A Fast Surface Method to Model Skin Effect in Transmission Lines with Conductors of Arbitrary Shape or Rough Profile

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Abstract—Accurate models for on-chip interconnects are required to study signal integrity issues in electronic systems. This paper presents a fast surface method to calculate the impedance of transmission lines with conductors of arbitrary shape, including those with irregular cross-sectional areas. The presented technique is over 140 times faster than finite elements, which makes it an ideal tool to perform sensitivity analyses. The method is used to study the impact of conductors’ geometrical variations and conductors’ surface roughness on the impedance parameters.

I. INTRODUCTION

Accurate transmission line models of interconnects are required to study and mitigate signal integrity issues such as crosstalk and distortion in high-speed electronic systems [1]. To create transmission line models, one requires an accurate calculation of the per-unit length (p.u.l.) impedance and admittance parameters of the interconnects, often over a wide frequency range from DC up to several GHz. Over such a wide frequency range, skin effect leads to large changes in the impedance, which are not trivial to predict.

In the literature, two classes of techniques to calculate impedance parameters have been explored: volumetric and surface methods. Volumetric techniques based on the finite element method (FEM) [2], conductor partitioning [3], or volumetric integral equations [4] require a fine discretization of the entire cross-section of the conductor in order to capture skin effect. Such techniques lead to a large number of unknowns and long computation times. Therefore, volumetric methods become impractical for certain problems, for example when multiple impedance evaluations are required, as in the case of sensitivity [5] or stochastic analysis [6]. Surface methods such as [7]–[10] or stochastic analysis [6]. Surface methods such as [7]–[10] achieve the same result by taking only the fields on the conductors’ boundary as unknown. An explicit discretization of the conductors’ volume is thus avoided. These techniques are more efficient than volumetric techniques, as they only introduce unknowns along the boundary of the conductors. However, these techniques also require the calculation of several kernel matrices which are dependent on the volumetric mesh. Recently, a surface method that altogether avoids volumetric discretization was proposed in [11]. This technique can efficiently calculate the impedance parameters of transmission lines made up of conductors of irregular shapes, such as an on-chip interconnect with trapezoidal lines, valley-shaped microstrip lines [12], and conformal interconnects [13]. This approach generalizes the surface admittance operator concept [14], which was previously restricted to conductors with canonical shapes, such as those with circular [14], [15], tubular [16], rectangular [14], or triangular [17] cross section.

In this paper, we make three contributions. First, we show how the method of [11] can be used to robustly compute the eigenfunctions of the Helmholtz operator for arbitrary conductors. Such functions provide an insight on which current distributions are most relevant for capturing skin and proximity effect in a conductor. The eigenvalue decomposition also demonstrates how the presented method generalizes the surface admittance formulation of [14]. Second, we show how the proposed method can be used to quickly assess the impact of geometrical variations on the line impedance. Such analyses are becoming increasingly relevant in signal integrity engineering, due to the growing impact of manufacturing variations on transmission lines performance. Unfortunately, when performed with existing volumetric methods like FEM, sensitivity analyses are notoriously time consuming, as they require many impedance evaluations for different parameter values. Finally, we demonstrate that the ability of predicting skin effect in conductors of arbitrary shape can be effectively used to assess the impact of surface roughness on attenuation [18] over a broad frequency range.

II. IMPEDANCE CALCULATION BASED ON THE SURFACE ADMITTANCE OPERATOR

First, we briefly review the method proposed in [14] and then generalized in [11] for impedance computation. We consider a multiconductor transmission line composed of $P$ conductors of arbitrary shape with conductivity $\sigma$, permittivity $\varepsilon$, and permeability $\mu$. The conductors are assumed to be longitudinally invariant and are surrounded by a stratified medium. Our goal is to calculate the p.u.l. resistance $R(\omega)$ and inductance $L(\omega)$ that appear in the Telegrapher’s equation [19]

$$\frac{\partial V}{\partial z} = -[R(\omega) + j\omega L(\omega)]I,$$ (1)
where $\mathbf{V} = [V_1 \ V_2 \ \ldots \ V_p]^T$ and $\mathbf{I} = [I_1 \ I_2 \ \ldots \ I_p]^T$ contain, respectively, the scalar potential $V_p$ and current $I_p$ in each conductor. The impedance parameters will be calculated based on the surface admittance operator [14] obtained through the contour integral method [11].

A. Surface Admittance Operator Obtained Using the Contour Integral Method

We discuss the surface admittance formulation by considering $p$-th conductor of an arbitrary shape enclosed by contour $\gamma(p)$, as shown in Fig 1. In order to calculate the p.u.l. impedance parameters, we expand the electric field on $\gamma(p)$ using pulse basis functions as follows

$$E_z^p(\vec{r}) = \sum_{n=1}^{N_p} e_n^p \Pi_n^p(\vec{r}),$$

(2)

where $\Pi_n^p(\vec{r})$ is the $n$-th pulse basis function that is equal to one on the $n$-th segment of the $p$-th conductor, and is zero otherwise. Likewise, we also expand the tangential magnetic field on the contour $\gamma(p)$ using pulse basis functions

$$H_t^p(\vec{r}) = \sum_{n=1}^{N_p} h_n^p \Pi_n^p(\vec{r}).$$

(3)

To simplify the notation, we collect the expansion coefficients in (2) and (3) into vectors $\mathbf{E}^p = [e_1^p \ e_2^p \ \ldots \ e_{N_p}^p]^T$ and $\mathbf{H}^p = [h_1^p \ h_2^p \ \ldots \ h_{N_p}^p]^T$.

Under the quasi-TM assumption, the electric field $E_z^p(\vec{r})$ and tangential magnetic field $H_t^p(\vec{r})$ on $\gamma(p)$ are related by the contour integral equation (CIM) [20]

$$E_z^p(\vec{r}) = \frac{j}{2} \oint_{\gamma(p)} \left[ \frac{\partial G(\vec{r},\vec{r}')}{\partial n'} E_z^p(\vec{r}') - j\omega \mu G(\vec{r},\vec{r}') H_t^p(\vec{r}') \right] d\vec{r}',$$

(4)

where $G(\vec{r},\vec{r}')$ is the Green’s function of the two-dimensional Helmholtz equation and is given by

$$G(\vec{r},\vec{r}') = C_0 J_0(kd) - jY_0(kd)$$

(5)

where $k = \sqrt{\omega \mu (\omega c - j\sigma)}$ is the wavenumber inside the conductor, and $J_0(.)$ and $Y_0(.)$ are the zeroth order Bessel and Neumann functions [21], respectively. The distance between $\vec{r}$ and $\vec{r}'$ is denoted as

$$\vec{d} = \vec{r}' - \vec{r},$$

(6)

and $d = |\vec{d}|$. Theoretically, CIM in (4) is valid for any value of $C_0$. However, for numerical robustness, we use $C_0 = 10^6$ for low frequency and $C_0 = 1$ for high frequency, as discussed in [11].

Next, we substitute (2) and (3) into (4), following which we apply the method of moments with point-matching [22] to obtain

$$\mathbf{P}^{-1} \mathbf{U} \mathbf{E}^p = \mathbf{H}^p.$$  (7)

The equation above relates the discretized electric and magnetic fields through matrices $\mathbf{P}$ and $\mathbf{U}$ that can be calculated from (4), as shown in [11].

Next, we apply the equivalence theorem [23] to replace the conductor by the surrounding medium, and the equivalent current density

$$J_s^p(\vec{r}) = \sum_{n=0}^{N_p} j_n^p \Pi_n^p(\vec{r})$$

(8)

on the contour $\gamma(p)$ in order to ensure that the field outside the contour $\gamma(p)$ remains unchanged [14]. According to the equivalence principle [23], we set

$$J_s^p(\vec{r}) = H_t^p(\vec{r}) - \tilde{H}_t^p(\vec{r}),$$

(9)

where $\tilde{H}_t^p(\vec{r})$ is the tangential magnetic field on $\gamma(p)$ after the conductor is replaced by the surrounding medium, i.e. for the configuration shown in the right panel of Fig. 1. Magnetic field $\tilde{H}_t^p(\vec{r})$ is also discretized using pulse basis functions as follows

$$\tilde{H}_t^p(\vec{r}) = \sum_{n=0}^{N_p} \tilde{h}_n^p \Pi_n^p(\vec{r}).$$

(10)

The expansion coefficients of equivalent current $J_s^p(\vec{r})$ and tangential magnetic field $\tilde{H}_t^p(\vec{r})$ are collected into vectors $\mathbf{J}^p = [j_1^p \ j_2^p \ \ldots \ j_{N_p}^p]^T$ and $\tilde{\mathbf{H}}^p = [\tilde{h}_1^p \ \tilde{h}_2^p \ \ldots \ \tilde{h}_{N_p}^p]^T$. The vector counterpart of (9) is given by

$$\mathbf{J}^p = \mathbf{H}^p - \tilde{\mathbf{H}}^p.$$  (11)

We again relate the electric field $E_z^p(\vec{r})$ and magnetic field $\tilde{H}_t^p(\vec{r})$ using the CIM (4), except we replace $k$ in (5) by $k_0$ which is the wavenumber of the surrounding layer. Following the method of moments procedure, we obtain

$$\tilde{\mathbf{P}}^{-1} \mathbf{U} \mathbf{E}^p = \tilde{\mathbf{H}}^p,$$  (12)

which is analogous to (7) for the equivalent configuration in the right panel of Fig. 1. Next, we substitute (7) and (12)
into (11) to relate the equivalent current to the electric field on the contour γ(p)

\[ \mathbf{J}^{(p)} = \mathbf{Y}_s^{(p)} \mathbf{E}^{(p)}, \]  

(13)

where

\[ \mathbf{Y}_s^{(p)} = \mathbf{P}^{-1} \mathbf{U} - \mathbf{P}^{-1} \mathbf{\tilde{U}} \]  

(14)

is the so-called surface admittance operator introduced in [14]. It is important to emphasize that the above derivation is applicable to any conductor shape, unlike prior works in the literature which are only applicable to round [14], [16], rectangular [14], and triangular conductors [17].

By applying the above procedure to each conductor in the system, we obtain the following global surface admittance relationship

\[ \mathbf{J} = \mathbf{Y}_s \mathbf{E}, \]  

(15)

where \( \mathbf{J} \) and \( \mathbf{E} \) collect equivalent currents and electric fields on the contour of each conductor. The discretized surface operator \( \mathbf{Y}_s \) is a block diagonal matrix with each block being \( \mathbf{Y}_s^{(p)} \).

B. Electric Field Integral Equation

Next, we relate the electric field and equivalent currents to the p.u.l. impedance parameters by the electric field integral equation [14]

\[ \mathbf{E}^{(p)}(\mathbf{r}) = j\omega \mu_0 \sum_{q=1}^{P} \int_{\gamma_q} \mathbf{j}_s^{(q)}(\mathbf{r}') \mathbf{G}_0(\mathbf{r}, \mathbf{r}') d\mathbf{r}' - \frac{\partial V_p}{\partial z}, \]  

(16)

where \( \mathbf{G}_0(\mathbf{r}, \mathbf{r}') \) is the 2D Green’s function of the surrounding medium.

Following the procedure outlined [11], we substitute (2), (8) and (1) into (16), and then apply the method of moments to obtain

\[ \mathbf{E} = j\omega \mu_0 \mathbf{G}_0 \mathbf{J} + \mathbf{Q} [\mathbf{R}(\omega) + j\omega \mathbf{L}(\omega)] \mathbf{I}, \]  

(17)

where \( \mathbf{Q} \) is a block diagonal matrix with \( P \) blocks, each block of size \( N_p \times 1 \) and entries set to 1. The current \( \mathbf{I} \) and current density \( \mathbf{J} \) are also related by

\[ \mathbf{I} = \mathbf{Q}^T \mathbf{WJ}, \]  

(18)

where \( \mathbf{W} \) is a diagonal matrix with entries equal to the width of segment of each pulse basis function. As shown in [11], by substituting the surface operator (15) into (17) and manipulating the resulting equation we obtain the partial p.u.l. resistance

\[ \mathbf{R}(\omega) = \Re \left\{ \left[ \mathbf{Q}^T \mathbf{W} (1 - j\omega \mu_0 \mathbf{Y}_s \mathbf{G}_0)^{-1} \mathbf{Y}_s \mathbf{Q} \right]^{-1} \right\}, \]  

(19)

and the partial p.u.l. inductance

\[ \mathbf{L}(\omega) = \omega^{-1} \Im \left\{ \left[ \mathbf{Q}^T \mathbf{W} (1 - j\omega \mu_0 \mathbf{Y}_s \mathbf{G}_0)^{-1} \mathbf{Y}_s \mathbf{Q} \right]^{-1} \right\}. \]  

(20)

III. EIGENFUNCTION EXPANSION OF THE SURFACE ADMITTANCE OPERATOR

The method in [11] is based on the surface admittance operator theory that was originally developed in [14] for canonical conductor geometries, whose Helmholtz operator eigenfunctions are known analytically. In this section, we show how to calculate the eigenfunctions for an arbitrary geometry in a robust way, starting from the surface admittance operator from [11].

A. Eigenvalue Decomposition

We apply the eigenvalue decomposition on the surface admittance operator \( \mathbf{Y}_s^{(p)} \), so that

\[ \mathbf{Y}_s^{(p)} = \mathbf{A} \mathbf{D} \mathbf{A}^{-1}, \]  

(21)

where \( \mathbf{A} = [\mathbf{A}_1 \mathbf{A}_2 \ldots \mathbf{A}_{N_p}] \) contains the eigenvectors and the diagonal matrix \( \mathbf{D} \) contains the corresponding eigenvalues \( \lambda_i \).

Substituting (21) into (13) and left multiplying the resulting equation by \( \mathbf{A}^{-1} \), we obtain

\[ \mathbf{A}^{-1} \mathbf{J}^{(p)} = \mathbf{D} \mathbf{E}^{(p)}. \]  

(22)

We now define the following transformations

\[ \mathbf{J}^{(p)} = \mathbf{A} \mathbf{\tilde{J}}^{(p)}, \]  

(23)

\[ \mathbf{E}^{(p)} = \mathbf{A} \mathbf{\tilde{E}}^{(p)}, \]  

(24)

so that we can rewrite (22) as

\[ \mathbf{\tilde{J}}^{(p)} = \mathbf{D} \mathbf{\tilde{E}}^{(p)}, \]  

(25)

which is of the same form as the surface admittance operator (13). Transformations (23) and (24) suggest that the electric field on the boundary \( \gamma^{(p)} \) and inside the \( p \)-th conductor can be represented as a superposition of the eigenfunctions of the Helmholtz operator. For canonical shapes, these eigenfunctions can be obtained analytically [14]. For arbitrary shapes, their computation is not a trivial task, and is prone to numerical errors. The proposed procedure provides a way to obtain them in a robust way, even for conductors with sharp angles and thin cross section.

B. Eigenfunctions of a Round Conductor

We apply the eigenvalue decomposition on the surface admittance matrix \( \mathbf{Y}_s \) calculated at \( f = 2 \) MHz for a single round copper conductor of radius \( a = 1 \) mm and conductivity \( \sigma = 5.8 \times 10^7 \) S/m. The eigenfunctions corresponding to the
the top three dominant eigenvectors are shown in Fig. 2. These eigenfunctions resemble those obtained analytically in [14].

It is observed that for the dominant mode the electric field is uniform along the boundary. In absence of proximity effects, only this mode is excited, and leads to an exact modeling of skin effect inside the conductor. In the presence of other conductors, proximity effect exists, and higher order eigenfunctions are excited.

C. Eigenfunctions of a Square Conductor

Next, we carry out the same analysis at 1 MHz for a square copper conductor with side length 1 mm. The eigenfunctions associated with the four most-dominant eigenvalues are shown in Fig. 3. In [14], it is shown that the electric field distribution at the edges of a square conductor can be expanded into a series of sinusoids with different harmonic frequency. The eigenfunctions in Fig. 3 clearly exhibit the same behavior, which confirms how the proposed method generalizes [14].

D. Eigenfunctions of a Trapezoidal Conductor

Next, we consider the case of a trapezoidal copper conductor with top edge length of 0.5 mm and bottom edge length of 1 mm. The eigenfunctions of the Helmholtz operator for a trapezoidal conductor are not known analytically. However, we numerically extracted these eigenfunctions using the presented method. The electric field distribution corresponding to the four dominant eigenfunctions are shown in Fig. 4. The first eigenfunction corresponds to a uniform electric field on the boundary, while higher-order eigenfunctions describe field variations along the edges, and are important to capture the effect of corners, as well as proximity effect.

E. Eigenfunctions of a Valley-shaped Conductor

Finally, Fig. 5 shows the electric field corresponding to four relevant eigenfunctions for a valley-shaped conductor whose dimensions are given in Sec. IV-A. It can be seen from Fig. 5 that none of the eigenfunctions lead to a uniform electric field distribution on the edges. Among the eigenfunctions in the figure, two describe a current flowing in the two upper arms, while the other two describe the current in the V-shaped region. The functions in the left panels have a relatively smooth variation over space, while those in the right panels have a null at the midpoint of the upper arms and of the V-shaped region, respectively.

IV. Numerical Results

In this section, we consider two examples to demonstrate the accuracy, computational speed, and the practical usefulness of a technique that can handle arbitrary cross sections.
performing such a sensitivity analysis with FEM or another volumetric method is very time consuming, as it would have required 71 hours with FEM to perform this study compared to less than 30 minutes required with the proposed technique.

### B. Two Rectangular Conductors with Triangular Roughness Profile

Finally, we consider two rectangular conductors, each of dimension $127 \, \mu m \times 15 \, \mu m$, spaced $330 \, \mu m$ apart. This example is a simplified version of the test case presented in [18] to study roughness. We use a simple triangular roughness profile to study the impact of surface roughness on the attenuation constant of the transmission line. A sample roughness profile of one of the conductors is shown in Fig. 10. We let $T = 9.07 \, \mu m$ and we vary the roughness amplitude $a$ from $0 \, \mu m$ to $3.4 \, \mu m$, as roughness can introduce peaks with amplitude of several

![Fig. 6. Valley microstrip line considered in Sec. IV-A. All dimensions are in micrometers. We consider three parameter variations: $l$, $w$ and $h$ which are as labelled in this figure. The nominal values for these parameters are $l = 0 \, \mu m$, $w = 0 \, \mu m$, and $h = 3 \, \mu m$.](image1)

![Fig. 7. P.u.l. resistance and inductance of the valley-shaped interconnect considered in Sec. IV-A for $l = [-1 \, \mu m, 1 \, \mu m]$, $w = 0 \, \mu m$, and $h = 3 \, \mu m$. The FEM results are marked by ($\times$).](image2)

![Fig. 8. P.u.l. resistance and inductance of the valley-shaped interconnect considered in Sec. IV-A for $w = [0, 1.5 \, \mu m]$, $l = 0 \, \mu m$, and $h = 3 \, \mu m$. The FEM results are marked by ($\times$).](image3)

![Fig. 9. P.u.l. inductance of the valley-shaped interconnect considered in Sec. IV-A for $h = [2 \, \mu m, 4 \, \mu m]$, $w = 0 \, \mu m$, and $l = 0 \, \mu m$. The FEM results are marked by ($\times$).](image4)

![Fig. 10. Rectangular conductor with triangular roughness profile considered in Sec. IV-B](image5)
micrometers [18]. We calculated the impedance parameters for different values of \( a \) using the presented technique. The line capacitance was calculated with FEM [24]. Figure 11 shows that the amplitude of the triangular grooves significantly influences the attenuation constant of the transmission line, especially at high frequency. Each simulation took an average of 2.16 s per frequency point, which is very reasonable given the irregular shape of the conductor.

V. Conclusion

We presented a fast surface method to model skin effect in conductors of arbitrary cross section, and accurately estimate the per-unit-length impedance over a broad frequency range. The method relies on the surface admittance operator to efficiently model how current redistributes inside a conductor as skin effect develops. Previously, such method was restricted to canonical conductor shapes (circular, rectangular, triangular), for which the Helmholtz equation eigenfunctions are known analytically. Using a contour integral formulation, we demonstrated that the eigenfunctions can be reliably computed numerically for any cross section, and provide an interesting insight on the most relevant current distributions inside a conductor in presence of skin and proximity effect. The computational efficiency of the proposed method was demonstrated through a sensitivity analysis, which otherwise would have been very time consuming if performed with existing methods such as finite elements. Finally, the presented method was applied to a conductor with irregular profile, to estimate how attenuation increases with surface roughness. This latter test demonstrated the usefulness of the novel formulation, which can handle conductors of arbitrary profile.

REFERENCES