Lorentz Reciprocity Theorem

Reciprocity is an important concept in antennas because it produced implications when we reverse the role of transmitting antennas and receiving antennas. A formal derivation of the Lorentz Reciprocity Theorem begins by considering a volume containing two sets of sources, $J_1$ and $J_2$, which each produce fields $E_1, H_1$ and $E_2, H_2$, respectively, as shown in Figure 1.

![Figure 1: Volume containing two electric sources](image)

Consider the quantity
\[
\nabla \cdot (E_1 \times H_2 - E_2 \times H_1),
\]
which is expandable using a vector identity as
\[
(\nabla \times E_1) \cdot H_2 - (\nabla \times H_2) \cdot E_1 - (\nabla \times E_2) \cdot H_1 + (\nabla \times H_1) \cdot E_2.
\]

From Maxwell’s curl equations,
\[
\nabla \times E_1 = -j\omega\mu H_1 \tag{3}
\]
\[
\nabla \times H_1 = j\omega\epsilon E_1 + J_1 \tag{4}
\]
\[
\nabla \times E_2 = -j\omega\mu H_2 \tag{5}
\]
\[
\nabla \times H_2 = j\omega\epsilon E_2 + J_2. \tag{6}
\]

Therefore,
\[
\nabla \cdot (E_1 \times H_2 - E_2 \times H_1) = -j\omega\mu H_1 \cdot H_2 - j\omega\epsilon E_2 \cdot E_1 - J_2 \cdot E_1 \\
+ j\omega\mu H_2 \cdot H_1 + j\omega\epsilon E_1 \cdot E_2 + J_1 \cdot E_2 \\
= J_1 \cdot E_2 - J_2 \cdot E_1. \tag{7}
\]

Since we took the divergence of a quantity in (1), let us now integrate the divergence over the volume of interest:
\[
\iiint_V \nabla \cdot (E_1 \times H_2 - E_2 \times H_1) \, dv' = \iiint_V (J_1 \cdot E_2 - J_2 \cdot E_1) \, dv'. \tag{9}
\]

Applying the Divergence Theorem to the left hand side:
\[
\ssint_S (E_1 \times H_2 - E_2 \times H_1) \cdot ds' = \iiint_V (J_1 \cdot E_2 - J_2 \cdot E_1) \, dv'. \tag{10}
\]
A more useful form of this theorem, applicable to antennas, is found by noticing that for electric and magnetic fields observed a large distance from a source (e.g., a sphere of infinite radius surrounding an antenna),

- \( \mathbf{E} \times \mathbf{H} \) points in the radial direction normal to the sphere, \( \hat{n} \).
- \( \mathbf{E} \) and \( \mathbf{H} \) are related through \( \mathbf{H} = (\hat{n} \times \mathbf{E})/\eta \).

Using the latter relation, the integrand on the left hand side of (10) can be re-written as

\[
\begin{align*}
(E_1 \times H_2 - E_2 \times H_1) \cdot \hat{n}dS &= (\hat{n} \times E_1) \cdot H_2 - (\hat{n} \times E_2) \cdot H_1 \\
&= \eta H_1 \cdot H_2 - \eta H_2 \cdot H_1 \\
&= 0
\end{align*}
\]

Hence,

\[
\iiint V J_1 \cdot E_2 dv' = \iiint V J_2 \cdot E_1 dv'
\]

This is the form of the Reciprocity Theorem that is used in the analysis of receiving antennas.