon_0} = \iiint_V \frac{\rho_v(\vec{r}, t)}{\epsilon_0} dv, \quad (4)$$

where  $Q(t)$  is the total charge enclosed by the surface  $S$ , or conversely the volume  $V$ . In simple media, Gauss' law is often written as

$$\oiint_S \vec{D}(\vec{r}, t) \cdot d\vec{s} = Q(t) = \iiint_V \rho_v(\vec{r}, t) dv, \quad (5)$$

Gauss' law states that the electric flux  $\vec{D}(\vec{r}, t)$  passing through any closed surface  $S$  is equal to the total charge enclosed by the surface. The total charge enclosed by the surface can be evaluated by performing a volume integral of the volumetric charge density  $\rho_v(\vec{r}, t)$  of the volume of the surface.

The *divergence theorem* can also be used to calculate the total enclosed charge to the electric flux density:

$$\underbrace{\oiint_S \vec{D}(\vec{r}, t) \cdot d\vec{s}}_{\text{closed surface integral}} = \underbrace{\iiint_V \nabla \cdot \vec{D}(\vec{r}, t) dv}_{\text{volume integral}}. \quad (6)$$

The divergence theorem allows a closed surface integral to be replaced by a volume integral, or vice versa, depending on what is more convenient to evaluate. Equating the integrand of Equation (4) with that of Equation (6), we have the first of Maxwell's equations in so-called "differential" or "point" form:

$$\nabla \cdot \vec{D}(\vec{r}, t) = \rho_v(\vec{r}, t). \quad (7)$$

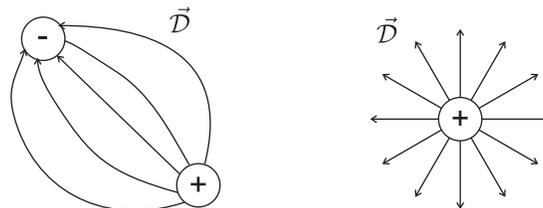
Succinctly, Gauss' Law says that electric flux densities always diverge away from electric sources (charges).

### 3 "Gauss' Law for Magnetic Fields"

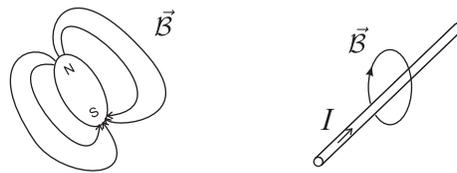
Having derived Gauss' law for electrostatics it is a natural question if the same thing exists for magnetic fields. We know that the magnetic flux  $\Phi$  is given by

$$\Phi = \iint_S \vec{B}(\vec{r}, t) \cdot d\vec{s}. \quad (8)$$

$\vec{B}(\vec{r}, t)$  is the magnetic equivalent of flux density for electric fields  $\vec{D}(\vec{r}, t)$ . Gauss' law says that lines of electric flux must begin/terminate on positive/negative charges, respectively, as demonstrated in the illustration below. However, since free charge is allowed to exist the divergence of an electric field (or electric field density) can be nonzero.



For magnetic fields, no such singular source of magnetic fields lines is known to exist: even the simplest source of a magnetic field produces a magnetic field  $\vec{H}(\vec{r}, t)$  (or proportionately,  $\vec{B}(\vec{r}, t)$ ) in concentric circles that do not diverge, as shown in the illustration below. Similarly, an elementary magnetic field source such as an infinite line current produces a magnetic field that encircles the current filament, as shown.



Hence,

$$\oiint_S \vec{B}(\vec{r}, t) \cdot d\vec{s} = 0 \quad (9)$$

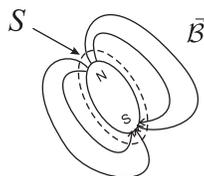
and using the Divergence Theorem

$$\oiint_S \vec{B}(\vec{r}, t) \cdot d\vec{s} = \iiint_V \nabla \cdot \vec{B}(\vec{r}, t) dv' = 0, \quad (10)$$

which yields

$$\nabla \cdot \vec{B}(\vec{r}, t) = 0. \quad (11)$$

This equation says that if one encloses a source of magnetic flux, the net flux density through the enclosing surface must be zero. This is demonstrated graphically the drawing below, which shows an arbitrary closed surface enclosing a source of magnetic field lines.



The divergence across the surface is zero, since for each magnetic field line that exits the surface, there is a corresponding re-entrance of the field line at some other point on the surface, yielding a net divergence of zero over the surface. Succinctly, this second law states that magnetic field lines *always* formed closed loops, i.e., they do not diverge.

## 4 Faraday's Law

You will recall in electrostatics that

$$\oint_C \vec{E}(\vec{r}) \cdot d\vec{\ell} = 0. \quad (12)$$

The right hand side of this equation is zero due to the *conservative* nature of a static electric field. Like a gravitational field, the amount of net work done in moving a charge/object along a closed path in the presence of a conservative field is zero.

Recall that Stokes' theorem can be used to transform a closed line integral to a surface integral:

$$\oint_C \vec{E}(\vec{r}) \cdot d\vec{\ell} = \iint_S (\nabla \times \vec{E})(\vec{r}) \cdot d\vec{s}', \quad (13)$$

so

$$\nabla \times \vec{\mathcal{E}}(\vec{r}) = 0 \quad (14)$$

for electrostatics.

You will recall that in electrodynamics, the only difference with Faraday's law is that the right hand side of Equation (14) is no longer 0. Instead, in a closed circuit, a time-varying magnetic field can induce an *electromotive force (EMF)* or voltage in the circuit equal to

$$\oint_C \vec{\mathcal{E}}(\vec{r}, t) \cdot d\vec{\ell} = -\frac{d\Phi(t)}{dt}, \quad (15)$$

where we note the time dependence in the equation now, and recall that the (-) sign is due to Lenz' Law. Using Equation (8), we can subsequently write

$$\frac{d\Phi}{dt} = \frac{d}{dt} \iint_S \vec{\mathcal{B}}(\vec{r}, t) \cdot d\vec{s}. \quad (16)$$

We can move the time derivative inside the integral, bearing in mind that  $\vec{\mathcal{B}}(\vec{r}, t)$  is also a function of position<sup>1</sup>, yielding

$$\frac{d\Phi}{dt} = \iint_S \frac{\partial \vec{\mathcal{B}}(\vec{r}, t)}{\partial t} \cdot d\vec{s}. \quad (17)$$

Applying Stokes' theorem,

$$-\iint_S \frac{\partial \vec{\mathcal{B}}(\vec{r}, t)}{\partial t} \cdot d\vec{s} = \oint_C \vec{\mathcal{E}}(\vec{r}, t) \cdot d\vec{\ell} = \iint_S [\nabla \times \vec{\mathcal{E}}(\vec{r}, t)] \cdot d\vec{s}. \quad (18)$$

We can equate the integrands to obtain

$$\nabla \times \vec{\mathcal{E}}(\vec{r}, t) = -\frac{\partial \vec{\mathcal{B}}(\vec{r}, t)}{\partial t}, \quad (19)$$

which is Maxwell's third equation.

## 5 Ampere's Law

Maxwell's fourth equation for electrodynamics comes from Ampere's Law, which you may recall for statics is:

$$\oint_C \vec{\mathcal{H}}(\vec{r}) \cdot d\vec{\ell} = I = \iint_S \vec{\mathcal{J}}(\vec{r}) \cdot d\vec{s}. \quad (20)$$

Again, Stokes tells us that

$$\oint_C \vec{\mathcal{H}}(\vec{r}) \cdot d\vec{\ell} = I = \iint_S [\nabla \times \vec{\mathcal{H}}(\vec{r})] \cdot d\vec{s} = \iint_S \vec{\mathcal{J}}(\vec{r}) \cdot d\vec{s}. \quad (21)$$

Therefore,

$$\nabla \times \vec{\mathcal{H}}(\vec{r}) = \vec{\mathcal{J}}(\vec{r}) \quad (22)$$

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<sup>1</sup>Note partial derivative

is the point form of Ampere's Law.

Let's assume there is time variation now, and take the divergence of both sides of the equation. This gives

$$\nabla \cdot \nabla \times \vec{\mathcal{H}}(\vec{r}, t) = \nabla \cdot \vec{\mathcal{J}}(\vec{r}, t) = 0, \quad (23)$$

where we have zero because the divergence of a curl is zero (this is a vector identity). However, recall the equation of continuity:

$$\nabla \cdot \vec{\mathcal{J}}(\vec{r}, t) = -\frac{\partial \rho_v(\vec{r}, t)}{\partial t}, \quad (24)$$

which tells us that  $\nabla \cdot \vec{\mathcal{J}}(\vec{r}, t)$  cannot equal zero in general, unless the charge density itself is constant with time or zero. Hence, Equation (23) is not generally suitable when time variations are present and it must be modified.

A suitable modification is to let

$$\nabla \times \vec{\mathcal{H}}(\vec{r}, t) = \vec{\mathcal{J}}(\vec{r}, t) + \vec{\mathcal{G}}(\vec{r}, t), \quad (25)$$

where  $\vec{\mathcal{G}}(\vec{r}, t)$  is some additional current density to be determined. Then,

$$\nabla \cdot \vec{\mathcal{G}}(\vec{r}, t) = \frac{\partial \rho_v(\vec{r}, t)}{\partial t} = \frac{\partial}{\partial t} [\nabla \cdot \vec{\mathcal{D}}(\vec{r}, t)] \quad (26)$$

which implies that

$$\vec{\mathcal{G}}(\vec{r}, t) = \frac{\partial \vec{\mathcal{D}}(\vec{r}, t)}{\partial t}. \quad (27)$$

We call this new quantity  $\vec{\mathcal{G}}(\vec{r}, t)$  a *displacement current density* since it results from a time-varying displacement (flux) density. A suitable analogy is the well-known "non-conduction" current that flows through a capacitor under time-varying conditions.

Maxwell's fourth equation is then written as:

$$\nabla \times \vec{\mathcal{H}}(\vec{r}, t) = \vec{\mathcal{J}}(\vec{r}, t) + \frac{\partial \vec{\mathcal{D}}(\vec{r}, t)}{\partial t}. \quad (28)$$

## 6 Summary of Maxwell's Equations

We write the general form of Maxwell's equations as follows.

$$\nabla \cdot \vec{\mathcal{D}}(\vec{r}, t) = \rho_v(\vec{r}, t) \quad (29)$$

$$\nabla \cdot \vec{\mathcal{B}}(\vec{r}, t) = 0 \quad (30)$$

$$\nabla \times \vec{\mathcal{E}}(\vec{r}, t) = -\frac{\partial \vec{\mathcal{B}}(\vec{r}, t)}{\partial t} \quad (31)$$

$$\nabla \times \vec{\mathcal{H}}(\vec{r}, t) = \vec{\mathcal{J}}(\vec{r}, t) + \frac{\partial \vec{\mathcal{D}}(\vec{r}, t)}{\partial t} = \vec{\mathcal{J}}_{src}(\vec{r}, t) + \sigma \vec{\mathcal{E}}(\vec{r}, t) + \frac{\partial \vec{\mathcal{D}}(\vec{r}, t)}{\partial t}. \quad (32)$$

In the last equation, the current density  $\vec{J}(\vec{r}, t)$  has been split into a source term  $\vec{J}_{src}(\vec{r}, t)$ , which is imposed by an external source, and a conduction current density term  $\sigma\vec{E}(\vec{r}, t)$  which results from conduction losses in the material. An important observation about these equations is that the terms  $\rho_v(\vec{r}, t)$  and  $\vec{J}_{src}(\vec{r}, t)$  are the ultimate *sources* or driving terms for all the other quantities in Maxwell's equations. In fact, the equation of continuity (24) relates  $\rho_v(\vec{r}, t)$  and  $\vec{J}_{src}(\vec{r}, t)$ , so ultimately, *electric currents* are responsible for producing all of the field quantities in Maxwell's equations. An antenna is simply a conducting structure along which electric currents are formed to produce fields that propagate on their own into space.

## 7 Harmonic Time Dependence: Phasor Form of Maxwell's Equations

Very often, we are interested in the behaviour of Maxwell's equations (or other equations) at a single frequency in the steady state. Under these conditions, fields take on a sinusoidal form; for example, considering an electric field with sinusoidal (harmonic) time dependence could be represented as

$$\vec{E}(\vec{r}, t) = \vec{A}(\vec{r}) \cos(\omega t + \phi), \quad (33)$$

where:

- $\vec{A}$  is the vector amplitude associated with the field at position  $(\vec{r})$ ,
- $\omega$  is the frequency of the field in rad/s, and
- $\phi$  is the phase of the field.

The equation

$$\cos(\omega t + \phi) \quad (34)$$

can be written as

$$\text{Re} \{ e^{j(\omega t + \phi)} \} = \text{Re} \{ e^{j\omega t} \cdot e^{j\phi} \}. \quad (35)$$

We write the phasor form of  $\vec{E}$  by dropping the harmonic term and removing the  $\text{Re}()$  to create a general complex quantity:

$$\mathbf{E}(\vec{r}) = \vec{A}(\vec{r}) e^{j\phi}. \quad (36)$$

Note the use of the bold Roman symbol to denote phasor-vector quantities. To convert back to the general real time-varying vector field, simply multiply by  $e^{j\omega t}$  and take the real part. Think of this as a sort of "phasor transform":

$$\vec{E}(\vec{r}, t) = \text{Re} [\mathbf{E}(\vec{r}) e^{j\omega t}]. \quad (37)$$

In phasors, we usually drop the explicit dependence on position  $\vec{r}$ .

A quantity with harmonic time dependence is quite easy to take the derivative of. For example,

$$\frac{\partial \vec{E}(\vec{r}, t)}{\partial t} = \text{Re} \left[ \frac{\partial}{\partial t} \mathbf{E}(\vec{r}) e^{j\omega t} \right] = \text{Re} [j\omega \mathbf{E}(\vec{r}) e^{j\omega t}]. \quad (38)$$

Therefore, we simply replace  $\frac{\partial}{\partial t}$  by  $j\omega$  when we have harmonic time dependence. Maxwell's equations in phasor form are:

$$\nabla \cdot \mathbf{D}(\vec{r}) = \rho_v(\vec{r}) \quad (39)$$

$$\nabla \cdot \mathbf{B}(\vec{r}) = 0 \quad (40)$$

$$\nabla \times \mathbf{E}(\vec{r}) = -j\omega \mathbf{B}(\vec{r}) \quad (41)$$

$$\nabla \times \mathbf{H}(\vec{r}) = j\omega \mathbf{D}(\vec{r}) + \sigma \mathbf{E}(\vec{r}) + \mathbf{J}_{src}(\vec{r}). \quad (42)$$

This is a real simplification since time derivatives are replaced with simple multiplications by scalars. We will be working primarily with phasor quantities in this course, therefore, we will use the bold Roman symbol notation for vectors, and also use that notation for other vectors such as position vectors.

You may be wondering what happens in the case of non-sinusoidal, finite-bandwidth signals like what we would encounter in a real-life system. A version of Maxwell's equations resembling the phasor form are still valid because of the linearity in time of Maxwell's equations. Hence, we can use a Fourier Transform to represent field in the frequency domain. For example, the electric field can be represented in the frequency domain using a Fourier Transform as

$$\vec{E}(\vec{r}, \omega) = \int_{-\infty}^{\infty} \vec{\mathcal{E}}(\vec{r}, t) e^{-j\omega t} dt. \quad (43)$$

The corresponding inverse Fourier transform is

$$\vec{\mathcal{E}}(\vec{r}, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \vec{E}(\vec{r}, \omega) e^{j\omega t} d\omega. \quad (44)$$

Consider the evaluation of  $\nabla \times \vec{\mathcal{E}}(\vec{r}, t) = -\frac{\partial \vec{\mathcal{B}}(\vec{r}, t)}{\partial t}$  using the frequency domain representation of the fields and an inverse Fourier transform:

$$\nabla \times \frac{1}{2\pi} \int_{-\infty}^{\infty} \vec{E}(\vec{r}, \omega) e^{j\omega t} d\omega = -\frac{1}{2\pi} \frac{\partial}{\partial t} \int_{-\infty}^{\infty} \vec{B}(\vec{r}, \omega) e^{j\omega t} d\omega \quad (45)$$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} [\nabla \times \vec{E}(\vec{r}, \omega) e^{j\omega t}] d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} [-j\omega \vec{B}(\vec{r}, \omega)] e^{j\omega t} d\omega. \quad (46)$$

Equating integrands,

$$\nabla \times \vec{E}(\vec{r}, \omega) = -j\omega \vec{B}(\vec{r}, \omega), \quad (47)$$

demonstrates that the resulting form of Maxwell's third equation is similar to the phasor form. The same can be shown for the other three equations.

## 8 Fields in Lossy Media

A general medium may have some nonzero conductivity  $\sigma$ . The permittivity of the medium in the frequency domain can then expressed as a complex quantity

$$\epsilon = \epsilon' - j\epsilon'' \quad (48)$$

where  $\epsilon'' = \frac{\sigma}{\omega} > 0^2$ . To see how this comes about we consider Maxwell's curl equation for  $\mathbf{H}$  in a source-free region ( $\mathbf{J}_{src} = 0$ ). Since  $\sigma$  is nonzero now, a conduction current density  $\mathbf{J} = \sigma \mathbf{E}$  will be produced, and the equation becomes:

$$\nabla \times \mathbf{H}(\vec{r}) = j\omega \mathbf{D}(\vec{r}) + \mathbf{J}(\vec{r}) \quad (49)$$

$$= j\omega \epsilon \mathbf{E}(\vec{r}) + \sigma \mathbf{E}(\vec{r}) \quad (50)$$

$$= j\omega \left( \epsilon - j \frac{\sigma}{\omega} \right) \mathbf{E}(\vec{r}) \quad (51)$$

$$\equiv j\omega (\epsilon' - j\epsilon'') \mathbf{E}(\vec{r}). \quad (52)$$

$$(53)$$

## 9 Poynting Vector

Let's consider a quantity quantity  $\mathbf{P}(\vec{r})$ , called the *time average Poynting vector*, as follows<sup>3</sup>:

$$\mathbf{P}(\vec{r}) = \frac{1}{2} \mathbf{E}(\vec{r}) \times \mathbf{H}^*(\vec{r}). \quad (54)$$

Since the units of  $\mathbf{E}$  are V/m, and the units of  $\mathbf{H}$  are A/m, we can see the units of  $\mathbf{P}$  are W/m<sup>2</sup>. Hence,  $\mathbf{P}(\vec{r})$  represents the *power density* of the fields in question<sup>4</sup>.

Knowing the power density, the total time-average power leaving a closed surface  $S$  can be found as

$$W = \frac{1}{2} \oiint_S (\mathbf{E}(\vec{r}) \times \mathbf{H}^*(\vec{r})) \cdot d\mathbf{s} = \oiint_S \mathbf{P}(\vec{r}) \cdot d\mathbf{s}. \quad (55)$$

The concept of power and power density is very important in RF systems because the performance of the communication system is directly determined by the received signal *power* at the receiver. The reason for this is that the presence of noise in all systems degrades signal quality, and we know that noise is described in terms of power densities. Hence, to overcome a given noise power, we must have sufficient signal power, in order to accurately detect a signal.

**Example:** Show that the time-average Poynting vector for a uniform plane wave travelling in free space in the  $+z$  direction is<sup>5</sup>

$$\mathbf{P} = \frac{|E_0|^2}{2\eta_0} \hat{\mathbf{z}}. \quad (56)$$

<sup>2</sup>Usually, in advanced texts,  $\epsilon''$  is used to account for the damping of the vibrating dipole moments (complex electric susceptibility) produced in lossy dielectrics, and ohmic losses produced by  $\sigma$  treated separately. However, here we will assume the bulk of the loss originates from the medium's conductivity, rather than damping.

<sup>3</sup>In ECE320, time-average Poynting vector only considered the real part of  $\mathbf{P}(\vec{r})$ , but in general, power flux density can have both real and imaginary parts.

<sup>4</sup>Note that  $\mathcal{S}$  is used in some books (e.g. Ulaby) to denote power density. Since we will be carrying out surface integrals over a surface denoted by  $S$ , we choose  $\mathbf{P}$  for clarity.

<sup>5</sup>You may need to consult the notes on uniform plane waves first.

$$\mathbf{P} = \frac{1}{2} \mathbf{E} \times \mathbf{H}^* \quad (57)$$

$$= \frac{1}{2} \mathbf{E}_0 e^{-j\mathbf{k}\cdot\mathbf{r}} \times \left[ \frac{1}{\eta} \hat{\mathbf{n}} \times \mathbf{E}_0 e^{-j\mathbf{k}\cdot\mathbf{r}} \right]^* \quad (58)$$

For example, if  $\mathbf{k} = k_0 \hat{\mathbf{z}}$ ,  $\hat{\mathbf{n}} = \hat{\mathbf{z}}$ , and  $\mathbf{E}_0 = E_x \hat{\mathbf{x}}$ . Then,

$$\mathbf{P} = \frac{1}{2} E_x \hat{\mathbf{x}} e^{-j\mathbf{k}\cdot\mathbf{r}} \times \left[ \frac{1}{\eta} \hat{\mathbf{z}} \times E_x^* \hat{\mathbf{x}} e^{+j\mathbf{k}\cdot\mathbf{r}} \right] \quad (59)$$

$$= \frac{1}{2} E_x \hat{\mathbf{x}} e^{-j\mathbf{k}\cdot\mathbf{r}} \times \frac{E_x^*}{\eta} \hat{\mathbf{y}} e^{+j\mathbf{k}\cdot\mathbf{r}} \quad (60)$$

$$= \frac{|E_x|^2}{2\eta} \hat{\mathbf{x}} \times \hat{\mathbf{y}} \quad (61)$$

$$= \frac{|E_x|^2}{2\eta} \hat{\mathbf{z}}. \quad (62)$$

This result shows that the power delivered by a uniform plane wave is in the same direction as its k-vector.