# Lorentz Reciprocity Theorem

*Reciprocity* is an important concept in antennas because it produced implications when we reverse the role of transmitting antennas and receiving antennas. A formal derivation of the *Lorentz Reciprocity Theorem* begins by considering a volume containing two sets of sources, $\boldsymbol{J}_1$ and $\boldsymbol{J}_2$, which each produce fields $\boldsymbol{E}_1, \boldsymbol{H}_1$ and $\boldsymbol{E}_2, \boldsymbol{H}_2$, respectively, as shown in Figure 1.

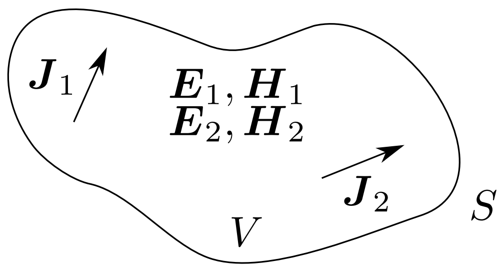


Figure 1: Volume containing two electric sources

Consider the quantity

$$\boldsymbol{\nabla} \cdot (\boldsymbol{E}_1 \times \boldsymbol{H}_2 - \boldsymbol{E}_2 \times \boldsymbol{H}_1), \tag{1}$$

which is expandable using a vector identity as

$$(\boldsymbol{\nabla} \times \boldsymbol{E}_1) \cdot \boldsymbol{H}_2 - (\boldsymbol{\nabla} \times \boldsymbol{H}_2) \cdot \boldsymbol{E}_1 - (\boldsymbol{\nabla} \times \boldsymbol{E}_2) \cdot \boldsymbol{H}_1 + (\boldsymbol{\nabla} \times \boldsymbol{H}_1) \cdot \boldsymbol{E}_2. \tag{2}$$

From Maxwell's curl equations,

$$\begin{aligned} \boldsymbol{\nabla} \times \boldsymbol{E}_1 &= -j\omega\mu \boldsymbol{H}_1 & (3)\\ \boldsymbol{\nabla} \times \boldsymbol{H}_1 &= j\omega\epsilon \boldsymbol{E}_1 + \boldsymbol{J}_1 & (4)\\ \boldsymbol{\nabla} \times \boldsymbol{E}_2 &= -j\omega\mu \boldsymbol{H}_2 & (5)\\ \boldsymbol{\nabla} \times \boldsymbol{H}_2 &= j\omega\epsilon \boldsymbol{E}_2 + \boldsymbol{J}_2. & (6) \end{aligned}$$

Therefore,

$$\begin{aligned} \boldsymbol{\nabla} \cdot (\boldsymbol{E}_1 \times \boldsymbol{H}_2 - \boldsymbol{E}_2 \times \boldsymbol{H}_1) &= -j\omega\mu \boldsymbol{H}_1 \cdot \boldsymbol{H}_2 - j\omega\epsilon \boldsymbol{E}_2 \cdot \boldsymbol{E}_1 - \boldsymbol{J}_2 \cdot \boldsymbol{E}_1 & (7)\\ &\quad + j\omega\mu \boldsymbol{H}_2 \cdot \boldsymbol{H}_1 + j\omega\epsilon \boldsymbol{E}_1 \cdot \boldsymbol{E}_2 + \boldsymbol{J}_1 \cdot \boldsymbol{E}_2 \\ &= \boldsymbol{J}_1 \cdot \boldsymbol{E}_2 - \boldsymbol{J}_2 \cdot \boldsymbol{E}_1. & (8) \end{aligned}$$

Since we took the divergence of a quantity in (1), let us now integrate the divergence over the volume of interest:

$$\iiint_V \boldsymbol{\nabla} \cdot (\boldsymbol{E}_1 \times \boldsymbol{H}_2 - \boldsymbol{E}_2 \times \boldsymbol{H}_1) dv' = \iiint_V (\boldsymbol{J}_1 \cdot \boldsymbol{E}_2 - \boldsymbol{J}_2 \cdot \boldsymbol{E}_1) dv' \tag{9}$$

Applying the Divergence Theorem to the left hand side:

$$\oiint_S (\boldsymbol{E}_1 \times \boldsymbol{H}_2 - \boldsymbol{E}_2 \times \boldsymbol{H}_1) \cdot d\boldsymbol{s}' = \iiint_V (\boldsymbol{J}_1 \cdot \boldsymbol{E}_2 - \boldsymbol{J}_2 \cdot \boldsymbol{E}_1) dv'. \tag{10}$$

A more useful form of this theorem, applicable to antennas, is found by noticing that for electric and magnetic fields observed a large distance from a source (e.g., a sphere of infinite radius surrounding an antenna),

- $\boldsymbol{E} \times \boldsymbol{H}$ points in the radial direction normal to the sphere, $\hat{\boldsymbol{n}}$.
- $\boldsymbol{E}$ and $\boldsymbol{H}$ are related through $\boldsymbol{H} = (\hat{\boldsymbol{n}} \times \boldsymbol{E})/\eta$.

Using the latter relation, the integrand on the left hand side of (10) can be re-written as

$$
\begin{aligned}
(\boldsymbol{E}_1 \times \boldsymbol{H}_2 - \boldsymbol{E}_2 \times \boldsymbol{H}_1) \cdot \hat{\boldsymbol{n}} dS &= (\hat{\boldsymbol{n}} \times \boldsymbol{E}_1) \cdot \boldsymbol{H}_2 - (\hat{\boldsymbol{n}} \times \boldsymbol{E}_2) \cdot \boldsymbol{H}_1 && (11) \\
&= \eta \boldsymbol{H}_1 \cdot \boldsymbol{H}_2 - \eta \boldsymbol{H}_2 \cdot \boldsymbol{H}_1 && (12) \\
&= 0 && (13)
\end{aligned}
$$

Hence,

$$\iiint_V \boldsymbol{J}_1 \cdot \boldsymbol{E}_2 dv' = \iiint_V \boldsymbol{J}_2 \cdot \boldsymbol{E}_1 dv' \tag{14}$$

This is the form of the Reciprocity Theorem that is used in the analysis of receiving antennas.