Abstract—Multiscale electromagnetic problems occur frequently in practical applications. Unfortunately, the Finite Difference Time Domain (FDTD) method can lose its remarkable efficiency when small and large geometrical features are present simultaneously. Small geometrical features impose a fine spatial grid, increasing memory consumption and CPU time. Moreover, because of the Courant-Friedrichs-Lewy (CFL) stability condition, time step has to be also reduced, further increasing the overall computational cost. We propose a new method to accelerate FDTD simulations with local grid refinements. The FDTD equations for the fine region are first compressed using model order reduction. The stability limit of the resulting equations is extended to match the stability limit of the coarse grid using a suitable perturbation approach. Finally, the reduced models of the fine regions are embedded into the main coarse grid. Owing to their enhanced stability, the whole simulation can be run at the large time step supported by the coarse grid. The proposed technique thus provides a way to perform stable FDTD simulations with subgridding at the CFL limit of the coarse grid. A numerical example demonstrates the remarkable efficiency of the proposed method with respect to FDTD and spatial subgridding.

I. INTRODUCTION

The Finite Difference Time Domain (FDTD) method is widely used to solve Maxwell’s equations numerically. The popularity of FDTD stems from its easy implementation, versatility, and low computational cost per iteration. FDTD indeed uses an explicit time integration scheme which avoids expensive matrix factorizations. As a consequence of its explicit nature, the FDTD time-step \( \Delta t \) must satisfy the Courant-Friedrichs-Lewy (CFL) stability condition

\[
\Delta t < \frac{1}{\sqrt{\sigma_e \mu} \left( \frac{1}{\Delta x^2} + \frac{1}{\Delta y^2} + \frac{1}{\Delta z^2} \right)}.
\]

The CFL limit can significantly reduce FDTD’s efficiency for multiscale problems, where small and large objects are simultaneously involved. Two factors contribute to the efficiency loss:

1) a fine grid is needed to properly resolve small details. The consequent raise in the number of unknowns increases memory consumption and cost per iteration;
2) if the grid is refined, the time-step \( \Delta t \) has to be also reduced according to (1). This increases the number of iterations needed to reach the desired final time.

Several approaches have been proposed to mitigate these issues. Implicit methods such as ADI-FDTD are not bound by the CFL limit, but have an increased computational cost per iteration with respect to FDTD. In subgridding, a fine grid is used for the regions containing small objects, and a coarse grid is used elsewhere. While this reduces the overall computational cost, the FDTD updates to the fine region are still bound by the restrictive CFL of the fine mesh. Even if an implicit method is used for the fine region, the whole simulation cannot be run at the large time step supported by the coarse grid.

Recently, several techniques have also been devised to overcome the CFL barrier without resorting to implicit schemes, based on spatial filtering, modal projection, and model order reduction. In this paper, we show how FDTD simulations with subgridding can be significantly accelerated using a variation of the CFL extension method presented in \( \cite{8} \). First, the FDTD equations for the fine grid alone are assembled and reduced using model order reduction. Then, the equations are made stable at the time-step of the coarse grid which surrounds the fine regions using a suitable perturbation approach. Finally, the obtained equations are coupled with those of the coarse grid and the simulation of the whole structure is performed at the large time-step supported by the coarse grid. This method improves previous works \( \cite{9} \) that used CFL extension techniques to run FDTD subgridding above the CFL limit of the fine grid. Compared to the method in \( \cite{9} \), the proposed method is able to run the whole simulation at the time-step of the coarse grid even when the fine region is not homogeneous. A numerical example will demonstrate the remarkable efficiency of the proposed technique in comparison to FDTD and spatial subgridding.

II. PROPOSED METHOD

We consider a FDTD setup where a fine grid is used in the region with small details, and a coarse grid is used elsewhere. Although, for simplicity, we consider only one fine region, the proposed method can be applied to multiple refined regions. The FDTD equations that update the \( E \) and \( H \) nodes in the fine mesh can be written in the following matrix form \( \cite{10} \).

\[
\left[ \begin{array}{cc}
\frac{D_e}{\Delta t} + \frac{D_m}{2} & 0 \\
-K^T & \frac{D_h}{\Delta t} + \frac{D_m}{2}
\end{array} \right]
\left[ \begin{array}{c}
E_{n+1} \\
H_{n+1/2}
\end{array} \right] =
\left[ \begin{array}{cc}
\frac{D_e}{\Delta t} - \frac{D_m}{2} & -K \\
0 & \frac{D_h}{\Delta t} - \frac{D_m}{2}
\end{array} \right]
\left[ \begin{array}{c}
E_n \\
H_{n+1/2}
\end{array} \right] + T_{FB} E_B|^{n+1}.
\] (2)
where $E[n]$ and $H[n+\frac{1}{2}]$ are column vectors containing all the electric and magnetic field unknowns in the fine grid. The diagonal matrices $D_e$, $D_\mu$, $D_{\sigma_e}$, and $D_{\sigma_m}$ contain the permittivity, permeability, electric conductivity and magnetic conductivity values for each cell, respectively. The matrix $K$ represents the discretized curl operator. For simplicity, we have assumed that all sources are in the coarse region. The fine-coarse mesh refinement is assumed to be along E nodes. The E field values at this boundary are collected in the vector $E_B[n+1]$, and used to update the fine grid nodes after a suitable interpolation. This operation is described by the matrix $T_{FB}$. An analogous matrix provides the fine grid values needed to update the coarse grid nodes near the boundary, as in standard FDTD subgridding [4].

The order of (2) is reduced using the model order reduction method presented in [8]. A projection matrix $V$ is computed using SPRIM [11] and used to approximate the full vector of unknowns in (2), through reduced unknown vectors $E[n]$ and $H[n+\frac{1}{2}]$. Moreover, because of the block-diagonal structure of the projection matrix $V$, the obtained reduced model is also in the form (2), but has “compressed” blocks [8].

\[
\begin{align*}
\bar{D}_e &= V_1^T D_e V_1 \\
\bar{D}_\mu &= V_2^T D_\mu V_2 \\
\bar{D}_{\sigma_e} &= V_1^T D_{\sigma_e} V_1 \\
\bar{D}_{\sigma_m} &= V_2^T D_{\sigma_m} V_2 \\
\bar{K} &= V_1^T K V_2 \\
\bar{T}_{FB} &= V^T T_{FB}
\end{align*}
\]

Being in the same form of the original FDTD equations, the obtained model for the fine grid can be seamlessly connected back to the coarse grid, and solved in a leap-frog manner [8]. As it is, the obtained reduced model is stable only up to the restrictive CFL limit of the fine grid. With a perturbation approach, we extend the CFL limit of the obtained model in order to reach the CFL limit of the coarse grid. Following [8], we compute the singular values $\sigma_i$ of $\bar{D}_e^{-\frac{1}{2}} \bar{K} \bar{D}_e^{-\frac{1}{2}}$. The singular values that do not satisfy $\sigma_i < \beta \frac{\Delta t}{\Delta x}$ with $\beta < 1$ are responsible for the instability of the FDTD equations for time-steps greater than $\Delta t$. Therefore, such singular values and the corresponding singular vectors are removed from the reduced model. This procedure leads to a compact FDTD model for the fine region which is stable at the maximum time step $\Delta t$ supported by the coarse grid. In all numerical tests we ran, a value of $\beta = 0.9$ was sufficient to ensure stable late-time results even for high grid refinement ratios. Further research is currently under way to determine an optimal value for $\beta$. With the obtained reduced models, the entire FDTD simulation can be run at the CFL limit of the coarse grid, which to our knowledge is a novel result.

### III. Numerical Example

We demonstrate the performance of the proposed technique through the 2D waveguide shown in Fig. 1. The waveguides operates in the TM mode and is $0.7 \times 4 \text{ m}$ large. Two very thin PEC irises with aperture size of $0.2 \text{ m}$ are placed along the waveguide at a distance of $1 \text{ m}$. Most of the structure is meshed with a coarse mesh with $\Delta x = \Delta y = 5 \text{ cm}$ and $14 \times 80$ cells. In order to resolve well the thin irises, and the fields around them, the grid is refined around both apertures in two subregions of size $0.5 \times 0.5 \text{ m}$. The waveguide is terminated at both ends on 4th-order matched absorbers with thickness of $10 \text{ cells}$. A line of current sources is placed at $y = 0.9 \text{ m}$ and a line of probes is placed at $y = 3.1 \text{ m}$. The waveguide is excited with a Gaussian pulse with maximum frequency $f_{max} = 0.4 \text{ GHz}$. At this frequency, one cell of the coarse mesh corresponds to $15 \text{ wavelengths}$.

The structure was analysed using the proposed method, FDTD subgridding [4], a standard FDTD with a coarse mesh everywhere, and a standard FDTD with a fine mesh everywhere. The latter case, that we denote as “full refinement”, is taken as reference. All simulations were performed for grid refinement ratios $GR$ of 3, 9, and 15. Figures 2 and 3 show the transmission coefficient extracted from the time domain results for $GR = 3, 9$ and 15. Without any grid refinement, the frequency response deviates from the reference one. This confirms the need for a finer grid at least around the irises. The proposed method ensured stable simulations for all tested refinement ratios. The results produced by the proposed algorithm are in very good agreement with the reference data, demonstrating that the extension of the CFL limit did not compromise accuracy. Table I gives the total number of unknowns used by each method for different refinement ratios.
of the coarse grid, with significant time savings with respect to standard FDTD and spatial subgridding. For a 2D waveguide with irises, the proposed method led to speed-ups of up to 256X with respect to FDTD, and up to 40X with respect to spatial subgridding. The developed technique shows significant promise in making FDTD more efficient for multiscale problems, where otherwise the CFL limit can be quite restrictive.

REFERENCES


Fig. 3: As in the right panel of Fig. 2 but for $GR = 9, 15$.

![Fig. 3](image-url)

IV. CONCLUSION

We proposed a new way to accelerate FDTD simulations in presence of local grid refinements. The FDTD equations for the fine regions are first compressed through model order reduction. Then, their CFL stability limit is extended with a perturbation approach in order to match the CFL limit of the coarse grid. In this way, a stable FDTD simulation of the whole problem can be performed at the time-step

![Fig. 4](image-url)