Solution to the Nonhomogenous Wave Equation

In the discussion of solving radiation problems, we considered the vector potential produced by a point current source at the origin. This involved solving the nonhomogeneous Helmholtz or wave equation,

$$\boldsymbol{\nabla}^2 \boldsymbol{A} + k^2 \boldsymbol{A} = -\mu \boldsymbol{J}. \tag{1}$$

Note that k can be used synonomously with β here since for the purposes of this course, we will mostly deal with antennas in lossless media, so $\gamma = \cancel{\alpha} + j\beta = jk$.

Let's consider solving the scalar wave equation (i.e. consider solving for one of the components of $\boldsymbol{A}$, with the other two components zero) for a simpler discussion. In the notes on vector potential we show how we can generalize the solution to the vector form. As per those notes, we will try to solve the equation

$$\nabla^2 A_z + k^2 A_z = -\mu J_z. \tag{2}$$

We will begin first by examining solutions to the homogenous version of this equation. We will use these relations to draw comparisons between static and time-varying solutions. Expanding the Laplacian in spherical coordinates, the homogenous vector potential equations becomes

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial A_z}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial A_z}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 A_z}{\partial \phi^2} + k^2 A_z = 0. \tag{3}$$

A point source cannot produce a potential that varies with θ or ϕ; therefore,

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dA_z}{dr}\right) + k^2 A_z = 0. \tag{4}$$

Let's assume the solution is of the form

$$A_z(r) = \frac{f(r)}{r}, \tag{5}$$

which reduces the equation to

$$\begin{aligned} \frac{1}{r^2}\frac{d}{dr}\left[r^2\left(\frac{1}{r}\frac{df}{dr} - \frac{f}{r^2}\right)\right] + k^2 \frac{f}{r} &= 0 \\ \frac{1}{r^2}\left[\frac{df}{dr} + r\frac{d^2 f}{dr^2} - \frac{df}{dr}\right] + k^2 \frac{f}{r} &= 0 \\ \frac{d^2 f}{dr^2} + k^2 f = 0. \end{aligned} \tag{6}$$

One solution to this equation is

$$f = C_1 e^{-jkr} \tag{7}$$

so

$$A_z(r) = C_1 \frac{e^{-jkr}}{r} \tag{8}$$

Re-introducing $e^{j\omega t}$ time dependence illustrates that this is a spherical wave propagating radially away from the origin. We aim to find the solution for the constant C_1 for the nonhomogeneous version of the wave equation.

Next, we note that the static solution ($k = 0$) is simply

$$A_z(\boldsymbol{r}) = \frac{C_1}{r} \tag{9}$$

revealing that the time-harmonic solution is simply found by multiplying by e^{-jkr}.

Now we consider the solution to the nonhomogeneous equation

$$\nabla^2 A_z^{\text{static}} = -\mu J_z. \tag{10}$$

This equation is recognized as Poisson's equation; in statics, the more recognizable form is

$$\nabla^2 \phi = -\frac{\rho_v}{\epsilon} \tag{11}$$

where ϕ is electrostatic potential. It has a solution

$$\phi = \frac{1}{4\pi\epsilon} \iiint_V \frac{\rho_v(\boldsymbol{r'})}{r} dv'. \tag{12}$$

If we consider a point source in electrostatics, then $\rho_v(\boldsymbol{r}) = \delta(x)\delta(y)\delta(z)$ and

$$\phi(\boldsymbol{r}) = \frac{1}{4\pi\epsilon r}, \tag{13}$$

the familiar electric potential associated with a point unit charge at the origin. Comparing to (10), then, we expect the corresponding point-source solution for A_z to be

$$A_z^{\text{static}}(\boldsymbol{r}) = \frac{\mu}{4\pi r} \tag{14}$$

which is of the same form as (13) whereby ϕ is replaced with A_z and ϵ by $1/\mu$. This demonstrates the *dual* nature of Maxwell's equations. In any case, the comparison to (10) reveals that the constant $C_1 = \frac{1}{4\pi}$. Using the rule we established earlier for re-introducing time dependence, we can write

$$A_z = \frac{\mu}{4\pi} \frac{e^{-jkr}}{r} \tag{15}$$

which is sometimes called a *retarded potential* because it is retarded in time.

We can now write the general solution for the solution to the inhomogeneous wave equation. Thanks to superposition, we simply integrate over a volume of point sources and use superposition to arrive at the general solution for $\boldsymbol{A}$,

$$\boldsymbol{A}(x, y, z) = \frac{\mu}{4\pi} \iiint_V \boldsymbol{J}(x', y', z') \frac{e^{-jkR}}{R} dv', \tag{16}$$

or in terms of a position vectors

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu}{4\pi} \iiint_V \boldsymbol{J}(\boldsymbol{r'}) \frac{e^{-jkR}}{R} dv', \tag{17}$$

where $\boldsymbol{R}$ is a vector point from the source point defined by position vector $\boldsymbol{r'}$, to the field point defined by positon vector $\boldsymbol{r}$, $\boldsymbol{R} = \boldsymbol{r} - \boldsymbol{r}'$.