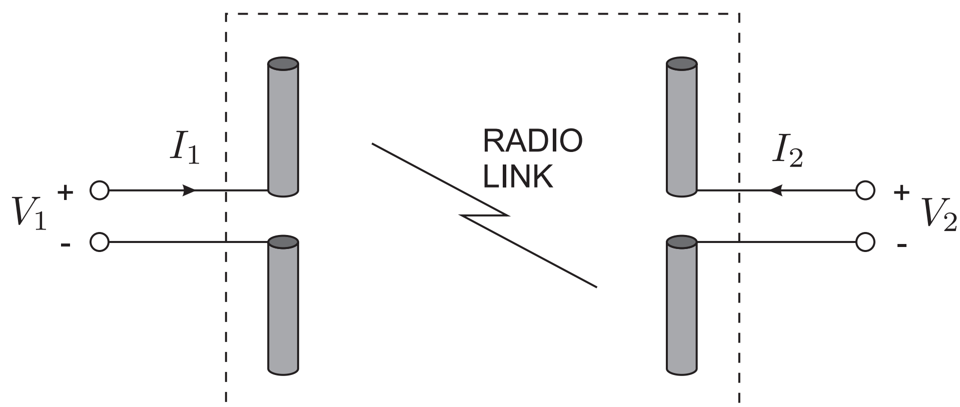
# Receiving Antennas

So far, we have been examining the properties of antennas when they are *transmitting* signals but have not spoken much of what happens when they received signals. For example, what is the receiving pattern of an antenna? How do we predict the voltage produced across the terminals of a receiving antenna? We aim to answer such questions next.

## 1 Equivalent Circuit Model of a Radio Link

To study the behaviour of a receiving antenna, we will consider a link consisting of two antennas: one transmitting and one receiving. We wish to understand the behaviour of such a system. Since an antenna is a one-port device, the analysis is facilitated by considering the system of antennas to be a 2-port "black box" with unknown internal characteristics (for now).




Recall from circuit theory than an unknown "black box" with two ports is fully characterized if we know the terminal voltage $(V_1, V_2)$ and currents $(I_1, I_2)$ of the device. These quantities are related through

$$V_1 = Z_{11}I_1 + Z_{12}I_2 \quad (1)$$
$$V_2 = Z_{21}I_1 + Z_{22}I_2, \quad (2)$$

or simply,

$$[V] = [Z][I], \quad (3)$$

where $[Z]$ is the *impedance matrix* of the two-port network. $Z_{11}$ and $Z_{22}$ are called *self-impedances* of the system while $Z_{12}$ and $Z_{21}$ are called *mutual impedances* of the network. Each of the impedances can be found through open-circuiting ports of the network, such that

$$Z_{mn} = \left.\frac{V_m}{I_n}\right|_{I_k=0 \text{ for } k\neq n} \quad (4)$$

for an arbitrary network.

Note that the input impedance seen looking into one port is a function of the loading on the second antenna. For example, the input impedance seen looking into port 1 is

$$Z_1 = \frac{V_1}{I_1} = \frac{Z_{11}I_1 + Z_{12}I_2}{I_1} = Z_{11} + \frac{Z_{12}I_2}{I_1}. \tag{5}$$

Hence, the current flowing on antenna 2 does influence the input impedance seen looking into antenna 1. However, if antenna 2 is placed very far away, the influence of the second antenna should be negligible ($Z_{12} \approx 0$), yielding $Z_1 \approx Z_{11}$, the self-impedance of antenna 1.

We are most concerned with the mutual impedance terms of our circuit model, since they describe the coupling between antenna 1 and antenna 2 (and vice versa, if antenna 2 was connected to the source). But first, we must consider a very important theorem that describes the behaviour of the system we have described when the antennas are immersed in a homogeneous, linear, passive, and isotropic medium (like free space).

# 2 Reciprocity

One of the most important electromagnetic concepts is that of *reciprocity*, which is the behaviour of an electromagnetic system in a simple medium. Consider the situation shown below, where we have a volume containing to sources, $\boldsymbol{J}_1$ and $\boldsymbol{J}_2$, which each produce fields $\boldsymbol{E}_1, \boldsymbol{H}_1$ and $\boldsymbol{E}_2, \boldsymbol{H}_2$, respectively, as shown in Figure 1.

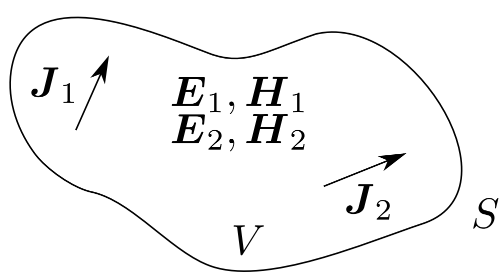



Figure 1: Volume containing two electric sources

Mathematically, reciprocity was developed by Lorentz[1] but a useful form of that theorem for our purposes states that

$$\iiint_V \boldsymbol{J}_1 \cdot \boldsymbol{E}_2 dv' = \iiint_V \boldsymbol{J}_2 \cdot \boldsymbol{E}_1 dv'. \tag{6}$$

Now consider the two-antenna (two-source) problem we considered originally, but the antennas located a large distance apart such that the assumptions above hold. Let's say the two antennas are ideal dipole antennas, driven by ideal current generators (with infinite source impedances).

Thinking of the structure of the antenna, $\boldsymbol{E}_2$, the field produced at antenna 1 by antenna 2 (source $\boldsymbol{J}_2$) only has a nonzero projection with $\boldsymbol{J}_1$ across the antenna terminals, since $\boldsymbol{J}_1$ is only nonzero along the conducting part of the antenna, and $\boldsymbol{E}_2$ is only nonzero across the gap because

[1] See the note on the Reciprocity Theorem for more details.

of PEC boundary conditions. If we assume that the current $\boldsymbol{J}_1$ can be represented as a linear current (existing only over a contour instead of a volume) and furthermore the current is uniform across the gap,

$$\iiint_V \boldsymbol{E_2} \cdot \boldsymbol{J}_1 = \int_C E_2 I_1 d\ell = I_1 \int_C E_2 d\ell = -V_1^{oc} I_1, \tag{7}$$

where $V_1^{oc}$ is the open-circuited voltage produced at antenna 1 as a result of the incident field produced by antenna 2 ($\boldsymbol{E}_2$). Similarly, at antenna 2,

$$\iiint_V \boldsymbol{E_1} \cdot \boldsymbol{J}_2 = -V_2^{oc} I_2 \tag{8}$$

According to the Reciprocity Theorem (6), equations (7) and (8) must be equal:

$$V_1^{oc} I_1 = V_2^{oc} I_2 \tag{9}$$

or

$$\frac{V_1^{oc}}{I_2} = \frac{V_2^{oc}}{I_1}. \tag{10}$$

From our equivalent circuit model of a two-antenna link, we know that

$$Z_{21} = \left.\frac{V_2}{I_1}\right|_{I_2=0} = \frac{V_2^{oc}}{I_1} \tag{11}$$

and

$$Z_{12} = \left.\frac{V_1}{I_2}\right|_{I_1=0} = \frac{V_1^{oc}}{I_2}, \tag{12}$$

therefore,

$$Z_{12} = Z_{21}. \tag{13}$$

This is the fundamental definition of a reciprocal two-port circuit, because we see that if we excite port 1 with a current source of amplitude $I$, the open circuit-voltage at port 2 is $Z_{21}I$ while if we flip the current source to port 2, the open-circuit voltage at port 1 is $Z_{12}I = Z_{21}I$ which is the same result as with the current source at port 1. That is, if we drive port 1 with an ideal current source having amplitude $I$, the open circuit voltage at port 2 is

$$V_2^{oc} = Z_{21} I. \tag{14}$$

If we flip the current source to port 2, then the open circuit voltage at port 1 is

$$V_1^{oc} = Z_{12} I. \tag{15}$$

Hence, in a reciprocal system, both cases should produce the same open-circuit voltage, regardless of which antenna is transmitting and which is receiving.

The consequence of this on antennas is found as follows. Consider an experiment where we measure the transmit pattern of antenna (1), using a second receiving antenna (2) moving about a circle of fixed radius about the transmitting antenna while remaining co-polarized with the transmission, as shown in Figure 2.

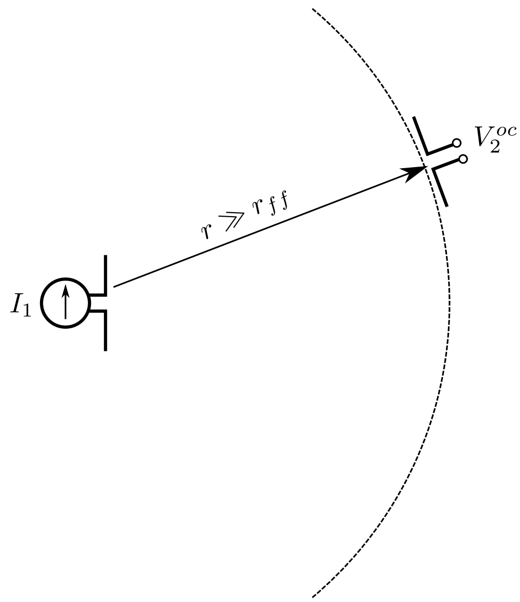



Figure 2: Antenna 1 transmits, antenna 2 receives

If we measure the open-circuit voltage at antenna 2, we know it is equal to

$$V_2^{oc}(\theta) = Z_{21}(\theta) I. \tag{16}$$

Now consider a second experiment where antenna (2) is used as the transmitter and antenna (1) as the receiver, as shown in Figure 3. Antenna (2) moves in an identical manner as the first experiment, while this time we measure the open-circuit voltage at the terminals of antenna 1, which is equal to

$$V_1^{oc}(\theta) = Z_{12}(\theta) I = Z_{21}(\theta) I. \tag{17}$$

Hence we see the "open-circuit voltage" pattern measured by both experiments is identical. Therefore, we conclude that **the transmit and receive patterns of an antenna are the same**.

# 3 Vector Effective Length

A *receiving antenna* is used to collect electromagnetic waves and extract power from them. The concept of the *effective length* of an antenna is used to determine the voltage induced on the open-circuited terminals of the antenna when a wave impinges on it. This effective length is a vector quantity and is defined as

$$\boldsymbol{\ell}_{eff}(\theta, \phi) = \ell_\theta(\theta, \phi)\, \hat{\boldsymbol{\theta}} + \ell_\phi(\theta, \phi)\, \hat{\boldsymbol{\phi}}. \tag{18}$$

It is related to the far-zone electric field radiated by the antenna through

$$\boldsymbol{E}_{rad} = -\frac{j\omega\mu I}{4\pi} \frac{e^{-jkr}}{r} \boldsymbol{\ell}_{eff}(\theta, \phi). \tag{19}$$

In general, for a transmitting antenna, vector effective length can be defined as follows:

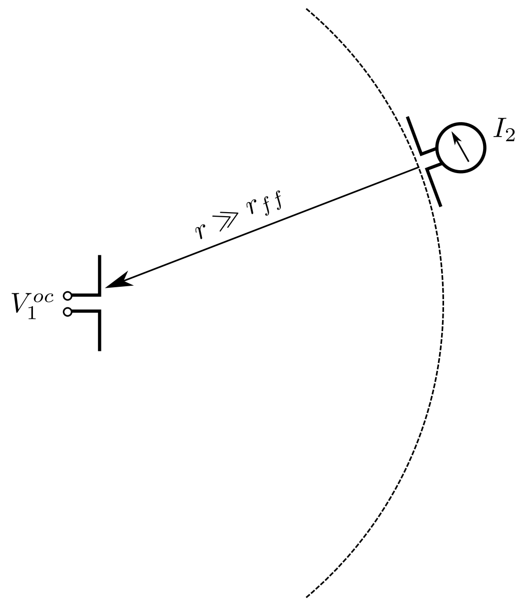



Figure 3: Antenna 2 transmits, antenna 1 receives

> The vector effective length of an antenna is the length and orientation of a uniform current required to produce the same electric field as the antenna under consideration.

If the antenna is receiving, the open-circuit voltage developed across the antenna terminals is

$$V^{oc} = \boldsymbol{E}^i \cdot \boldsymbol{\ell}_{eff} \qquad \text{(note units are correct)} \tag{20}$$

We note that the complex conjugate is used to change the reference direction for the receiver (this only applies to antennas with complex vector lengths: those with elliptical or circular polarization). When $\boldsymbol{\ell}_{eff}$ and $\boldsymbol{E}^i$ are linearly polarized, $\boldsymbol{\ell}_{eff}$ can be thought of as the vector length of a linear antenna that the open circuit voltage is being induced into by $\boldsymbol{E}^i$.

The radiated electric field of an ideal dipole oriented along the $z$-axis is, as we know,

$$\boldsymbol{E} = \frac{j\omega\mu I \Delta z}{4\pi} \frac{e^{-jkr}}{r} \sin\theta\, \hat{\boldsymbol{\theta}}. \tag{21}$$

Comparing Equations (21) and (19),

$$\boldsymbol{\ell}_{eff} = -\Delta z \sin\theta\, \hat{\boldsymbol{\theta}} \tag{22}$$

produces $\boldsymbol{E} = \frac{j\omega\mu I \Delta z}{4\pi} \frac{e^{-jkr}}{r} \sin\theta\, \hat{\boldsymbol{\theta}}$. Hence, Equation (22) represents the vector effective length of an ideal dipole oriented along the $z$-axis.

**Example:** Half-wave dipole along the $z$-axis.

$$
\begin{aligned}
\boldsymbol{E} &= j\omega\mu \frac{2I}{4\pi k} \frac{e^{-jkr}}{r} \frac{\cos(\frac{\pi}{2}\cos\theta)}{\sin\theta} \hat{\boldsymbol{\theta}} \\
&= \frac{j\mu\omega I}{4\pi} \frac{e^{-jkr}}{r} \left( \frac{2}{k} \frac{\cos(\frac{\pi}{2}\cos\theta)}{\sin\theta}\, \hat{\boldsymbol{\theta}} \right) \\
\Rightarrow \boldsymbol{\ell}_{\mathit{eff}} &= -\frac{\lambda}{\pi} \frac{\cos(\frac{\pi}{2}\cos\theta)}{\sin\theta}\, \hat{\boldsymbol{\theta}}.
\end{aligned}
$$

Comparing the maximum effective length of a $\lambda/2$ dipole to that of an ideal dipole (i.e. for $\theta = 90°$), we find that

$$
\frac{|\ell_{\mathit{eff},\lambda/2}|}{\ell_{\mathit{eff,ideal}}} = \frac{\lambda}{\pi} = \frac{2}{\pi}\frac{\lambda}{2},
$$

indicating that the effective length of a half-wave dipole corresponds to only $(2/\pi = 63.7\%)$ of its actual length (in contrast to an ideal dipole having $\ell_{\mathit{eff}} = \Delta\ell$).

**Example:** What is the induced voltage in an ideal dipole along the $z$-axis if the incident electric field is that of a plane wave travelling such that its k-vector makes an angle $\theta$ with the $z$-axis, and E-field is contained in the $yz$-plane and points towards the $+z$ axis?

$$
\begin{aligned}
V^{oc} &= \boldsymbol{E}^i \cdot \boldsymbol{\ell}_{\mathit{eff}} \\
&= (-E_\theta\, \hat{\boldsymbol{\theta}} + \cancel{E_\phi\, \hat{\boldsymbol{\phi}}}) \cdot (-\Delta z \sin\theta)\, \hat{\boldsymbol{\theta}} \\
&= E_\theta \Delta z \sin\theta
\end{aligned}
$$

We note the sign of $V^{oc}$ is correct with respect to terminal conventions.

**Example:** What is the open circuit voltage magnitude developed at an ideal dipole that is perfectly aligned with the incident field, and aligned for maximum output?

$$
\begin{aligned}
V^{oc}_{max} &= \mathsf{max}(|\boldsymbol{E}^i \cdot \boldsymbol{\ell}_{\mathit{eff}}|) \\
&= \mathsf{max}(E^i \ell_{\mathit{eff}}) \\
&= E^i \Delta\ell\ \mathsf{max}(\sin\theta) \\
&= E^i \Delta\ell
\end{aligned}
$$

Hence, for a perfectly aligned ideal dipole the terminal voltage is simply the product of the incident electric field along the *physical* length of the antenna. This is only true for ideal dipoles; hence, we could define vector effective length as the length of an ideal dipole that relates the open circuit voltage and incident field through this simple relationship.

We will use this result in the derivation of the relationship between an antenna's effective area and its gain.

To conclude, the concept of the *vector effective length* of an antenna is useful for two things:

1. It allows us to relate an incident electric field on any antenna to the open-circuit voltage developed at its terminals; and
2. It is a useful tool in determining the effect of polarization mismatch between the incident field and the antenna, which will be discussed later in the course.