Self consistent, efficient and parametric macromodels for high-speed interconnects design

Settore disciplinare: Elettrotecnica (ING/IND-31)

Piero Triverio (matr. 136193)

Maggio 2009
Summary

The evolution of electronics in terms of miniaturization, operating speed, materials and technologies has made the design of modern electronic systems a very challenging task, that can be successfully accomplished only with suitable Computer Aided Design (CAD) techniques. Among the most difficult analyses we certainly have Electromagnetic Compatibility and Signal Integrity assessments in interconnect networks, an essential task for many designers because of the higher and higher operating frequencies used in electronics. These analyses call for broadband, accurate and efficient simulation models for any component of the interconnect. Their creation is a challenging task, with several problems of both theoretical and numerical nature still open. This thesis develops new techniques for interconnects modeling and simulation, focusing on three key aspects: physical consistency, efficiency and parameterization.

We first perform a complete investigation of the important properties of causality, stability and passivity, together with the development of accurate algorithms for their verification in models and data used in CAD. These results improve the reliability and robustness of real modeling and simulation processes, preventing failures commonly encountered even with state of the art algorithms. Then, we devise new macromodeling algorithms for the inclusion of design parameters and propagation delays in macromodels. These innovative characteristics are essential for the efficient simulation and optimization of long and complex interconnect links.
Acknowledgements

A valid supervisor is the most relevant factor in making a Ph.D. an important step towards a good scientific and professional career, as well as an engaging opportunity of higher education and research. I am very grateful to my supervisor, Prof. Stefano Grivet-Talocia, for making my Ph.D. meet these expectations. Working with the Electromagnetic Compatibility research group at Politecnico di Torino has been an enriching experience, and I am grateful for this to all its members, with a special mention to Prof. Flavio Canavero, Michelangelo Bandinu, Alessandro Chinea, Ramiro Serra, and Andrea Ubolli Macco.

During my Ph.D. I’ve been a visiting student at the CAD group at Carleton University in Ottawa, Canada, having the enviable opportunity to work with a brilliant Professor as Michel Nakhla. I am very grateful to him for this opportunity and I also kindly acknowledge the AEIT association from Milan, for supporting this internship with the “Isabella Sassi Bonadonna” scholarship.
# Contents

Summary

Acknowledgements

1 High-speed interconnects
   1.1 Design complexity .............................................. 2
   1.2 Macromodeling .................................................. 5
      1.2.1 General concepts ......................................... 5
      1.2.2 Macromodeling of high-speed interconnects ............. 7
      1.2.3 Transmission lines ........................................ 8
      1.2.4 Discontinuities .......................................... 8
      1.2.5 Logic devices ............................................ 10
   1.3 Objectives of this work ..................................... 10

I Self consistency of macromodels

2 Physical properties of macromodels
   2.1 Motivation ...................................................... 15
   2.2 Time domain ................................................... 17
      2.2.1 Causality ................................................... 18
      2.2.2 Stability ................................................... 19
      2.2.3 Passivity .................................................. 20
   2.3 Laplace domain ............................................... 22
      2.3.1 Two-sided Laplace transform .............................. 23
      2.3.2 Causality ................................................... 24
      2.3.3 Stability ................................................... 25
      2.3.4 Passivity .................................................. 26
   2.4 Frequency domain ............................................. 28
      2.4.1 Stability ................................................... 28
      2.4.2 Causality ................................................... 29
      2.4.3 Passivity .................................................. 31
   2.5 Examples ..................................................... 32
      2.5.1 An analytic example ...................................... 32
Chapter 1

High-speed interconnects

Every electronic system is made by a set of discrete or integrated components and by a complex interconnect network that allows for electronic signals exchange and supply power distribution. The interconnect network includes board traces, via holes, power/ground planes, board-to-board connectors, wires, chip packages, i.e. all elements found between the intelligent components of the system. Complexity of interconnects can be very high, especially in modern IT, automotive, aerospace and industrial products, where thousands of different signal paths are present. The aim of interconnect networks is very simple: provide the electrical connection between the different system components delivering signals with negligible distortion. Depending on the path length and the signal bandwidth, this aim can be very easy or very difficult to achieve by designers.

In low-speed systems, where interconnects length is much smaller than the minimum signal wavelength resulting in negligible propagation times, interconnects behave almost as ideal wires, with minor effects on the delivered signals. Their presence is therefore often neglected by designers, who focus on the functional system components. However, when the maximum operating frequency is such that the wavelength is comparable with the interconnect size their behavior becomes significantly richer and complex. Many effects like crosstalk, signal reflection, attenuation, dispersion, radiation and coupling of external fields arise and may compromise the operating of the whole electronic system, unless a careful interconnect design is performed. Interconnects falling in this second category are commonly denoted in the literature as high speed interconnects [12–14], and are the subject of this doctoral work.

Figure 1.1 shows the IBM BladeCenter ® QS22 server, an example of modern electronic system with several high speed interconnect components: the cpu-memory bus on the PCB\(^1\), the backplane connectors, the CPU chip packages. All these elements run at data rates of several Gbps. The multichip module shown in Fig. 1.2 is another example of a complex interconnect device.

\(^1\)PCB: Printed Circuit Board
1. Design complexity

The design of a modern electronic system is a complex task, which can be successfully accomplished only with Computer Aided Design (CAD) software that can perform system level simulations taking into account all relevant phenomena taking place into the
circuit components as well as in the interconnect networks. The range of required simulations is very broad, encompassing single transient simulations, eye-diagram computations, optimizations to maximize system performance, statistical analyses to check the design robustness against non-deterministic factors such as manufacturing tolerances or external agents. When the operating frequency is high, these CAD tools must simulate not only the components behavior, but also properly account for all the electromagnetic phenomena that take place in high-speed interconnects. Unfortunately, this is a difficult task, and even state of the art commercial CAD softwares are not satisfactory from this viewpoint. The development of better CAD techniques for systems with high-speed interconnects is the main objective of this work.

The numerical simulation of electric systems with high-speed interconnects is intrinsically difficult. We illustrate this concept focusing on the “canonical” structure of Figure 1.3, representing two chips connected by an interconnect link.

Two different regions can be identified in this system: the interconnect portion, denoted in the Figure with A, and the logical devices, denoted with B. The large difference in the characteristics of these two parts, summarized in Table 1.1, is the key reason for the difficulty of the numerical simulation of high-speed interconnects. In fact, simulation techniques available for one part may not be fit for the other, and viceversa, as evident if we discuss the applicability of standard simulation algorithms to the system in Figure 1.3:

**Pure frequency domain techniques** Frequency domain is the most natural and convenient to analyze interconnects. Unfortunately, these techniques cannot be applied to the simulation of the system in Figure 1.3 because of the nonlinearity of the logical ICs.
Frequency domain techniques for non-linear systems Techniques such as Harmonic Balance [15] and shooting methods [16] have been devised to account for nonlinearities in frequency domain analyses, and are very popular for RF systems design. Application to high-speed interconnects is however not feasible since Harmonic Balance is efficient only for weakly nonlinear circuits operating in nearly sinusoidal conditions. Shooting methods instead have limitations in handling distributed components [17].

Time domain techniques Since the system is non-linear, time domain is the most appropriate domain for simulations. Several algorithms have been developed for either circuits or electromagnetic systems. Unfortunately, circuit simulators cannot be directly applied to this system since, in most cases, have very limited models for transmission lines, and no high frequency models for other interconnect components such as connectors, packages, vias. In a few words, they are not suited to account for the electromagnetic effects present in distributed circuits. On the contrary, electromagnetic simulators have this capability, but have instead troubles dealing with circuits. The most limiting factor is however their computational complexity, strongly dependent on the structure size. While suitable for the simulation of a single component (e.g. a board connector), their computational cost is way too high for system-level simulations.

It is now clear that many simulation methods successfully employed by electronic designers fail when applied to systems with high-speed interconnects, and alternative strategies must be devised, as discussed in the next Section.

Table 1.1. Comparative summary of the properties of the interconnect and ICs part of the circuit in Figure 1.3.
1.2 Macromodeling

1.2.1 General concepts

The main difficulty in the simulation of high-speed interconnects is the coexistence of two different worlds, the circuit part and the electromagnetic part. Several methodologies have been proposed to simulate these heterogeneous systems, establishing a bridge between the two worlds. These approaches can be viewed in the general setting of macromodeling, illustrated in Figure 1.4. Macromodeling consists of building, for every component of the system, a mathematical model that accurately represents its behavior. The prefix macro emphasizes that just the macroscopic behavior of the system as seen from its inputs/outputs is described, while no information is retained on its internal working and composition. The combination of different macromodels leads to a global system model, suitable for system-level simulations.

Macromodeling involves the steps illustrated in Figure 1.5: first the component is characterized with a measurement or a numerical evaluation of its behavior, either in time or in frequency domain. Typically, the frequency response is used or, in time domain, the output response to selected input waveforms. The obtained characterization is then used to create the macromodel. This step is called identification and aims to minimize the error between the given data and the macromodel response with a suitable choice of the model coefficients. Finally, the model is used into a simulation environment to perform the analyses required by the design. The steps in Figure 1.5 are the most common but
other approaches exist. For example, macromodeling may start from the a geometrical description of the device instead of a characterization of its behavior as, for example, in Partial Element Equivalent Circuit (PEEC) methods, briefly described in Section 1.2.4.

The main advantages of the macromodeling philosophy are:

**Efficiency** Macromodels represent only the component behavior as seen from its input/output ports, with the most compact mathematical formulation possible. Thus, they retain only the information strictly needed for simulation purposes, avoiding unnecessary details. This often results in an important reduction of complexity with respect to conventional models, and in a large simulation speedups.

**Flexibility** The proposed approach is very flexible because of its modular nature. Once available, macromodels can be easily combined, providing global models even for very large and heterogeneous systems, like the one in Figure 1.3. This characteristic of partitioning the original problem is a key advantage for heterogeneous systems and also helps to reduce the original problem into smaller subproblems, easier to treat. These features are becoming more and more important in engineering, due to the increasing complexity of modern systems and the integration of several different technologies (e.g., system-in-package, optoelectronic devices, MEMS).

**Compatibility** Macromodels are derived to be compatible with already existing CAD tools and simulation methods. Integration with existing models, design flows, and projects is thus very easy.

**Model reuse** Once a macromodel is built, it can shared among designers and repeatedly used in all projects involving the real component.

**Intellectual property protection** Since macromodels hide the internal details of the original component, companies can use them to provide accurate simulation models for their devices without disclosing proprietary information.

Macromodeling has also some limitations and drawbacks:

---

2MEMS: Micro Electro-Mechanical System.
• Designers must choose how to partition the original system, and select the appropriate modeling algorithm to be applied to each portion.

• Since the original system is divided into blocks connected only through a certain set of inputs and outputs, other interactions are neglected. Designers discretion is therefore needed to evaluate all interactions that must be taken into account. In Chapter 5 we will reduce this limitation for long interconnect chains by presenting a technique that can model a whole link avoiding division into blocks.

• If a component is modified during the design, a new macromodel must be computed from scratch. This drawback is particularly relevant when many different system configurations must be simulated in order to optimize the system performance. In Chapter 4 we will solve this open problem showing how to introduce symbolic parameters into macromodels.

1.2.2 Macromodeling of high-speed interconnects

After this general introduction to macromodeling, we focus on its application to high-speed interconnects, describing the state of the art in this field. For macromodeling purposes, the components of a system such as the one in Figure 1.3 are usually divided into three groups, as depicted in Figure 1.4: transmission lines, discontinuities, and logical devices. For each group, dedicated macromodeling techniques have been developed, and the most important will be mentioned in Sections 1.2.3, 1.2.4, and 1.2.5. Before this we discuss the features that all these techniques should achieve.

Accuracy Clearly, macromodels must be accurate representations of the original components in order to lead to trustworthy results. Accuracy is usually defined in terms of the error between the model response and the original device characterization.

Numerical robustness and cost Model identification must have acceptable computational cost and memory consumption. It must also be numerically robust, i.e. provide consistent performance and stability on different application scenarios. Both requirements are not trivial, since model identification for a complex device may require the solution of a very large and potentially ill-conditioned system of equations.

Physical consistency A major requirement is the respect of the fundamental physical properties of the original structure. This constraint is important not only for a theoretical consistency of the model with the real world, but also to achieve meaningful simulation results. In Chapter 2 we will demonstrate that three properties are crucial for interconnects macromodels, namely stability, passivity and causality, and that, when not satisfied, misleading results or even simulation failures may be obtained.

Efficiency Delivered macromodels must be relatively compact, made by a reduced number of equations. Otherwise, when all models are assembled together for a system level simulation, complexity will be too large and simulation unfeasible.
Compatibility Integration with standard circuit simulators and leading CAD software must be possible. This requirement is satisfied with a suitable choice of the mathematical expression for the macromodel, that can be cast as an equivalent electric circuit.

1.2.3 Transmission lines

Long interconnects with a uniform cross-section can be analyzed and modelled using the theory of transmission lines [18] as theoretical background. The available macromodeling methods belong to two categories: those that lead to a purely lumped model, and those able to produce distributed macromodels.

The simplest method in the first category is lumped segmentation: the line is chopped into small sections of length lower than the minimum signal wavelength [12, 18, 19]. Each section can therefore be considered lumped, and represented by a simple RLGC circuit. The obtained model is a cascade of RLGC blocks and may have very high dynamical order if the line is long. Although very simple, it is inefficient for simulation of complex interconnect networks. In order to overcome this drawback more advanced techniques have been developed. Matrix Rational Approximation [20, 21], Finite Difference methods [22], and Integrated Congruence Transform [23] are the most representatives. They provide simpler models of reduced complexity when compared to lumped segmentation. There is however a fundamental efficiency limitation in all these techniques. Because of the lumped nature of the macromodel, propagation delays cannot be represented directly, but must be approximated by many lumped elements. This issue is particularly relevant in long transmission lines, where propagation delays are large.

Overcoming this limit is possible if distributed elements are included in the macromodel, especially delay elements that can directly represent propagative effects in transmission lines. This approach has two important advantages. First, the macromodel efficiency is significantly improved, since a delay element may replace several lumped components. Second, the model complexity does not depend anymore on the line length. The generalized Method of Characteristics (MoC) [24] is the basis for several modeling techniques with delay extraction [25–28], that ensure very high simulation efficiency even for long lines. Passivity is unfortunately not preserved by construction, and this issue was only recently addressed [29–31]. DEPACT [32] is another modeling strategy that combines Matrix Rational Approximation with delay extraction and has the remarkable property of leading to passive models.

1.2.4 Discontinuities

Typical interconnects present also, in addition to straight transmission lines, other elements with a more complex geometry, that do not fit in the transmission lines theory. In this category we can find all kinds of discontinuities, such as connectors, chip packages, via holes, bends, and junctions. These components are often modelled with algorithms for the identification of linear macromodels. The most popular solution in the state of the art is Vector Fitting [33, 34], an algorithm that can fit complex tabulated frequency responses
with a rational approximation. Vector Fitting’s popularity stems from its robustness, accuracy, the automatic enforcement of model stability and its low computational cost. All these features make it suitable even for devices with very high dynamical order or port count. This algorithm has improved and replaced older methods such as [35, 36]. Several extensions of Vector Fitting (VF) have been proposed, most notably its time domain counterpart [37], and modified versions with different basis functions [38, 39], improved noise robustness [40, 41], lower computational cost [42]. This algorithm will be often referred in this work as it is a widely diffused tool and is the reference point for linear macromodeling algorithms. For convenience of the Reader we included in Appendix A a short description of this algorithm.

A major issue in these fitting methods is the enforcement of model passivity, not guaranteed by VF. The main reason why passivity is not enforced is the intrinsic complexity of passivity conditions, that would increase too much the fitting computational cost, making the algorithm useless for practical modeling problems. This lack led to several techniques for an a posteriori correction of passivity violations. Linear or quadratic programming [43–45], Hamiltonian matrices [46, 47], and convex optimization [48] are the mathematical concepts that have been exploited to address this issue. The direct generation of passive macromodels from tabulated data is still an open problem.

PEEC\(^3\) techniques are another approach to model discontinuities. These methods [49–54] start from the geometrical description of the structure and the electromagnetic characteristics of the materials, and represents Maxwell’s equations with an equivalent circuit. They therefore operate a discretization of Maxwell’s partial differential equations in terms of standard circuit elements. Among the advantages of this approach we certainly have generality and flexibility. The main issue is the very large components count of the equivalent circuit, due to the discretization of a three-dimensional structure. Therefore, PEEC methods are often coupled with order reduction techniques.

Model order reduction algorithms are quite popular and useful for interconnects simulation, as they reduce the dynamic order of a given model while retaining the accuracy over the bandwidth of interest. They are therefore applied after a model was already constructed with one of the methods described above. The most common methods used for interconnects are based on either moment matching or truncated balanced realization. Among the first group we have explicit methods like Asymptotic Waveform Evaluation [55], Complex Frequency Hopping [56], and the more appealing implicit methods, based on the Arnoldi [57] or Lanczos [58] process. PRIMA [14] is the most representative algorithm based on moment-matching, and couples low computational cost, high robustness with the preservation of passivity.

Truncated balanced realization was developed for control systems [59] and was then applied to circuit simulation, in a modified version that preserves passivity [60]. Compared to PRIMA, this algorithm guarantees smaller models, nearly optimal, but has higher computational cost. Therefore, it is often applied after a first model reduction has been performed with PRIMA.

\(^3\)PEEC: Partial Element Equivalent Circuit.
1.2.5 Logic devices

Models for logic devices range from transistor-level models to functional-only models, working in pure digital conditions. The first provide the best possible accuracy but may be extremely inefficient due to the large transistors count in a typical IC. The second are instead very efficient but may be too simplistic. There is thus the need for better models, simultaneously accurate and efficient, at least for the blocks of the chips that interface the internal logic devices with the outside analog world, i.e. the transmitter and receiver stages. These blocks must be accurately modelled, reproducing the correct analog behavior seen at IC ports, in order to get trustworthy results from interconnects simulation. Resorting to transistor level models for these parts is often not an option, since their complexity is still too high for a system level simulation with several chips and interconnect elements. This issue brought researchers to develop macromodeling techniques also for the transmitter and receiver stages of logical ICs. These models interface with the logical signals inside the chip on one side, and with the analog signals on the interconnect from the other side.

A popular method for I/O buffer macromodeling is IBIS [61], that produces simple equivalent circuits representing the chip behavior at ports. Although simple, this approach suffers from some limitations, in terms of accuracy and flexibility, and is not suitable to accurately model the most advanced digital devices. More powerful modeling techniques have therefore been proposed, based on nonlinear identification methods [62]. The proposed methods make use of radial/sigmoidal basis functions [63], neural networks [64, 65], splines [65], and composite local-linear state space relations [66] for the mathematical formulation of the model, to cite just a few popular approaches. These techniques have been briefly mentioned here for completeness, but will not be considered in this work.

1.3 Objectives of this work

Despite the large research effort on the simulation of high-speed interconnects, several problems are still unsolved or call for better solutions in order to meet industry needs. The ultimate objective of this doctoral work is to improve the state of the art in this field, addressing several open problems related to the theoretical and physical foundations of macromodels as well as developing new macromodeling algorithms with innovative features.

Part I of this thesis will be dedicated to the physical consistency of macromodels and macromodeling data. This study is motivated by two strong reasons. First, physical consistency is a crucial aspect in CAD, since it can significantly impact the accuracy, effectiveness and reliability of all simulation and macromodeling algorithms. Second, this analysis will provide the necessary theoretical background for the solution of some challenging macromodeling problems, as those treated in Part II.

Unfortunately, the literature of the circuit simulation community lacks an adequate understanding of the physical properties that macromodels shall respect and of their impact in “real engineering life”. Chapter 2 aims to definitively fill this gap, providing a thorough study of the physical properties of causality, stability and passivity, in order
1.3 – Objectives of this work

to give to the community a solid and complete understanding of these fundamental aspects. This analysis will underscore the paramount importance of physical consistency in CAD, and provide a clear explanation to some common failures faced by designers in real macromodeling and simulation scenarios.

In Chapter 3, we will focus on the physical consistency of measurement and simulation data used for macromodeling and simulation tasks. Data consistency may be lost because of several factors, ranging from measurement and simulation errors, imperfect instruments calibration, numerical errors, unphysical assumptions. When this happens even state of the art macromodeling and simulation algorithms fail, as shown in Chapter 2. Therefore, reliable algorithms to promptly detect these inconsistencies in the raw data are highly desirable to harden macromodeling and simulation tasks. Since literature lacks such techniques, in Chapter 3 we will propose two excellent algorithms for the verification of causality and passivity in tabulated frequency data. This aim will be reached thanks to a new numerical method for the computation of the generalized Hilbert transform, that overcomes the limitation of all previously published approaches. This method will be also exploited in a causality-constrained interpolation routine suitable to reconstruct missing frequency samples not obtainable from electromagnetic solvers or measurement equipment.

Capitalizing on the solid background developed in Part I, the second part of this thesis will tackle two open macromodeling problems. We will show how to include design parameters or delay elements in macromodels. Although these problems may seem uncorrelated, they can be traced back to the same root problem: finding a suitable macromodel formulation that is general enough to support the insertion of new terms such as symbolic parameters or delay elements, while being still compatible with most circuit solvers and easy to fit to tabulated data with a low complexity algorithm.

In Chapter 4 we will work on the creation of parametric macromodels, i.e. on macromodels that retain the dependence on some design parameters in a symbolic form. This feature is very useful to speed up lengthy optimizations, Monte Carlo, design centering, what-if analyses where interconnects are involved. Standard macromodels are of partial help for this task, since at every run the system changes and the entire macromodeling workflow must thus be repeated. If instead parametric macromodels are available, one generation process is enough, and then repeated runs can be performed in rapid sequence by just updating the parametric model coefficients. Chapter 4 will present two innovative algorithms for parametric macromodeling, with results also on the non-trivial problem of uniform stability.

The focus will then move in Chapter 5 to another hot open problem, the generation of distributed macromodels with delay elements. As already discussed, lumped macromodels for long structures have very low efficiency. While some solutions are available for transmission lines, no fitting algorithms exist for a general structure such as a link chain with several discontinuities. We will therefore propose an identification technique that can synthesize highly efficient models with delay elements, starting simply from measurement or simulation data. The proposed technique enforces the model stability by construction and will be shown to have an excellent potential, being able to cut simulation times for long structures by at least one order of magnitude.
1 – High-speed interconnects
Part I

Self consistency of macromodels
Chapter 2

Physical properties of macromodels

Macromodels are abstract mathematical representations of the behavior of a physical component, expressed in a form which is particularly useful for numerical simulation purposes. Is is intuitive that an accurate match between the macromodel response and the real system behavior is necessary for trustworthy results. At first, this may seem the only requirement. Actually it is not, and accuracy is not even the most important requirement for macromodels, because other properties, related to the physics of the real component, happen to be more important and essential to obtain correct simulation results.

The aim of this Chapter is to identify all physical properties that are relevant for the macromodeling of electrical components, with a particular focus on interconnect elements. It will turn out that causality, stability, and passivity are the three key attributes. Clearly owned by the original system, they may be lost during the characterization or identification stages of the macromodeling process, due to measurement errors, imperfect instruments calibration, numerical errors, noise, wrong assumptions. We will show that this occurrence is very dangerous, not only because the physical consistency with the original system is lost, but also because it may cause serious modeling and simulation problems. Section 2.1 will further motivate this statement with an example taken from a real macromodeling scenario.

Sections 2.2, 2.3 and 2.4 will define and analyze the properties of stability, causality and passivity in the three domains most commonly used in engineering, namely the time, Laplace and Fourier domain. This material will then be used in Section 2.5 to clarify some common problems encountered in practical macromodeling activities, and will also constitute a rich and sound theoretical background for the rest of this thesis. This Chapter is a slightly modified version of [1,11].

2.1 Motivation

This section considers a simple interconnect example for which the generation of a macromodel fails. We present the example under the standpoint of the design engineer, who
knows the physical geometry of the interconnect and is required to generate a SPICE-compatible model in order to carry out the design. The structure is a three-conductor transmission line (courtesy of Dr. I. Kelander, Nokia). Its scattering matrix is first computed over a bandwidth of 4 GHz using a commercial frequency-domain 3D field solver. Then, the various scattering matrix entries are processed by Vector Fitting (VF) in order to produce a lumped model for the structure (see Appendix A). It turns out that VF fails to provide a reasonably accurate model. As an example, we report in Figure 2.1 the original return loss $S_{11}$ and the corresponding response of a rational model with 20 poles. Although the raw data are quite smooth, the model is very inaccurate.

![Figure 2.1. Vector Fitting-generated model (20 poles, all with negative real part). Raw $S_{11}$ response (solid line) and model response (dashed line).](image)

The first solution one can think of in order to improve the accuracy is to increase the number of model poles or the number of VF iterations. Table 2.1 reports the resulting fitting error for up to 40 poles. The table clearly shows that even if the model order or the number of VF iterations are increased, the accuracy remains poor. The rational fitting scheme does not seem to converge.

<table>
<thead>
<tr>
<th>Model order</th>
<th>VF iterations</th>
<th>Maximum error</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>4</td>
<td>$1.4 \times 10^{-2}$</td>
</tr>
<tr>
<td>20</td>
<td>4</td>
<td>$1.1 \times 10^{-2}$</td>
</tr>
<tr>
<td>30</td>
<td>4</td>
<td>$1.0 \times 10^{-2}$</td>
</tr>
<tr>
<td>30</td>
<td>15</td>
<td>$1.1 \times 10^{-2}$</td>
</tr>
<tr>
<td>40</td>
<td>15</td>
<td>$1.2 \times 10^{-2}$</td>
</tr>
</tbody>
</table>

Table 2.1. Accuracy of the rational model generated by Vector Fitting. Poles with positive real part are not allowed.

The standard VF algorithm avoids the presence of unstable poles by changing the sign of their real part whenever they occur during the iterations. Our next try is to disable this feature and to let VF choose the best poles placement in the entire complex
Table 2.2. Accuracy of the rational model generated by Vector Fitting. Poles with positive real part are allowed.

<table>
<thead>
<tr>
<th>Model order</th>
<th>Stable</th>
<th>Maximum error</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>NO</td>
<td>$7.3 \times 10^{-3}$</td>
</tr>
<tr>
<td>20</td>
<td>NO</td>
<td>$1.4 \times 10^{-4}$</td>
</tr>
<tr>
<td>30</td>
<td>NO</td>
<td>$1.2 \times 10^{-4}$</td>
</tr>
</tbody>
</table>

plane. Surprisingly, VF readily computes a highly accurate model even with few poles, as illustrated in Table 2.2. Unfortunately, this model will be useless for any practical purpose because, due to the presence of unstable poles, any time-domain simulation in a CAD environment will blow up exponentially. Moreover, it is quite unreasonable that the frequency response of a certainly passive structure requires the presence of unstable poles for its rational approximation.

It turns out that the raw frequency responses are flawed by causality violations, as we will demonstrate in Section 2.5. However, the symptoms of these inconsistencies arise only when trying to fit the data. Consequently, the main problem is difficult to identify, and even more difficult is to realize how to fix it. The theoretical results presented in Sections 2.2-2.4 will provide the background material that will allow, in Section 2.5, a complete explanation and interpretation of the VF results for this example.

### 2.2 Time domain

In this section the physical concepts of causality, stability and passivity are described and precisely defined by appropriate mathematical conditions. We restrict our attention to linear\(^1\) and time-invariant\(^2\) electrical \(n\)-port networks, with input and output denoted, respectively, by the \(n\)-elements vectors \(x(t)\) and \(w(t)\). Due to linearity and time-invariance, the system can be conveniently represented with a convolution [67] relating the input \(x(t)\) and output \(w(t)\),

\[
w(t) = h(t) * x(t) = \int_{-\infty}^{+\infty} h(t-\tau)x(\tau)d\tau.
\]

The matrix \(h(t)\) is the system impulse response, with each element \(h_{ij}(t)\) being the response at port \(i\) when an ideal impulse (Dirac’s delta) is applied at port \(j\), with all other inputs set to zero. We will consider different representations of electrical \(n\)-port networks,

\(^1\)A system is linear if the response to a linear combination of two inputs

\[
x(t) = c_1 x_1(t) + c_2 x_2(t)
\]

is

\[
w(t) = c_1 w_1(t) + c_2 w_2(t),
\]

where \(w_1(t)\) and \(w_2(t)\) are the outputs corresponding to each input \(x_1(t)\) and \(x_2(t)\), respectively.

\(^2\)The time-invariance property identifies those systems that do not change their behavior with time. If \(w(t)\) is the output excited by input \(x(t)\), then \(w(t-\tau)\) is the output for the delayed input \(x(t-\tau)\).
including impedance ($x$ being currents and $w$ voltages), admittance ($x$ being voltages and $w$ currents), and scattering (both $x, w$ being power waves).

### 2.2.1 Causality

It is part of our real world experience that an effect cannot precede its cause. This intuitive concept is the fundamental principle of causality [68], that every physical system has to respect. For example, if two inputs $x_1(t)$ and $x_2(t)$, equal up to $t = t_0$, are applied to a causal system, their respective outputs are expected to be equal up to $t = t_0$. If this is not the case (see Figure 2.2), the system is non-causal, because it forecasts a difference in the inputs before it actually occurs.

The precise definition of causal system that follows is just the formal writing of this intuitive consideration.

**Definition 2.1 (Causality [69]).** A system is causal if and only if for all input pairs $x_1(t)$ and $x_2(t)$ such that

$$x_1(t) = x_2(t), \quad t \leq t_0 \quad \forall t_0$$

the two corresponding outputs satisfy

$$w_1(t) = w_2(t), \quad t \leq t_0.$$

From this general definition, a simpler condition for the causality of linear systems can be stated [69].

**Theorem 2.1.** A linear system is causal if and only if for every input $x(t)$ that vanishes for $t < t_0$, the corresponding output $w(t)$ vanishes too for $t < t_0$.

Finally, we derive the important constraint imposed by causality on the impulse response $h(t)$ of linear, time-invariant (LTI) systems [67].
**Theorem 2.2.** A LTI system is causal if and only if all the elements $h_{ij}(t)$ of its impulse response matrix $h(t)$ are vanishing for $t < 0$, i.e.

$$h(t) = 0, \quad t < 0. \tag{2.2}$$

**Proof.** For the sake of simplicity, we consider a scalar impulse response $h(t)$. Condition (2.2) is necessary for causality because, if a Dirac’s delta $\delta(t)$ is taken as input, the output is $w(t) = h(t) * \delta(t) = h(t)$. Since input vanishes for $t < 0$ then, for Theorem 2.1, the output $h(t)$ must vanish too for $t < 0$.

Condition (2.2) is also sufficient to guarantee causality. In this case (2.1) becomes

$$w(t) = \int_{-\infty}^{t} h(t - \tau)x(\tau)d\tau$$

and, due to the upper integration limit, causality follows from Theorem 2.1.

**Remark 2.1.** The above definitions of causality are general and apply to both lumped and distributed systems. In the latter case, however, it may be important to adopt a more stringent definition by explicitly considering the propagation delays due to the finite propagation speed of signals [68],

$$h_{ij}(t) = 0, \quad t < T_{ij}, \quad T_{ij} \geq 0 \quad \forall i,j.$$  

This holds, e.g., in any transmission-line network. The identification of models that take into account these propagation delays is indeed an active research area [24–32, 70–77] and will be considered in Chapter 5. Throughout this thesis, we will adopt the delay-free definition of causality of Theorem 2.2. Therefore, we will always refer to causality meaning zero-delay causality.

### 2.2.2 Stability

The concept of stability is related to the boundedness of the system responses. In fact, engineers always verify that their circuits are stable in order to be sure that no inputs can drive them beyond operating limits. For this reason, although several different definitions of stability are available, herewith we consider the so-called Bounded Input Bounded Output (BIBO) definition of stability [78].

**Definition 2.2 (Stability).** A system is stable if the output $w(t)$ is bounded$^3$ for all bounded inputs $x(t)$.

The BIBO stability is guaranteed in a LTI system if and only if all elements of $h(t)$ are such that [78]

$$\int_{-\infty}^{+\infty} |h_{ij}(t)| dt < +\infty. \tag{2.3}$$

The above condition applies to both lumped and distributed systems.

---

$^3$ A vector $w(t)$ is bounded if any of its components $w_i(t)$ is such that $|w_i(t)| < M, \forall t$. 

19
Example 2.1. Let us consider a $RC$ circuit, as shown in Figure 2.3a. Since this circuit is made of real components, causality should be assumed a priori. Under this hypothesis, the impedance impulse response is

$$h(t) = Ae^{pt}u(t), \quad p = -\frac{1}{RC}, \quad A = \frac{1}{C} \quad (2.4)$$

where $u(t)$ denotes the unit step (Heaviside) function. Stability depends on the sign of $R$ and $C$. If both are positive (as is true for real resistors and capacitors), then $p < 0$, the integral in (2.3) is bounded, and the system turns out to be BIBO stable. Conversely, if for example $R$ is negative (as can be obtained with an active device, at least within a given voltage range), $h(t)$ grows for large $t$, and stability does not hold.

Example 2.2. We consider the physical circuit shown in Figure 2.3b. The impulse response in the admittance representation is

$$h(t) = A\cos(\omega_0 t)u(t), \quad \omega_0 = \frac{1}{\sqrt{LC}}, \quad A = \frac{1}{L} \quad (2.5)$$

and violates (2.3). The system is thus not BIBO stable. This is further confirmed by choosing the bounded input $x(t) = \sin(\omega_0 t)u(t)$, which produces the unbounded output $w(t) = \frac{1}{\omega_0} \sin(\omega_0 t)u(t)$. This is the well-known principle of (lossless) resonance, which will be discussed in more detail in Section 2.3.

2.2.3 Passivity

A physical system is denoted as passive when it is unable to generate energy. The precise mathematical definition of passivity depends on the representation adopted for the $n$-port network. For impedance or admittance representations we have [79]

**Definition 2.3** (Passivity). An $n$-port network is said to be passive if

$$\int_{-\infty}^{t} v^T(\tau)i(\tau)d\tau \geq 0 \quad (2.6)$$

for all $t$ and all admissible port voltages $v(t)$ and currents $i(t)$. 
2.2 – Time domain

For scattering representations, the passivity definition is similar, with (2.6) replaced by
\[ \int_{-\infty}^{t} [a^T(\tau)a(\tau) - b^T(\tau)b(\tau)] d\tau \geq 0 \] (2.7)
where \(a(t)\) and \(b(t)\) are respectively the incident and reflected power waves at the ports. The above definitions apply to both lumped and distributed systems.

Integrals in (2.6) and (2.7) represent the cumulative net energy absorbed by the system up to instant \(t\). This energy has to be positive for all \(t\) in any passive system. This requirement is satisfied if two conditions hold: (i) the system absorbs more energy than it generates; (ii) the possible generation occurs after absorption. A non-causal system that first generates energy and then absorbs it, even to a larger extent, is thus considered non-passive. With this consideration in mind, it is not surprising that all passive systems are causal [67, 79].

**Theorem 2.3.** If a LTI system is passive, then it is also causal.

**Proof.** We prove this important result for the scattering representation; a similar result holds for the impedance/admittance representations [79]. For simplicity, we focus on a one-port system with input \(a(t)\) and output \(b(t)\). The proof establishes causality by verifying that, for passive systems, Theorem 2.1 always holds. We choose an arbitrary input signal \(a(t)\) that vanishes for \(t < t_0\). The passivity definition (2.7), requires that
\[ \int_{-\infty}^{t} b^2(\tau)d\tau \leq 0, \quad t < t_0. \]
The integrand function is non-negative by construction. Therefore, the above inequality holds for all \(t < t_0\) only if the output \(b(t)\) vanishes for \(t < t_0\). Hence, the system is causal.

**Example 2.3.** Figure 2.4 depicts an ideal transmission line (characteristic impedance \(R_0\), propagation constant \(\beta\), length \(l\)) terminated by a load resistor \(R\). If \(R_0\) is assumed as the reference port impedance, the reflected power wave \(b(t)\) turns out to be
\[ b(t) = \Gamma_R a(t - 2t_0), \quad \Gamma_R = \frac{R - R_0}{R + R_0} \] (2.8)
where \( t_0 = \beta l \) is the one-way time-of-flight of the line. Obviously, this system is passive if \( R \) is positive. We prove this by applying the passivity condition (2.7), which in this case becomes

\[
\int_{-\infty}^{t} \left[ a^2(\tau) - \Gamma_R^2 a^2(\tau - 2t_0) \right] d\tau = (1 - \Gamma_R^2) \int_{-\infty}^{t-2t_0} a^2(\tau) d\tau + \int_{t-2t_0}^{t} a^2(\tau) d\tau \geq 0 \tag{2.9}
\]

In the above inequality, both integrals are positive for any possible input signal \( a(t) \). Therefore, the sign of the entire expression depends only on the factor \( (1 - \Gamma_R^2) \), that is positive if

\[
|\Gamma_R| = \left| \frac{R - R_0}{R + R_0} \right| \leq 1 \Leftrightarrow R \geq 0.
\]

So, if \( R \geq 0 \), (2.7) holds and the system is passive as expected.

**Remark 2.2.** Theorem 2.3 has two important consequences. First, since all passive systems are causal, any non-causal system cannot be passive. Second, any macromodeling algorithm that enforces model passivity will also guarantee model causality. Conversely, a model that violates causality will violate passivity too, as pointed out in the next example.

**Example 2.4.** We consider again the structure of Example 2.3, but with a negative delay, i.e., \( t_0 < 0 \). This system is clearly non-causal, since the output \( b(t) = \Gamma_R a(t - 2t_0) \) is an anticipated version of the input \( a(t) \). Of course, a physical equivalent does not exist. However, it is an interesting illustration of the fact that non-causal systems are also non-passive. We show the lack of passivity by noting that the passivity condition (2.7), for an input \( a(\tau) \) that vanishes for \( \tau \in (-\infty, t] \), reads

\[
-\Gamma_R^2 \int_{t}^{t-2t_0} a^2(\tau) d\tau \geq 0.
\]

This condition is never satisfied because of the negative sign in front of the certainly positive integral. This example also shows that any passivity violation can be highlighted or detected by choosing an appropriate input that results in a negative absorbed energy.

### 2.3 Laplace domain

The Laplace transform is the natural tool for the analysis of LTI systems, since it transforms differential time-domain operators into algebraic \( s \)-domain operators. In Laplace domain, (2.1) becomes

\[
W(s) = H(s)X(s) \tag{2.10}
\]

where \( H(s) \) represents the system transfer function.

In this section, we derive the conditions for stability, causality and passivity in the Laplace domain. However, we should be careful in using the appropriate definition of the transform. In fact, the widely used one-sided Laplace transform, defined as

\[
\mathcal{L}\{f(t)\} = \int_{0}^{+\infty} f(t)e^{-st}dt \tag{2.11}
\]
2.3 – Laplace domain

The two-sided Laplace transform is defined as [78]:

$$ F(s) = \mathcal{L}_b\{f(t)\} = \int_{-\infty}^{+\infty} f(t)e^{-st}dt $$

(2.12)

where $s = \sigma + j\omega$. The key difference between two-sided and one-sided Laplace transform is the importance of the region of convergence (ROC), i.e., the set of $s$ values for which the integral in (2.12) converges absolutely. We illustrate this via a simple example.

**Example 2.5.** Consider the two distinct functions $f_1(t) = e^{pt}u(t)$ and $f_2(t) = -e^{pt}u(-t)$, where $p$ is a real quantity. A direct calculation from (2.12) leads to

$$ \mathcal{L}_b\{f_1(t)\} = F_1(s) = \frac{1}{s-p} , \quad \text{ROC: Re}\{s\} > p $$

(2.13)

$$ \mathcal{L}_b\{f_2(t)\} = F_2(s) = \frac{1}{s-p} , \quad \text{ROC: Re}\{s\} < p $$

(2.14)

so that the actual transformed functions take identical expressions. Therefore, the only way to discriminate them is the knowledge of their respective ROCs, depicted in Figure 2.5.

We will show that the ROC plays a fundamental role for the characterization of both causality ($f_1$ is causal and $f_2$ is not) and stability. For completeness, we report four general ROC properties, clearly verified for the above example:

---

4 Two-sided Laplace transform (2.12) has the same properties of the one-sided transform [78], except for the transform of a differentiated signal, that turns out to be $\mathcal{L}_b\{\frac{d}{dt}f(t)\} = sF(s)$.
ROC is always, in the complex $s = \sigma + j\omega$ plane, a strip parallel to the imaginary axis;

- if a function $f(t)$ vanishes for $t < t_0$, its ROC is a half plane open on the right, i.e., $\Re\{s\} > \sigma_0$ for some $\sigma_0$;

- $F(s)$ is analytic inside its ROC;

- ROC is bounded on its left and right by the singularities of $F(s)$ (poles for lumped systems).

Although inversion of (2.12) can be computed via line integration within the ROC [78], inverse Laplace transform is usually obtained (at least for lumped systems) by partial fractions decomposition, as shown in the following example.

**Example 2.6.** The function $F(s) = \frac{1}{(s+1)(s+2)}$ with ROC $-2 < \Re\{s\} < -1$ is decomposed as

$$
\frac{1}{(s+1)(s+2)} = \frac{1}{s+1} - \frac{1}{s+2}
$$

The regions of convergence of the two partial fractions have to be chosen such that their intersection is the ROC of $F(s)$. According to the above mentioned properties, the possible ROCs for the first term $\frac{1}{s+1}$ are $\Re\{s\} < -1$ and $\Re\{s\} > -1$ and, for the second one, are $\Re\{s\} < -2$ and $\Re\{s\} > -2$. The ROCs of each fraction in (2.15) are the only combination which is compatible with the ROC of the original function. The inverse Laplace transform is then $f(t) = -e^{-t}u(-t) - e^{-2t}u(t)$.

### 2.3.2 Causality

In a causal system, since each element of $h(t)$ is vanishing for negative time, the ROC for each of the elements of $H(s)$ is a half plane open on the right. However, this condition is not sufficient for causality. The following theorem [80] provides a precise characterization

**Theorem 2.4.** A signal $h(t)$ is vanishing for $t < 0$ if and only if its two-sided Laplace transform:

1. is defined and analytic in a half plane $\Re\{s\} > \sigma_0$;

2. grows not faster than a polynomial for $\Re\{s\} > \sigma_0$.

The importance of the two conditions stated by this Theorem is highlighted by the following two examples.

---

5A function $F(s)$ of a complex variable $s$ is analytic in a region $\Omega$ if it has no poles nor other singularities (e.g. branch points) in $\Omega$. 

24
2.3 – Laplace domain

Figure 2.6. When a causal transfer function has all singularities confined in the left hand plane, the ROC surely includes the imaginary axis.

**Example 2.7.** The Laplace transforms \( F_1(s) \) and \( F_2(s) \) of Example 2.5 clearly show that a ROC open on the right is necessary for causality. The first function is defined for \( \Re\{s\} > p \) and is associated to a time domain signal \( f_1(t) = e^{pt}u(t) \) that vanishes for \( t < 0 \). Conversely \( F_2(s) \), in spite of sharing the same mathematical expression of \( F_1(s) \), is defined in a completely different ROC and does not satisfy the conditions stated by Theorem 2.4. Its inverse Laplace transform \( f_2(t) = -e^{pt}u(-t) \) is thus a non-causal signal.

**Example 2.8.** The scattering matrix of the circuit in Figure 2.4 is

\[
S(s) = \Gamma_R e^{-2\zeta_0}, \quad \Gamma_R = \frac{R - R_0}{R + R_0}
\]

and is defined and analytic over the entire complex \( s \) plane. Consequently, it satisfies the first condition stated in Theorem 2.4 whatever \( t_0 \) is; however, \( S(s) \) is causal only if \( t_0 \geq 0 \), requiring that the exponential factor represents a true delay and not a non-causal anticipation. In the latter case with \( t_0 < 0 \), the second condition of Theorem 2.4 is obviously violated since \( S(s) \) grows exponentially for \( \Re\{s\} > 0 \).

2.3.3 Stability

The ROC associated to a system transfer function is important to ascertain stability. For lumped systems we have the following theorem [81]

**Theorem 2.5.** A system is stable according to Definition 2.2 if and only if (i) the ROC associated to its transfer matrix \( H(s) \) includes the imaginary axis, and (ii) \( H(\infty) \) is bounded.

This condition is quite different from the more practical rule normally employed by engineers, who usually test stability by checking that all the system poles have negative real part. For causal systems, both criteria are equivalent. In fact, since the ROC for any causal system is open on the right and is bounded on the left by the system singularities, when these singularities are confined to the left half plane, the ROC will necessarily include the imaginary axis, as shown in Figure 2.6. For non-causal systems, however, only the analysis of the region of convergence allows to prove stability, as shown in next example.
Example 2.9. Consider the impulse responses
\[ h_1(t) = e^{pt} u(t) \]
\[ H_1(s) = \frac{1}{s - p}, \quad \text{ROC: } \Re\{s\} > p \]
\[ h_2(t) = -e^{pt} u(-t) \]
\[ H_2(s) = \frac{1}{s - p}, \quad \text{ROC: } \Re\{s\} < p \]
As already discussed in Example 2.1, \( h_1(t) \) is stable only if \( p < 0 \), in which case the ROC includes the imaginary axis. Since \( h_1(t) \) is causal, the stability condition is thus equivalent to requiring that the system pole lies in left hand plane. When \( p > 0 \), the ROC does not include the imaginary axis and \( h_1(t) \) is unstable. We have a different situation for the non-causal impulse response \( h_2(t) \), which is stable when \( p > 0 \), i.e., when its ROC includes the imaginary axis.

Example 2.10. The admittance of the LC resonator depicted in Figure 2.3b is
\[ Y(s) = \frac{1}{L} \frac{s}{s^2 + \frac{1}{LC}} \]
where the ROC has been chosen open on the right in order to insure causality. In this case the ROC does not include the imaginary axis, where the two system poles \( s = \pm \frac{1}{\sqrt{LC}} \) are located. Therefore, according to Theorem 2.5, the system is not BIBO stable, as reported also in Example 2.2. This example confirms that systems with purely imaginary poles are a boundary case for stability, and the adopted definition of BIBO stability rules out these systems. Under a practical standpoint, we believe that this definition satisfies the theoretical need of a design engineer, since lossless resonant structures never occur in practice due to the unavoidable presence of losses. In addition, any model which has poles on the imaginary axis may become critical under certain excitations and should be carefully avoided.

2.3.4 Passivity

The passivity conditions in Laplace domain depend on the adopted representation. In the impedance or admittance cases we have \[67\]

Theorem 2.6. An impedance matrix \( Z(s) \) represents a passive linear system if and only if:

1. each element of \( Z(s) \) is defined and analytic in \( \Re\{s\} > 0 \);
2. \( Z^H(s) + Z(s) \) is a nonnegative-definite matrix⁶ for all \( s \) such that \( \Re\{s\} > 0 \);
3. \( Z(s^*) = Z^*(s) \).

⁶A complex Hermitian matrix \( A = A^H \) is nonnegative-definite if \( x^H A x \geq 0 \) for all complex vectors \( x \neq 0 \).
The superscripts $^*$ and $^H$ denote the complex conjugate and transpose conjugate, respectively. Note that the second condition generalizes for the matrix case the requirement that a passive one-port impedance must have positive real part. The third condition insures that the associated impulse response is real. The first condition is related to causality and stability, since it requires a ROC that is open on the right and touching the imaginary axis. In fact, it is possible to prove that the three above conditions for passivity always imply causality. The first condition also implies BIBO stability, provided that the system has no singularities on the imaginary axis (i.e. purely imaginary poles).

**Remark 2.3.** From the above discussion, it appears evident that passivity is the strongest requirement for the well-posedness and physical consistency of a given model, since passivity implies both causality and stability.

**Example 2.11.** The impedance of the RC circuit shown in Figure 2.3a (with positive $R$ and $C$) is

\[
Z(s) = \frac{1}{C} \left( \frac{1}{s + \frac{1}{RC}} \right) \quad \text{ROC: Re}\{s\} > -\frac{1}{RC} \tag{2.18}
\]

and clearly satisfies the first and third conditions reported in Theorem 2.6. The second condition reads

\[
Z(s) + Z^H(s) = 2 \frac{\sigma + \frac{1}{RC}}{|s + \frac{1}{RC}|^2} \geq 0
\]

where $s = \sigma + j\omega$ and is satisfied because, for $\text{Re}\{s\} = \sigma > 0$, all quantities are positive.

**Example 2.12.** A similar calculation proves that the admittance function (2.17) of the LC resonator in Figure 2.3b is passive. In fact, the second condition of Theorem 2.6 becomes

\[
Y(s) + Y^H(s) = 2 \frac{\sigma(\omega^2 + 1) + \frac{1}{LC}}{|s^2 + \frac{1}{LC}|^2} \geq 0
\]

and is satisfied for $\text{Re}\{s\} = \sigma > 0$.

For the scattering representation we have a similar result [67].

**Theorem 2.7.** A scattering matrix $S(s)$ represents a passive linear system if and only if

1. each element of $S(s)$ is analytic in $\text{Re}\{s\} > 0$;
2. $I - S^H(s)S(s)$ is a nonnegative-definite matrix for all $s$ such that $\text{Re}\{s\} > 0$;
3. $S(s^*) = S^*(s)$.

A matrix fulfilling these three conditions is said to be bounded real. Conditions 1) and 3) have the same meaning as in Theorem 2.6. Condition 2) is basically a bound for $S(s)$, which generalizes the basic condition on passive one-port networks having a reflection coefficient not larger than one. An alternative and equivalent condition requires that $\|S(s)\|_2$, i.e., the largest singular value of $S(s)$, does not exceed one in the right hand plane.
Example 2.13. The scattering coefficient (2.16) of the circuit depicted in Figure 2.4 satisfies the passivity constraints only if \( R > 0 \). In fact, conditions 1) and 3) of Theorem 2.7 hold independently of \( R \), while condition 2) holds only if \( R > 0 \), since it requires that

\[
|\Gamma_R|^2 e^{-4t_0 \sigma} \leq 1
\]

for all \( \sigma > 0 \). If \( R \) is negative, then \( |\Gamma_R| > 1 \) and the above inequality does not hold for small values of \( \sigma \).

2.4 Frequency domain

The Laplace-domain conditions reviewed in Section 2.3 are exhaustive but may be difficult to check, since they require testing the entire or at least half of the complex \( s \)-plane. However, these results may be restricted to the imaginary axis \( s = j \omega \) only, by considering the standard Fourier transform

\[
F(j\omega) = \mathcal{F}\{f(t)\} = \int_{-\infty}^{+\infty} f(t) e^{-j\omega t} dt \quad (2.19)
\]

instead of the Laplace transform. Of course, use of Fourier transform makes sense only if the integral (2.19) exists (converges). The resulting frequency-domain equivalent of (2.1) is

\[
W(j\omega) = H(j\omega) X(j\omega), \quad (2.20)
\]

and makes sense only when the Fourier transform exists for both the system impulse response \( h(t) \) and the excitation signal \( x(t) \). It is well known that \( H(j\omega) \) is directly related to the sinusoidal steady-state response and it can be directly measured.

2.4.1 Stability

Fourier analysis is always possible for stable systems, because if (2.3) is satisfied, the integral in (2.19) converges absolutely. Difficulties arise for unstable systems, as shown in the following example.

Example 2.14. The frequency-domain impedance representation of the circuit in Figure 2.3a is

\[
Z(j\omega) = A \frac{1}{j\omega - p}, \quad p = -\frac{1}{RC}, \quad A = \frac{1}{C} \quad (2.21)
\]

and exists only if \( p < 0 \). In fact, the system impulse response is \( Ae^{pt} u(t) \) and cannot be Fourier transformed via (2.19) if \( p > 0 \), consistently with the fact that a sinusoidal steady-state is never established in such an unstable system.

For unstable systems, Laplace transform is a more appropriate tool, since it can be defined regardless of stability, as discussed in Section 2.3. Two-sided Laplace and Fourier transforms can both be defined if the ROC includes the imaginary axis.
2.4 Frequency domain

2.4.2 Causality

Causality imposes strong conditions on the frequency response of a system. Denoting as $h(t)$ a causal impulse response (vanishing for $t < 0$), we have

$$h(t) = \text{sign}(t)h(t)$$

where sign$(t)$ is the sign function that equals 1 for $t > 0$ and $-1$ for $t < 0$. Applying Fourier transform\(^7\) one obtains

$$\mathcal{F}\{h(t)\} = \frac{1}{2\pi} \mathcal{F}\{\text{sign}(t)\} \ast \mathcal{F}\{h(t)\}$$

since the transform of the product $\text{sign}(t)h(t)$ leads to a convolution. A direct calculation shows that

$$H(j\omega) = \frac{1}{j\pi} \int \frac{H(j\omega')}{\omega - \omega'} d\omega'$$

(2.22)

where the integral converges despite of the integrand singularity for $\omega' = \omega$, because the principal value

$$f = \lim_{\epsilon \to 0^+} \left[ \int_{-\infty}^{\omega-\epsilon} + \int_{\omega+\epsilon}^{+\infty} \right]$$

(2.23)

is taken. In order to appreciate the strong implications of (2.22), it is useful to divide it into real and imaginary parts

$$U(\omega) = \frac{1}{\pi} \int \frac{V(\omega')}{\omega - \omega'} d\omega'$$

(2.24a)

$$V(\omega) = -\frac{1}{\pi} \int \frac{U(\omega')}{\omega - \omega'} d\omega'$$

(2.24b)

where $H(j\omega) = U(\omega) + jV(\omega)$. These equations, known as Kramers-Krönig dispersion relations [68, 82–85] or Hilbert transform [86, 87], are valid for every causal system and state that the frequency response real and imaginary parts are not independent. Kramers-Kröning relations are necessary and also sufficient for causality, as stated by the next theorem.

**Theorem 2.8.** If $h(t)$ admits a Fourier transform, the following facts are equivalent:

(i) $h(t) = 0$ for $t < 0$;

(ii) $H(j\omega)$ is the limit, as $\sigma \to 0$, of a function $H(s)$ defined in $\Re\{s\} > 0$ and here analytic and of polynomial growth;

(iii) $H(j\omega) = \mathcal{F}\{h(t)\}$ satisfies Kramers-Kröning relations.

This interesting result, due to Titchmarsh [85] and generalized by Beltrami [88], summarizes and relates the conditions for causality in time, Laplace and frequency domain.

\(^7\)Mathematically, this calculation has to be done with distributions.
Example 2.15. We consider the frequency response

\[ H(j\omega) = \frac{1}{j\omega - p} = \frac{-p}{\omega^2 + p^2} + j \frac{-\omega}{\omega^2 + p^2} \]  

with \( p \in \mathbb{R} \) and test whether its real part \( U(\omega) \) and imaginary part \( V(\omega) \) satisfy dispersion relations. The integral in equation (2.24a) can be computed by using the following decomposition

\[ \frac{-\omega'}{\omega^2 + p^2} = \frac{-\omega}{\omega^2 + p^2} - \frac{\omega'}{\omega^2 + p^2} + \frac{p^2}{\omega^2 + p^2} \frac{1}{\omega^2 + p^2}. \]

This allows us to write

\[ \int \frac{-\omega'}{\omega^2 + p^2} \frac{d\omega'}{\omega - \omega'} = \frac{p^2}{\omega^2 + p^2} \int \frac{d\omega'}{\omega^2 + p^2} \]

since the first two partial fraction terms above are odd-symmetric, thus leading to a vanishing principal value integral. Finally, we have

\[ \frac{1}{\pi} \int \frac{V(\omega')}{\omega - \omega'} d\omega' = \frac{|p|}{p^2 + \omega^2} \]  

This computed real part matches (2.25), i.e., the system is causal according to (2.24a), only if \( p < 0 \). Of course, this is consistent with the fact that the extension of \( H(j\omega) \) to a Laplace transform in the complex \( s \)-plane reads

\[ H(s) = \frac{1}{s - p} \text{ ROC: } \Re \{ s \} > p, \quad p < 0 \]

because the ROC must include the imaginary axis where the original frequency response is defined. Conversely, if \( p > 0 \) the frequency response is non-causal since

1. dispersion relations are not satisfied;
2. the extension of \( H(j\omega) \) to a Laplace transform in the complex \( s \)-plane reads

\[ H(s) = \frac{1}{s - p} \text{ ROC: } \Re \{ s \} < p, \quad p > 0, \]

and the ROC is not a half plane open on the right;
3. the associated time-domain signal is \( h(t) = -e^{pt}u(-t) \), which is obviously non-causal.
2.4.3 Passivity

The Laplace-domain passivity conditions stated by Theorems 2.6 and 2.7 have to be verified in the entire half plane \( \Re \{s\} > 0 \). We report here some passivity conditions that practically require testing only the imaginary axis \( s = j\omega \).

The following theorem [67] applies only to lumped systems, whose transfer functions are always rational.

**Theorem 2.9.** A rational matrix \( Z(s) \) is the impedance of a passive lumped system if and only if

1. each element of \( Z(s) \) is defined and analytic in \( \Re \{s\} > 0 \);
2. \( Z^H(j\omega) + Z(j\omega) \) is a nonnegative-definite matrix for all \( \omega \in \mathbb{R} \), except for simple poles \( j\omega_0 \) of \( Z(s) \), where the residue matrix must be nonnegative definite;
3. \( Z(-j\omega) = Z^*(j\omega) \);
4. asymptotically, \( Z(s) \to As \) in \( \Re \{s\} > 0 \), where \( A \) is a real, constant, symmetric, nonnegative-definite matrix.

A similar result holds for the admittance matrix \( Y(s) \).

**Example 2.16.** The impedance of the circuit depicted in Figure 2.3b is

\[
Z(s) = sL + \frac{1}{sC} \quad \text{ROC: } \Re \{s\} > 0. \tag{2.27}
\]

This is a rational function, so we can apply Theorem 2.9 to ascertain passivity. The first condition of the Theorem is satisfied because \( Z(s) \) is analytic in the whole complex plane except for \( s = 0 \); condition 2) holds because for \( s = j\omega \) one has \( Z^H(j\omega) + Z(j\omega) = 0 \). Condition 3) is satisfied as well as condition 4), since asymptotically \( Z(s) \to sL \).

The corresponding result of Theorem 2.9 for the scattering representation (valid for both lumped and distributed systems) reads [67]

**Theorem 2.10.** A scattering matrix \( S(j\omega) \) represents a passive linear system if and only if

1. dispersion relations (2.24) hold;
2. \( I - S^H(j\omega)S(j\omega) \) is a nonnegative-definite matrix for all \( \omega \);
3. \( S(-j\omega) = S^*(j\omega) \).

Note that this theorem involves only conditions restricted to the imaginary axis \( s = j\omega \).

It is remarkable that the combined conditions 1) (dispersion relations) and 2) (unitary boundedness) are sufficient to control the function behavior in \( \Re \{s\} > 0 \), i.e., passivity. Therefore, it appears that the scattering representation is more appealing for practical use in data and model verification. Of course, if \( S(s) \) is rational and analytic in \( \Re \{s\} > 0 \), only conditions 2) and 3) are necessary, since dispersion relations are automatically satisfied.
Example 2.17. The second condition of Theorem 2.10, written for the scattering parameter of the circuit in Figure 2.4

\[ S(j\omega) = \Gamma Re^{-2j\omega t_0} \quad \Gamma_R = \frac{R - R_0}{R + R_0} \]  \hfill (2.28)

reads

\[ I - S^H(j\omega)S(j\omega) = 1 - \Gamma_R^2 e^{2j\omega t_0} e^{-2j\omega t_0} = 1 - \Gamma_R^2 \geq 0. \]

When \( |\Gamma_R| \leq 1 \) this inequality is always satisfied, independently of \( t_0 \). However, if \( t_0 < 0 \) the system is not causal (as discusses in Example 2.4) and hence non-passive. This example points out that condition 2) (unitary boundedness) is not sufficient to ascertain the passivity of distributed systems. It is necessary, as stated by Theorem 2.10, to also ensure that dispersion relations are satisfied, i.e., that the system is causal.

2.5 Examples

2.5.1 An analytic example

We start with a simple analytic example. Despite its apparent triviality, this example is quite significant since it allows to pinpoint via analytic derivations a typical source of problems in real-life modeling and simulation tasks. We consider the scattering frequency response

\[ S(j\omega) = \frac{1}{1 + \omega^2} \]  \hfill (2.29)

and we want to derive an equivalent model that can be used in a time-domain simulation tool.

Causal and unstable model

The typical approach for macromodel derivation is to fit the data with a rational expression in the Laplace domain. In this case, an error-free fit is possible and leads to the expression

\[ S(s) = \frac{1}{1 - s^2} \]  \hfill (2.30)

In fact, Eq. (2.30) reduces exactly to Eq. (2.29) when evaluated for \( s = j\omega \). The model poles are \( p = \pm 1 \), with one unstable pole with positive real part. The corresponding impulse response is exponentially unstable. Figure 2.7 provides a further illustration of this instability by depicting (continuous line) the results of a computer simulation of the model (2.30) when a square pulse is applied as input. It is clear that any practical use of this model is impossible, even if the match to the data is perfect.
2.5 – Examples

Figure 2.7. Response of two different time-domain models of (2.29) to a square pulse. The continuous line refers to (2.30), the dashed line is obtained via inverse Fourier transform.

Stable and non-causal model

An alternative approach to compute the time-domain response of (2.29) under a given excitation \(x(t)\) is via inverse Fourier transform

\[
 w(t) = \mathcal{F}^{-1}\{S(j\omega)X(j\omega)\}, \quad X(j\omega) = \mathcal{F}\{x(t)\}. \tag{2.31}
\]

The result for a square pulse input is depicted with a dashed line in Figure 2.7. It is clear that the model is not causal but stable. Also in this case, if proper care is taken in computing the inverse Fourier transform, the computed response is virtually error-free.

Discussion

It appears that two models with very different behavior and characterized by different fundamental properties are compatible with (2.29). The main reason for this inconsistency is the lack of causality of the original data (2.29). In fact, this frequency response is not causal because it violates Theorem 2.8, whatever condition is considered:

(i) the inverse Fourier transform of (2.29) is

\[
 s(t) = \mathcal{F}^{-1}\{S(j\omega)\} = \frac{1}{2}e^{-|t|}
\]

which does not vanish for \(t < 0\);
(ii) the extension of $S(j\omega)$ to the entire $s$-plane,

$$S(s) = \frac{1}{1 - s^2} \quad \text{ROC: } -1 < \text{Re}\{s\} < 1$$

is not defined in the left half plane $\text{Re}\{s\} > 0$ (note that the ROC has to be defined so that it includes the imaginary axis);

(iii) the frequency response itself does not satisfy the dispersion relations (2.24). In fact, a straightforward calculation (as in Example 2.15) reveals that $S(j\omega)$ requires an associated imaginary part $-\frac{\omega}{1+\omega^2}$ in order to be causal.

A clear picture of the situation is provided by Figure 2.8. Each of the two models is characterized by a different ROC, and we know from section 2.3 that the ROC determines directly the stability and the causality properties of the model. Due to the singularity at $s = 1$, a simultaneous enforcement of stability and causality is not possible.

In summary, given a non-causal dataset as in (2.29), the objective of computing a causal and stable model becomes an ill-posed problem. It is of course possible to compute a rational approximation by constraining all poles to be stable, so that the ROC will be open on the right and include the imaginary axis. Unfortunately, such a fit will be very poor and the accuracy will not be under control. Table 2.3 reports the results of VF applied to this example, with a varying number of (strictly stable) poles. It is clear that the approximation does not converge to the data, as expected.

We conclude this example with a remark. Most time-domain circuit solvers implicitly assume causality in computing the transient response of a given network, especially to set up initial conditions. In fact, initial conditions computation is based on the value of all inputs and sources for $t < 0$, which is correct for causal circuits, but is clearly wrong when non-causal elements are present, and input variations for $t >= 0$ may have effects also for $t < 0$. 

Figure 2.8. Graphical illustration of the ROC for the two models of section 2.5.1.
2.5 – Examples

<table>
<thead>
<tr>
<th>Model order</th>
<th>Maximum error</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>0.523</td>
</tr>
<tr>
<td>8</td>
<td>0.416</td>
</tr>
<tr>
<td>12</td>
<td>0.417</td>
</tr>
<tr>
<td>16</td>
<td>0.422</td>
</tr>
<tr>
<td>20</td>
<td>0.422</td>
</tr>
</tbody>
</table>

Table 2.3. Failure of a causal and stable rational model fit to the non-causal dataset of section 2.5.1.

2.5.2 Revisiting the test case of Section 2.1

We reconsider now the example reported in Section 2.1, for which an accurate rational fit with stable poles only was not possible. This is actually the same scenario that was encountered in the simple example of Sec. 2.5.1. We may argue that the main reason is hidden in some causality violations of the raw frequency response. However, in this case only a set of tabulated frequency points are available, and no analytic expressions will help us in the verification of this hypothesis. This is the practical situation that a design engineer would actually face, with no source of information available other than the results of his/her measurement or field simulation.

Among the various conditions for testing causality that were reviewed in this Chapter, it appears that the direct application of the dispersion relations (2.24) is the only feasible in this case. Due to the singular nature of the integrals and to the availability of samples over a limited frequency band, the direct computation of (2.24) may be very inaccurate. This problem will be addressed in Section 3.2, where an accurate and robust methodology to ascertain the causality of tabulated frequency responses will be developed, based on a generalized form of the Hilbert transform. We applied this method to the available frequency data, and the results are shown in Figures 2.9 and 2.10. The solid lines represent the original data, while the gray shaded areas are frequency-dependent regions where the data should lay in order to satisfy causality. When a data point is outside these areas we are sure that a causality violation is present.

These results confirm that the failure reported in Section 2.1 is really due to causality violations in the data. However, the detection of such violations requires a sophisticated numerical tool. This example should illustrate how critical may be to handle a flawed dataset. The symptoms of data inconsistencies only appear when trying to build a macro-model, and it may not be clear to the engineer what is the real cause of the problems. It is also evident that any dataset should be certified for causality before attempting any macromodeling and simulation step.

2.5.3 A PCB interconnect

We consider in this section a coupled interconnect structure on a Printed Circuit Board (PCB), whose geometry is depicted in Figure 2.11 (courtesy of I. Stievano, C. Siviero, V. Teppati, G. Dassano, Politecnico di Torino). We built a test board and we performed
Figure 2.9. Check for causality of $S_{11}$, based on the computation of the Generalized Hilbert transform. Since the continuous line (representing raw data) is outside the gray area (computed Hilbert transform inclusive of a frequency-dependent error bar), the dataset is non-causal.
Figure 2.10. As in Fig. 2.9, but for $S_{24}$. 

---

2.5 – Examples
two sets of measurements of the 4-port scattering matrix of the interconnect. The first measured dataset is valid and accurate. The second dataset is instead flawed by an imperfect calibration. Hence, we denote the former dataset as “good” and the latter as “bad”. Magnitude and phase of two scattering matrix entries for the two measurements are compared in Figure 2.12. Figure 2.13 depicts the frequency-dependent maximum singular value of the two scattering matrices, showing that the “bad” dataset is also violating passivity (the maximum singular value is larger than one), whereas the “good” dataset is passive.

We now attempt the construction of a rational macromodel for these two datasets with a varying number of poles. The results obtained with VF are reported in Table 2.4, where three sets of models are compared. The models for the “good” and passive dataset are also passive and the VF error converges when the number of poles is increased. The

Figure 2.11. Layout of the PCB with the coupled interconnect structure under investigation. Port numbering is also specified.

Figure 2.12. Scattering parameters $S_{12}$ (Near-End Crosstalk) and $S_{13}$ (Transmission) for the two datasets.
models for the “bad” dataset are also convergent, but a passivity check, here performed using [46], shows that also the models are non-passive. This is expected, since a model that closely matches a non-passive dataset will almost surely be non-passive. Finally, the set of models in the last column is obtained from the “bad” dataset by enforcing passivity, as in [46]. All these passive models result inaccurate with respect to the original data. This is also expected, since a passive model cannot match a non-passive dataset better than a given threshold accuracy, which is of course related to the amount of passivity violation in the data. As a confirmation, the model accuracy seems to be limited to a value which is nearly the same as \( \max_\omega \{ ||S(j\omega)|| - 1 \} \) (see Figure 2.13). Note also that any other passivity enforcement algorithm, such as [43–45, 48] will produce similar results.

The measured data for this example are only available from a minimum frequency \( f_{\text{min}} = 130 \text{ MHz} \) and not from DC. This fact has two important implications, listed below.

- If we test data causality using the Kramers-Krönig dispersion relations (2.24), we need to take into account the unavoidable bias due to the missing samples. This is indeed the main motivation that led to the advanced algorithms developed in Section 3.2, which explicitly deal with missing samples by computing an equivalent “numeric resolution” of the causality check.

- Any model derived from this data can only be accurate where frequency samples are available. Therefore, the model behavior from DC up to \( f_{\text{min}} \) is not under control, unless some additional assumptions on the missing data are made. This may lead to spurious passivity violations in the model, which are located at frequencies where raw data samples are missing. An interpolation routine as the one to be presented in Section 3.4 can be used to reconstruct the missing bandwidth data.

In case only a “bad” non-passive dataset is available, we are faced with three possibilities:

\[ \text{Figure 2.13. Norm of the } S \text{ matrix: values greater than one denote passivity violations.} \]
• throw away the flawed dataset and repeat the measurement,
• accept the accurate but non-passive macromodel,
• accept a passive macromodel at the cost of a reduced accuracy with respect to the available data.

It turns out that the first choice is the right one. In fact, an accurate but non-passive model may lead to an unstable network even when suitable passive terminations are considered. This is actually not far from real life experience, and we show this for the non-passive model with 18 poles. When we terminate the interconnect model with identical loads at its four ports [89] with structure \( R_1 \| (R_2 + j\omega L) \) (component values: \( R_1 = 1.93 \, \text{k}\Omega \), \( R_2 = 1.29 \, \text{\Omega} \), \( L = 9.18 \, \text{nH} \)), we get the voltage waveform reported in Figure 2.14, which confirms the loss of the network stability, caused by a pair of complex-conjugate poles with positive real part, \( p_{1,2} = 0.037 \pm 5.40 \, \text{Grad/s} \). Thus, accepting non-passive models involves a necessary risk that the system-level simulations employing the model may fail. The third choice has also some drawbacks. In fact, even if the model is passive, we will never know how closely it represents the real interconnect (which is certainly a passive structure), since it is derived from a dataset that does not represents correctly the fundamental physics of the system.

<table>
<thead>
<tr>
<th>Model order</th>
<th>Passive data &amp; models</th>
<th>Non-passive data &amp; models</th>
<th>Non-passive data, passive models</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>0.240</td>
<td>0.280</td>
<td>0.23</td>
</tr>
<tr>
<td>12</td>
<td>0.020</td>
<td>0.020</td>
<td>0.11</td>
</tr>
<tr>
<td>18</td>
<td>0.012</td>
<td>0.017</td>
<td>0.13</td>
</tr>
</tbody>
</table>

Table 2.4. Model errors as a function of the number of poles.

### 2.6 Conclusions

We have reviewed in this Chapter the fundamental properties of stability, causality, and passivity, and we have shown all important interrelations between them in the time, Laplace, and Fourier domains. These concepts have been used to interpret and justify common difficulties encountered when trying to derive macromodels from tabulated frequency data. We have shown with several test cases that whenever raw frequency data do not fulfill these fundamental properties, a failure in the macromodeling process must be expected.

Two main conclusions can be drawn. First, any macromodeling algorithm must preserve such properties in order to avoid flawed simulation results. Second, it is mandatory that any dataset be certified to be self-consistent, causal and passive before proceeding to further modeling steps. This need strongly motivates the development of accurate physical consistency verification algorithms for tabulated data, to be pursued in the next Chapter.
Figure 2.14. Loss of stability of a non-passive model when connected to simple passive $RL$ loads.
Chapter 3

Physical consistency verification of macromodeling data

The previous Chapter demonstrated the importance of the physical consistency of data used for macromodeling and CAD simulations, identifying in causality, stability and passivity the three fundamental requirements. Since these characteristics can be compromised during measurement-based or simulation-based generation of frequency data, it is particularly important to include in CAD programs suitable tools to check the physical consistency of tabulated data.

Dispersion relations are the key tool to perform these checks. In fact, Theorem 2.8 shows that they are fully equivalent to causality. Moreover, they are also necessary for passivity verifications of frequency responses, as shown by Theorem 2.10. Dispersion relations have been discovered by Kramers [82] and Kröning [83] in the early 1900s, and extensively used in many fields of engineering and science because of their global validity. In electronics, for example, they have been exploited for measured data reconstruction [90], correction [91], extrapolation [92], time-domain inversion [93], and delay extraction [72], to cite just a few applications. A comprehensive set of bibliographic references on the subject can be found in [94], together with a brief history on the scientific developments on the subject.

Unfortunately, in spite of their widespread use, their accurate evaluation for practical data, that are tabulated and known over a limited bandwidth, is still an open problem considered so far only by few authors [94–98], and a satisfactory algorithm with general applicability is still missing. Moreover, a causality check based on dispersion relations also requires a rigorous bound for the necessary errors intrinsic in their evaluation for tabulated data. Only with this bound a causality check can be performed in an automatic and reliable way. In fact, one must be able to distinguish between those dispersion relations violations that exceed the worst case bound for errors, thus being symptoms of a real causality inconsistency, and other violations that could be due just to the unavoidable numerical errors.

In this Chapter we solve both problems by introducing a robust and accurate method
to evaluate the so-called generalized dispersion relations (also known as dispersion relations with subtractions or generalized Hilbert transform). The accuracy of this method is excellent even in presence of finite bandwidth data, as demonstrated by the derived closed-form estimates. Moreover these estimates allow to unbias causality and passivity checks from numerical errors, leading to highly robust and trustworthy algorithms.

The Chapter is taken from [2,5–7] and is organized as follows. In Section 3.1, the generalized dispersion relations are presented, together with a detailed analysis of all sources of errors involved in their numerical evaluation. This section includes a detailed comparison of proposed approach with existing techniques, showing how the state of the art is improved. In Section 3.2, a robust and accurate causality check scheme based on generalized dispersion relations is presented. In Section 3.3, the technique is applied to verify the passivity of tabulated scattering responses. Finally, in Section 3.4, a causality-constrained interpolation scheme is derived.

3.1 Generalized dispersion relations

3.1.1 Practical difficulties with standard dispersion relations

In Chapter 2, dispersion relations (2.22) and (2.24) have been introduced as necessary and sufficient conditions for causality in the frequency domain, as stated by Theorem 2.8. Before discussing the issues related to their numerical computation, we reinterpret them under a slightly different standpoint. Eq. (2.22) can be seen as the application of a (Hilbert-transform) reconstruction operator $R_0$

$$H_0(j\omega) = R_0H(j\omega) = \frac{1}{j\pi} \int \frac{H(j\omega')}{\omega - \omega'} d\omega'.$$

(3.1)

This operator maps any causal frequency response onto itself,

$$H(j\omega) \text{ causal} \iff H_0(j\omega) = R_0H(j\omega) = H(j\omega),$$

(3.2)

equivalently, $R_0$ becomes the identity operator when applied to causal responses. If instead we release the causality assumption on $H(j\omega)$, we obtain $H_0(j\omega) \neq H(j\omega)$, with a corresponding reconstruction error

$$\Delta_0(j\omega) = H_0(j\omega) - H(j\omega).$$

(3.3)

A non-vanishing reconstruction error indicates the presence of causality violations in the original frequency response. This will be the main numerical tool that we will further develop for causality detection and characterization.

Two main difficulties arise when trying to apply (3.1). First, the high-frequency behavior of $H(j\omega)$ for lumped or distributed networks may not be decreasing to zero. It may not even be bounded, as in the case of impedance/admittance representations. This implies difficulties in the definition of the Hilbert transform integral, which holds only when $H(j\omega)$ is square integrable [85].
3.1 – Generalized dispersion relations

Even if the Hilbert transform integral is well defined, a second practical problem must be faced in its numerical evaluation. In fact, in practical applications the frequency response is obtained either via numerical simulation or direct measurement and is available only over a set of discrete frequency samples up to a maximum frequency \( \omega_{\text{max}} \),

\[
H(j\omega_k), \quad k = 1, \ldots, \bar{k}, \quad \omega_k = \omega_{\text{max}}. \tag{3.4}
\]

We can further distinguish between the baseband case

\[
\omega_1 = 0 \tag{3.5}
\]

and the bandpass case

\[
\omega_1 = \omega_{\text{min}} > 0 \tag{3.6}
\]

with missing samples at low frequencies. In the following, we will focus on the baseband case (3.5), and we will denote the frequency range where data points are available as

\[
\Omega = [-\omega_{\text{max}}, \omega_{\text{max}}], \tag{3.7}
\]

with data for negative frequencies being recovered from basic spectrum symmetries. Full details on the bandpass case will be presented in Appendix B. Under these conditions, the usefulness of dispersion relations is subject to the availability of a robust and efficient algorithm for their numerical computation. This is a challenging task because of two reasons:

- Since the available data span a limited frequency range, the integrals in (2.24) have to be restricted to \( \Omega \), introducing a truncation error. Unfortunately, this error may be very large. In order to overcome this issue and achieve high accuracy, a more advanced form of dispersion relations will be introduced in Section 3.1.2.

- The discrete nature of the available data introduces a discretization error in the numerical evaluation of (2.24). Also, a dedicated quadrature algorithm must be devised due to the presence of the singular kernel \((\omega - \omega')^{-1}\), since standard techniques may lead to very poor accuracies.

To show the significance of these two errors, we numerically computed the dispersion relations (3.1) for the \( S_{11} \) parameter of a simple transmission line, tabulated from 0 up to 5 GHz at 40 frequency points. The numerical computation of the integral in (3.1) was performed with a conventional quadrature algorithm (trapezoidal rule), regardless of the singular nature of the integral. The integration interval was restricted to the available bandwidth \([-5.5]\) GHz. Since the \( S_{11} \) parameter is certainly causal, we would expect (3.2) to be satisfied, or equivalently, the reconstruction error \( \Delta_0(j\omega) \) to be vanishing. Numerical results are very different. As depicted in Fig. 3.1, the numerically computed reconstruction error is very large, because of both truncation and discretization errors. Although the discretization error can be somewhat controlled by a sufficiently fine frequency sampling, the truncation error can be very large. This strongly limits the usefulness of dispersion relations, unless a more careful formulation and implementation is devised. This is the subject of the next Section.
Figure 3.1. Numerically computed reconstruction error for the $S_{11}$ coefficient of a simple transmission line (per-unit-length parameters $L = 4.73 \, \text{nH/cm}, C = 3.8 \, \text{pF/cm}, R = 0.8 \, \Omega/cm, \rho = 0$, length $l = 10 \, \text{cm}$), tabulated up to 5 GHz (40 points). The contributions of truncation and discretization errors are shown separately.
3.1.2 Dispersion relations with subtractions

The main limitation of standard Kramers-Kröning relations (2.24) is their sensitivity to the high frequency data, which are not available in practice. To overcome this serious issue, a generalized formulation of dispersion relations named dispersion relations with subtractions can be used.

\[
H_n(j\omega) = R_nH(j\omega) = \mathcal{L}_H(j\omega) + \sum_{q=1}^{n} \frac{H(j\omega_q) - \mathcal{L}_H(j\omega_q)}{\prod_{p=1}^{n} \frac{\omega - \omega_p}{\omega_q - \omega_p}} \frac{d\omega'}{\omega - \omega'}, \quad (3.8)
\]

where the so-called subtraction points \(\{\omega_q\}_{q=1}^n\) are spread over the available bandwidth \(\Omega\). In (3.8), the term \(\mathcal{L}_H(j\omega)\) denotes the Lagrange interpolation polynomial [99] for \(H(j\omega)\)

\[
\mathcal{L}_H(j\omega) = \sum_{q=1}^{n} H(j\omega_q) \prod_{p=1}^{n} \frac{\omega - \omega_p}{\omega_q - \omega_p} \quad (3.9)
\]

with the subtraction points \(\{\omega_q\}_{q=1}^n\) used as interpolation knots. A complete derivation of these formulas can be found in [68,100]. Here, we just observe that (3.8) can be interpreted as the application of (2.22) to the auxiliary frequency response

\[
G(j\omega) = \frac{H(j\omega) - \mathcal{L}_H(j\omega)}{\prod_{q=1}^{n} \frac{\omega - \omega_q}{\omega_q - \omega}} \quad (3.10)
\]

which is constructed by subtracting the polynomial trend \(\mathcal{L}_H(j\omega)\) from \(H(j\omega)\) and dividing by the polynomial normalization factor at the denominator. Note that the singularities of \(G(j\omega)\) at the subtraction points are only apparent, due to the presence of the Lagrange polynomial \(\mathcal{L}_H(j\omega)\), which equals \(H(j\omega)\) for \(\omega = \omega_q\).

Equation (3.8), also known as generalized Hilbert transform [101], defines a generalized reconstruction operator \(R_n\). Clearly, Eq. (3.1) can be obtained as a particular case of (3.8) by setting \(n = 0\). Similarly, we can define the real operators \(\mathcal{R}_n, \mathcal{R}_n'\) that generalize (2.24) by extracting the real and the imaginary parts of (3.8)

\[
V_n(\omega) = \mathcal{R}_n'U(\omega), \quad U_n(\omega) = \mathcal{R}_n''V(\omega), \quad (3.11)
\]

where \(H_n(j\omega) = U_n(\omega) + jV_n(\omega)\). As for (3.1), we have that only causal frequency responses are mapped onto themselves by the reconstruction operator \(\mathcal{R}_n\), i.e.,

\[
H(j\omega) \text{ causal } \iff H_n(j\omega) = \mathcal{R}_nH(j\omega) = H(j\omega). \quad (3.12)
\]

The generalized reconstruction operator \(\mathcal{R}_n\) has two important advantages with respect to \(\mathcal{R}_0\). First advantage is generality, since \(\mathcal{R}_n\) results well-defined for any frequency response having a polynomial growth up to \(\omega^{n-1}\). Second, its sensitivity to the high-frequency behavior of \(H(j\omega)\) results drastically reduced. This is essentially due to the presence of the polynomial at the denominator in (3.8), which acts as a sort of “low-pass” filter. These considerations are made more precise in the following.
3.1.3 Truncation error

Integration in (3.8) is performed over the whole real line. However, application to bandlimited responses imposes a restriction of the integration interval to \( \Omega \) defined in (3.7). Therefore, only an approximation \( \hat{H}_n(j\omega) \) of the reconstructed frequency response can be evaluated, for \( \omega \in \Omega \), as

\[
\hat{H}_n(j\omega) = L_H(j\omega) + \frac{\prod_{q=1}^{n}(\omega - \bar{\omega}_q)}{\pi} \int_{\Omega} \frac{H(j\omega') - L_H(j\omega')}{\prod_{q=1}^{n}(\omega' - \bar{\omega}_q)} \frac{d\omega'}{\omega - \omega'}
\]

\[
+ \frac{\prod_{q=1}^{n}(\omega - \bar{\omega}_q)}{\pi} \int_{\Omega^C} \frac{-L_H(j\omega')}{\prod_{q=1}^{n}(\omega' - \bar{\omega}_q)} \frac{d\omega'}{\omega - \omega'}.
\]

(3.13)

The last term includes the contribution of the Lagrange interpolation polynomial \( L_H(j\omega) \) over the complement set \( \Omega^C = \mathbb{R} \setminus \Omega \) that identifies the band which is not spanned by the data. If this contribution is not included in the computation, the result turns out to be very inaccurate, thus wasting the effort in using the more sophisticated generalized dispersion relations, as will be shown in Sec. 3.1.7. Since the Lagrange polynomial is analytically known, the quantity \( C_n(j\omega) \) can be evaluated in closed form and reads

\[
C_n(j\omega) = \sum_{q=1}^{n} \frac{H(j\bar{\omega}_q)}{\omega - \bar{\omega}_q} \prod_{p=1, p \neq q}^{n} \frac{\omega - \bar{\omega}_p}{\bar{\omega}_p - \bar{\omega}_q}
\]

(3.14)

where

\[
\mathbb{L}(x) = \ln \left| \frac{\omega_{\text{max}} + x}{\omega_{\text{max}} - x} \right|.
\]

(3.15)

The expression (3.14) will be used for the numerical evaluation of (3.13) in Section 3.1.6. We now define the truncation error \( E_n(j\omega) \) by taking the difference between the bandlimited approximation (3.13) and (3.8),

\[
E_n(j\omega) = \hat{H}_n(j\omega) - H_n(j\omega) = \frac{\prod_{q=1}^{n}(\omega - \bar{\omega}_q)}{\pi} \int_{\Omega^C} \frac{-H(j\omega')}{\prod_{q=1}^{n}(\omega' - \bar{\omega}_q)} \frac{d\omega'}{\omega - \omega'}.
\]

(3.16)

This error is a function of number and position of subtraction points. A careful selection of these parameters allows to control this error almost up to arbitrary precision. In fact, when the number of subtractions \( n \) is increased, a smaller integrand is obtained in (3.16), resulting in a smaller truncation error \( E_n(j\omega) \). It turns out that a rigorous bound for \( E_n(j\omega) \) can be formally derived from (3.16). Under the hypothesis

\[
|H(j\omega)| \leq M |\omega|^\alpha \text{ for } \omega \in \Omega^C
\]

(3.17)

and for \( \alpha = 0,1,2,\ldots, \) it can be proved (see Appendix C) that

\[
|E_n(j\omega)| \leq T_n(\omega),
\]

(3.18)
3.1 – Generalized dispersion relations

where

\[
T_n(\omega) = \frac{M}{\pi} \sum_{q=1}^{n} (\bar{\omega}_q)^\alpha \left[ \ln \frac{\omega_{\text{max}} - \bar{\omega}_q}{\omega_{\text{max}} - \omega} - (-1)^{n+\alpha} \ln \frac{\omega_{\text{max}} + \bar{\omega}_q}{\omega_{\text{max}} + \omega} \right] \times \prod_{p=1 \atop p \neq q}^{n} \frac{\omega - \bar{\omega}_p}{\bar{\omega}_q - \bar{\omega}_p}, \tag{3.19}
\]

and that this bound is tight. The case \( \alpha = 0, M = 1 \) is particularly important since it corresponds to the scattering responses of passive networks, for which we have \(|H(j\omega)| \leq 1\).

From (3.19) we can easily verify that the truncation error is bounded between any pair of subtraction points and decreases when their number \( n \) is increased. Figure 3.2 confirms these statements by depicting the truncation error bound \( T_n(\omega) \) for \( n = 4,8,12 \).

![Figure 3.2. Truncation error bound (for \( \alpha = 0, M = 1 \)) as a function of the number \( n \) of subtraction points.](image)

3.1.4 Optimal displacement of subtraction points

The truncation error is frequency-dependent. It vanishes at the subtraction points, and it reaches a maximum between each pair of subtractions. The exact values of these maxima depend on the actual location of the subtraction points. It is clear that an optimal placement of these points is obtained when all these maxima are equal, so that the truncation error can be uniformly bounded by a constant throughout the bandwidth of interest. Such condition is approximately reached when the subtraction frequencies \( \{\bar{\omega}_q\}_{q=1}^{n} \) are placed according to a Chebyshev distribution [96]

\[
\bar{\omega}_q = -\omega_{\text{max}}(1 - \epsilon) \cos \left( \frac{(q - 1)\pi}{n - 1} \right), \quad q = 1, \ldots, n. \tag{3.20}
\]

This placement allows to minimize the truncation error bound uniformly in the bandwidth

\[
\Omega' = [-\omega_{\text{max}}(1 - \epsilon), \omega_{\text{max}}(1 - \epsilon)], \tag{3.21}
\]
Figure 3.3. Truncation error bound for uniform and Chebyshev distributions of \( n = 8 \) subtraction points \((\alpha = 0, M = 1)\).

i.e., in the full bandwidth \( \Omega \) except two arbitrarily small intervals at the bandwidth edges. Obviously, the truncation error diverges at the bandwidth edges due to the missing out-of-band samples. A comparison between the truncation error for uniform and Chebyshev distributions is depicted in Fig. 3.3. An intuitive justification for this result can be given noting that the truncation error increases when the Hilbert kernel gets close to the edges. Therefore, an increased density of subtractions near the edges guarantees a smaller truncation error. The optimal displacement of subtraction points for the bandpass case (3.6) is discussed in Appendix B.

3.1.5 Discretization error

We focus now on the unavoidable discretization error arising in the numerical evaluation of (3.13) via some quadrature rule. We denote as \( \tilde{H}_n(j\omega) \) the outcome from a given numerical quadrature rule, whereas \( D_n(j\omega) \) denotes the corresponding discretization error

\[
D_n(j\omega) = \tilde{H}_n(j\omega) - \hat{H}_n(j\omega).
\]  

(3.22)

The discretization error may be very large if no special care is taken in handling the singular kernel of the Hilbert transform. To regularize the integral, we adopt a singularity extraction procedure. The singular part of the integrand function is subtracted from the integral and added separately, as shown in the following equation

\[
\int_{\Omega} g(\omega') \frac{d\omega'}{\omega - \omega'} = \int_{\Omega} \left[ g(\omega') - g(\omega) \right] \frac{d\omega'}{\omega - \omega'} + g(\omega) \int_{\Omega} \frac{d\omega'}{\omega - \omega'} = \\
\int_{\Omega} \frac{g(\omega') - g(\omega)}{\omega - \omega'} d\omega' + g(\omega)L(\omega),
\]

(3.23)

The second term in (3.23) represents the contribution of the singularity and is evaluated in closed form using (3.15). The remaining integral is smooth and well behaved, since the
in the integrand function is now regular for $\omega' = \omega$ and can be computed with any quadrature routine.

In order to estimate the discretization error introduced by numerical integration, one can opt for two different strategies, depending on the application. The first strategy performs the computation twice using two different integration methods with different orders $\nu_1 < \nu_2$. An estimate of the discretization error is obtained by taking the difference of the two results,

$$\tilde{D}_n(\omega) = \left| \tilde{H}_n^{\nu_1}(j\omega) - \tilde{H}_n^{\nu_2}(j\omega) \right| \approx |D_n(j\omega)|,$$

(3.24)

under the reasonable assumption that the higher order quadrature rule provides a much better result, which can be used as the reference for the error estimate.

The above technique usually gives reasonable estimates but does not provide an upper bound of the discretization error. A possible alternative is to derive a worst-case bound for the adopted quadrature rule. As an example, conservative bounds for the Simpson’s quadrature rule have been derived in [102]. Using this second strategy will guarantee that the actual discretization error is always smaller than the error bound. The specific choice depends whether one prefers the results with the highest resolution (first strategy) or the worst-case scenario (second strategy). The latter will guarantee that no false positives are obtained in the causality test, to be presented in Section 3.2.

Without any a priori information, a low-order quadrature rule is preferable in order to build a robust numerical tool which is applicable also to noisy data. Throughout this Chapter, we use a combination of Simpson’s and trapezoidal rule, using (3.24) to estimate the discretization error.

### 3.1.6 Error-controlled evaluation of dispersion relations

The numerically reconstructed frequency response $\tilde{H}_n(j\omega)$ defined via the generalized dispersion relations is obtained by applying (3.13) to the set of discrete samples in (3.4). The first term in (3.13) is the Lagrange interpolation polynomial and is analytically known, see (3.9). The integral in the second term is computed using the singularity extraction procedure (3.23) combined with some quadrature rule. Finally, the last term in (3.13) is also known analytically and is given by (3.14).

Thanks to the systematic analysis of truncation and discretization errors, the worst-case error $E_n^{\text{tot}}(\omega)$ affecting the numerical result

$$\left| \tilde{H}_n(j\omega) - H_n(j\omega) \right| \leq E_n^{\text{tot}}(\omega)$$

(3.25)

is known and is given by

$$E_n^{\text{tot}}(\omega) = T_n(\omega) + \tilde{D}_n(\omega)$$

(3.26)

In addition, the accuracy in the reconstruction can be greatly enhanced by increasing the number of subtraction points $n$, thus lowering $E_n^{\text{tot}}(\omega)$ down to the limit represented by discretization error $\tilde{D}_n(\omega)$. The excellent accuracy of the proposed technique is demonstrated in the applications presented in the next Sections.
3.1.7 Comparison with existing techniques

Before proceeding any further, we compare our approach with previous works on dispersion relations, to show how the state of the art is improved.

- In [91, 96] the truncation error is minimized with an extrapolation of the available data beyond the maximum available frequency $\omega_{\text{max}}$. This may somehow improve the accuracy of the result, but does not allow neither a rigorous arbitrary minimization nor an estimation of the truncation error, as guaranteed by the proposed approach.

- Some earlier works on dispersion relations with subtractions [96, 97] neglect the Lagrange polynomial $L_H(j\omega)$ under the integral sign in (3.8). Unfortunately, without that term the integrand function in (3.8) turns out to be singular for $\omega = \bar{\omega}_q$, therefore its numerical evaluation can be inaccurate. This approximation might be acceptable only in very particular cases. For example, in case of highly resonant data as in [96, 97], the subtraction points $\{\bar{\omega}_q\}$ can be placed where $H(j\omega)$ is small, resulting in a small interpolation polynomial $L_H(j\omega)$. It is clear that this solution lacks generality and significantly limits the number and position of subtractions.

- In [94, 103], the integration interval in (3.8) is simply restricted to available bandwidth $\Omega$ without any special care. This approach introduces large error terms, called “artifacts” in [94, 103], which may compromise the benefits of adopting a generalized form of dispersion relations. We introduce instead (3.13), a new bandlimited version of (3.8), that includes the additional term $C_n(\omega)$, related to the out-of-band contribution of the Lagrange polynomial, which is extracted analytically. Only when this term is included one can rigorously prove that the truncation error (3.16) is bounded by (3.19) and can be arbitrarily minimized by increasing $n$. Fig. 3.4 displays how neglecting the term $C_n(\omega)$ in (3.13) increases the reconstruction error beyond the bound (3.19). If $C_n(j\omega)$ is instead considered the reconstruction error always remains below the predicted bound.

- We finally cite the interesting work [95] that derives rigorous bounds for the reconstruction error, with application to the dielectric permittivity and magnetic permeability. The approach is quite different from ours, and based on the properties of Stieltjes functions. The functions considered in [95] have finite limit for $\omega \to \infty$ and not a generic polynomial growth, as considered in this Chapter. An extension of [95] to this kind of functions is an interesting direction for future research on the topic, and will allow a comparison with the proposed approach.

3.2 Robust causality check for tabulated data

Measurement or simulation errors may destroy the causality of tabulated frequency responses, otherwise guaranteed by physical reasons. As discussed in Chapter 2 even small causality violations in the data may seriously compromise modeling and simulation tasks, due to the physical inconsistency of the flawed frequency samples. Therefore, a robust
3.2 – Robust causality check for tabulated data

Figure 3.4. Magnitude of the reconstruction error for the transmission line insertion loss \( S_{11} \) of Fig. 3.1, obtained when the term \( C_{n}(j\omega) \) in (3.13) is either considered (solid line, our approach), or neglected (dashed line, as in [94, 103]). The truncation error bound \( T_{n}(\omega) \) is also depicted (dash dot line).

and accurate procedure for causality verification of tabulated frequency data is highly desirable, in order to certify a given dataset for safe use in a CAD environment.

One possibility to infer causality from frequency-domain responses is to directly check condition (2.2) by computing the inverse Fourier transform of (3.4) via FFT\(^1\). This procedure turns out to be very unreliable. In fact, the bandlimited nature of the data may give rise to the well-known Gibbs phenomenon and to aliasing effects [91], which superimpose to the true impulse response \( h(t) \) a significant error term, thus distorting the causality check. Using the standard dispersion relations (2.22) is also unreliable, due to the possibly large truncation error in the evaluation of the Hilbert transform. In this Section, we present an accurate and reliable method to ascertain the causality of tabulated data, based on the generalized dispersion relations.

### 3.2.1 Theoretical derivation

A given frequency response \( H(j\omega) \) is causal if and only if the ideal reconstruction error

\[
\Delta_{n}(j\omega) = H_{n}(j\omega) - H(j\omega)
\]

(3.27)

is vanishing at all frequencies. However, in practice only the numerical estimate

\[
\tilde{\Delta}_{n}(j\omega) = \tilde{H}_{n}(j\omega) - H(j\omega)
\]

\[
= \Delta_{n}(j\omega) + E_{n}(j\omega) + D_{n}(j\omega)
\]

(3.28)

is available, differing from the ideal case because of truncation and discretization errors. In order to unbias the causality test from these terms and obtain a reliable identification

---

\(^1\)FFT: Fast Fourier Transform [104].
of causality violations, we explicitly take into account the bound in (3.26). Two situations may occur

- when
  \[ \exists k : |\tilde{\Delta}_n(j\omega_k)| > E^\text{tot}_n(\omega_k), \]
  \[ (3.29) \]
  we are confident that \( H(j\omega) \) is not causal, because the computed reconstruction error exceeds the bound that has been derived for all possible sources of numerical errors;
- when
  \[ |\tilde{\Delta}_n(j\omega_k)| \leq E^\text{tot}_n(\omega_k) \quad \forall k, \]
  \[ (3.30) \]
  any causality violation in the data is smaller than the numerical resolution affecting the calculations, hence it cannot be detected. Some control over the resolution is provided by the number of subtraction points \( n \), as discussed below. However, this resolution cannot be made arbitrarily small, being intrinsically limited by the finite number of frequency samples, known over a finite bandwidth.

We now provide some insight on the effectiveness of (3.29) in the detection of causality violations. To this end, we assume that the available data for \( H(j\omega) \) are composed by the true frequency response \( H_c(j\omega) \), which is certainly causal, and a perturbation term \( P(j\omega) \)

\[ H(j\omega) = H_c(j\omega) + P(j\omega). \]

(3.31)

This perturbation may be due, e.g., to measurement or simulation errors during the extraction of the raw frequency responses. Therefore, we define \( P(j\omega) \) as identically vanishing outside the available bandwidth. In general, we can split this perturbation as

\[ P(j\omega) = P_c(j\omega) + P_a(j\omega), \]

(3.32)

where \( P_c(j\omega) \) is causal and \( P_a(j\omega) \) is anti-causal, i.e., having an inverse Fourier transform \( p_a(t) = \mathcal{F}^{-1}\{P_a(j\omega)\} \) which is vanishing for \( t \geq 0 \). Applying now (3.27) and (3.28) to (3.31), and noting that the ideal reconstruction error for both \( H_c(j\omega) \) and \( P_c(j\omega) \) is vanishing, we get the following expression for the numerically computed reconstruction error

\[ \tilde{\Delta}_n(j\omega) = 2[L_{P_n}(j\omega) - P_n(j\omega)] + E_n(j\omega) + D_n(j\omega). \]

(3.33)

The above expression takes into account that any anti-causal function satisfies \( p_a(t) = -\text{sign}(t)p_a(t) \), which leads to anti-causal dispersion relations identical to (3.8), except for a sign change in front of the integral. Since

\[ |\tilde{\Delta}_n(j\omega)| \geq 2|P_a(j\omega) - L_{P_a}(j\omega)| - |E_n(j\omega) + D_n(j\omega)| \geq 2|P_a(j\omega) - L_{P_a}(j\omega)| - E^\text{tot}_n(\omega), \]

our proposed test (3.29) will detect the causality violation when the following condition holds

\[ \exists k : |P_a(j\omega_k) - L_{P_a}(j\omega_k)| > E^\text{tot}_n(\omega_k). \]

(3.34)
This expression involves only the anti-causal perturbation term and its associated Lagrange polynomial. In (3.34), the left hand side can be interpreted as the effective causality violation “seen” by the algorithm, while the right hand side as its “resolution”. Obviously, detection occurs when the violation is greater than the resolution $E_{n}^{\text{tot}}(\omega)$, which is given by (3.26). Increasing the number of subtractions $n$ improves the detection capabilities of the method because the truncation error $T_{n}(\omega)$ is decreased, thus enhancing the resolution up to the limit represented by the discretization error $\tilde{D}_{n}(\omega)$.

There is only one situation when a large number of subtractions does not lead to any advantage. This case occurs when $|P_{n}(j\omega) - \mathcal{L}P_{n}(j\omega)|$ decreases with $n$ faster than the truncation error, i.e., when the causality violation $P_{n}(j\omega)$ is very smooth. Standard Fourier analysis arguments show that the corresponding time-domain representation $p_{n}(t)$ has a narrow support or a fast decay rate away from $t = 0$. Such violation is intrinsically difficult to detect, independently on the adopted algorithm. Finally, we remark that the error bound (3.19) that we derived is the strictest possible (see Appendix C), and it provides the best resolution with a given bandwidth.

### 3.2.2 Example: detection of weak causality violations

We verified the above considerations with the following example, which highlights also the excellent resolution of the proposed method. The S-parameters of the line considered in Figure 3.1 have been perturbed with a Gaussian shaped term

$$P(j\omega) = A \exp\left\{- \left(\frac{\omega - \omega_{0}}{2\pi B}\right)^{2}\right\}$$

(3.35)

centered at $\omega = \omega_{0}$. The above perturbation is obviously non-causal, with $A$ and $B$ controlling respectively the amplitude and bandwidth of the induced causality violation. We applied the proposed causality check in order to detect this violation. In Figure 3.5, the norm\(^2\) of both frequency-dependent threshold $||E_{n}^{\text{tot}}(\omega)||_{\infty}$ and reconstruction error $||\Delta_{n}(\omega)||_{\infty}$ are plotted versus the number of subtractions $n$, for different perturbation amplitudes $A$. Detection occurs when $||\Delta_{n}(\omega)|| > ||E_{n}^{\text{tot}}(\omega)||$, i.e. when the solid curve emerges from the dashed one. As evident from the top panel, even very weak causality violations can be revealed by increasing the number of subtractions, due to the reduction of the detection threshold $E_{n}^{\text{tot}}(\omega)$. In the bottom panel the available frequency points have been reduced from 1000 to 250. The increased discretization error limits the reduction of $||E_{n}^{\text{tot}}(\omega)||$ to about $5 \times 10^{-4}$. Therefore, the detection of causality violations smaller than this baseline is not possible.

We focus now on the effect of the perturbation bandwidth $B$ on the causality check. If $B$ increases, the perturbation becomes wider and smoother in the frequency domain, and narrower in the time domain. As discussed above, such a causality violation is intrinsically harder to be detected, as confirmed by the curves depicted in Figure 3.6.

\(^2\)The adopted $\infty$-norm is defined as $||\cdot||_{\infty} = \max_{\Omega'} |\cdot|$ with $\Omega'$ given by (3.21).
3 – Physical consistency verification of macromodeling data

Figure 3.5. Norm of the frequency-dependent threshold $||E_n^{\text{tot}}(\omega)||_\infty$ and of the reconstruction error $||\Delta_n(\omega)||_\infty$ versus the number of subtractions $n$. The perturbation is centered at $f_0 = 2.5$ GHz, has a semi-bandwidth $B = 0.5$ GHz and different amplitudes $A = 10^{-2}, 10^{-3}, 10^{-4}$. The thickest solid line denotes the unperturbed case ($A = 0$). The S-parameters of the line have been computed at 1000 points (top panel) and 250 points (bottom panel).

3.2.3 Example: qualification of a measured dataset

We consider here a long interconnect link, whose scattering parameters have been measured with a VNA in the frequency range 10 MHz - 10 GHz (courtesy of C. Schuster and E. Klink, formerly IBM, and D. Kaller, IBM). We apply the proposed causality check technique to qualify the measurement results. In fact, we suspect the presence of some inconsistency in this dataset from the failure of a standard macromodeling process. More precisely, we tried to compute with the Vector Fitting algorithm (see Appendix A) a model for the dataset. Figure 3.7 shows the model error for different orders. Clearly, Vector Fitting is unable to improve the model accuracy beyond $10^{-2}$, even if the model order is significantly increased.
3.2 – Robust causality check for tabulated data

Figure 3.6. As in the top panel of Figure 3.5, but with constant perturbation amplitude $A = 10^{-3}$ and variable perturbation bandwidth $B = 0.5, 1.5$ GHz.

Figure 3.7. Model extraction from measured scattering data of a long interconnect. The residual error of the rational model generated by VF is plotted versus model order.

Application of the proposed methodology indicates the presence of causality violations in the data. In the top panel of Figure 3.8, the imaginary part of the $S_{12}$ parameter is depicted (solid line), together with its numerical reconstruction computed from the real part via dispersion relations. The latter is depicted in the plot with a gray shaded area, in order to take into account that the numerical reconstruction is only available with an associated numerical error. The thickness of the shaded area is related to the worst case error (3.26). The points which fall outside of this region are inconsistent because they do not respect (3.30). Small causality violations have been found also in other scattering parameters, as shown in the bottom panel of Figure 3.8 for $S_{22}$. As demonstrated in Chapter 2, causality violations in the raw data may seriously compromise the accuracy and convergence speed of Vector Fitting. This happens because VF, while minimizing
iteratively the error between the estimated macromodel and the given data, constraints the model poles to be in the left half plane, enforcing both model stability and causality. However, when the raw frequency data are not causal, this is an impossible task, since the frequency response of a macromodel that is causal by construction will never match with good accuracy noncausal data.

3.3 Robust passivity check for tabulated data

Passive components such as lumped elements (capacitors, resistors, inductors) and interconnect structures (wires, connectors, board and package lines) are unable to generate energy. However, when the frequency response of a passive component or subsystem is obtained through a measurement or a simulation, errors may compromise the passivity of the data, thus impairing the accuracy and the physical consistency of the characterization. Moreover, as we saw in Chapter 2, non-passive data may lead to non-passive models and consequently divergent simulations A passivity check procedure is therefore important to qualify frequency datasets for safe use in CAD design tools. The main issue in check routines already available in CAD tools is that they are based on a necessary only condition and therefore cannot detect all passivity violations. In this Section we propose a better
passivity check scheme based on necessary and sufficient conditions.

### 3.3.1 Theoretical derivation

The basic passivity conditions for S-parameters are given by Theorem 2.7, where the scattering matrix $S(s)$ is considered in the Laplace $s$-domain. Is it clear that they cannot be employed for practical purposes, since they imply the knowledge of the scattering matrix in the whole $\Re\{s\} > 0$ half plane. Alternative conditions on the $s = j\omega$ imaginary axis are provided by Theorem 2.10, and used here to develop a new passivity verification algorithm. Condition (i) of Theorem 2.10 requires $S(j\omega)$ to be causal, and can be numerically verified with the procedure discussed in Section 3.2. For condition (ii), one simply checks if all the singular values of $S(j\omega)$ are bounded by one $\forall \omega$. Finally, condition (iii) represents the spectrum symmetries valid for the Fourier transform of real valued signals, and is always assumed to be true.

The main advantage of the proposed algorithm is generality: it holds in fact for data coming from any LTI system, being either lumped or distributed. Moreover the conditions are the tightest possible, since they are necessary and sufficient. Therefore this algorithm can identify all possible passivity violations, as opposed to existing algorithms, that check only condition (ii).

### 3.3.2 Example

The proposed passivity check scheme was applied to the three coupled lines of Sections 2.1 and 2.5.2. First, we verified condition (ii) of Theorem 2.10. Since all singular values of the scattering matrix are uniformly bounded by one (see Figure 3.9), this condition is fulfilled. Therefore, the raw data would appear consistent and passive to most standard passivity checking routines. However, the data are not passive because condition (i) is violated, as shown in Figures 2.9 and 2.10, where the given $S_{11}$ and $S_{24}$ parameters are compared with the numerical reconstruction computed via dispersion relations. Inconsistencies are clearly visible at low frequency, denoting obvious causality and passivity violations.

### 3.4 Causality-constrained interpolation

In this Section we develop a causality-controlled interpolation scheme based on the generalized dispersion relations. The main advantage of the proposed algorithm is a superior accuracy with respect to standard interpolation schemes, with the additional guarantee of the causality in the result. As an application example, we show the usefulness of this technique in recovering a sound estimate of the system response at missing low-frequency samples, including the DC (zero-frequency) point. It is well-known that such points are usually not available via standard measurements or simulations.
3.4.1 Theoretical derivation

We consider a tabulated frequency response (3.6) with the aim of reconstructing the missing frequency data for $|\omega| < \omega_{\text{min}}$. This task can be easily accomplished by interpolating the real and imaginary parts of $H(j\omega) = U(\omega) + jV(\omega)$ with, for example, splines. Unfortunately, this simple solution fails to provide a physically consistent result, since the independent reconstruction of the real and imaginary parts does not preserve the relations imposed by causality.

Physical consistency can be achieved by combining interpolation and dispersion relations as follows. First, a reconstructed imaginary part is obtained as

$$\bar{V}_R(\omega) = SV(\omega) \quad |\omega| < \omega_{\text{min}},$$

(3.36)

where $S$ denotes a standard (e.g., linear or spline-based) interpolation scheme using the available samples $V(\omega_k)$. The result differs from the exact but unknown $V(\omega)$ by some interpolation error $\delta V_R(\omega)$

$$\bar{V}_R(\omega) = V(\omega) + \delta V_R(\omega) \quad |\omega| < \omega_{\text{min}}.$$  

(3.37)

This reconstructed dataset is used to fill the data gap as

$$V_R(\omega) = \begin{cases} 
\bar{V}_R(\omega) & |\omega| < \omega_{\text{min}} \\
V(\omega) & \omega_{\text{min}} \leq |\omega| \leq \omega_{\text{max}} 
\end{cases}.$$  

(3.38)

In a second stage, the missing portion of the real part in $[-\omega_{\text{min}}, \omega_{\text{min}}]$ is reconstructed from $V_R(\omega)$ using the generalized dispersion relations. More precisely, if we denote as $\tilde{R}_n''$ the numerical discretization of the real reconstruction operator in (3.11), we define

$$\tilde{U}_R(\omega) = \tilde{R}_n''V_R(\omega) \quad \omega \in [-\omega_{\text{min}}, \omega_{\text{min}}]$$

(3.39)
The complete reconstructed real part is thus obtained as

\[
U_R(\omega) = \begin{cases} 
\tilde{U}_R(\omega) & |\omega| < \omega_{\text{min}} \\
U(\omega) & \omega_{\text{min}} \leq |\omega| \leq \omega_{\text{max}} 
\end{cases} .
\]  
(3.40)

The reconstructed response

\[
H_R(\omega) = U_R(\omega) + jV_R(\omega)
\]
(3.41)

is causal by construction regardless of the interpolation error \(\delta V_R(\omega)\), since its presence is accounted for in the computation of the real part. In fact, \(U_R(\omega)\) includes its generalized Hilbert transform, since

\[
\tilde{U}_R(\omega) = U(\omega) + \tilde{R}_n'' \delta V_R(\omega) \quad |\omega| < \omega_{\text{min}}
\]
(3.42)

It must be noted that, although the interpolation error \(\delta V_R(\omega)\) vanishes outside \([-\omega_{\text{min}}, \omega_{\text{min}}]\), its Hilbert transform may not. However \(\tilde{R}_n'' \delta V_R(\omega)\) is very small outside the missing bandwidth, vanishing at all subtraction points. Therefore, it can be considered to be important only in the reconstructed frequency gap \([-\omega_{\text{min}}, \omega_{\text{min}}]\), without significantly affecting the causality of \(H_R(j\omega)\).

In order to maximize the accuracy in the missing bandwidth \([-\omega_{\text{min}}, \omega_{\text{min}}]\), a careful placement of the subtraction points is in order. Two conflicting constraints must be considered. On one hand, the subtraction points should be placed close to the edges of the missing bandwidth, so that the truncation error is minimized where needed. On the other hand, any pair of subtraction points should not be too close, since an excessive proximity of singularities in (3.13) increases the numerical discretization error. We found that the following rule leads to an appropriate placement of subtractions (\(n\) is supposed to be even)

\[
\{\bar{\omega}_q\}_{q=1}^n = \{\pm \omega_{\text{min}} (1 + \beta)^i, i = 1, \ldots, n/2\}
\]
(3.43)

with \(\beta = 0.2 \div 0.3\). The number of subtractions \(n\) is determined using the closed-form bound (3.19), in order to guarantee a truncation error in the missing bandwidth smaller than any prescribed tolerance.

### 3.4.2 Analytic example

We first demonstrate the performance of the proposed technique with an analytic example. We computed the \(S_{11}\) parameter of a simple transmission line (per unit length parameters: \(L = 5.2\) nH/cm, \(C = 1.1\) pF/cm, \(R = 1.3\) Ω/cm, \(G = 0\), length: \(l = 5\) cm) from 0.4 up to 10 GHz. Then, we reconstructed \(S_{11}\) in the missing bandwidth (from 0 to 0.4 GHz) using a standard spline interpolation for both real and imaginary parts, and the proposed causality-constrained technique. Figure 3.10 reports the results. The two interpolations for the imaginary part are identical, whereas the real part estimates are quite different. The proposed technique guarantees a significantly better accuracy and satisfies the bounds imposed by causality (shaded area). Spline interpolation fails to provide both a good accuracy and a causal result.
Figure 3.10. The exact $S_{11}$ parameter of a transmission line (solid line) is compared with the results of a standard spline-based interpolation (dash-dot line) and our proposed causality-constrained interpolation (dashed line). The bounds imposed by causality on the real part are also shown (shaded area).

The sensitivity of the reconstruction with respect to the extent of the data gap was also tested. We varied the amplitude of the missing bandwidth in the range $\omega_{\text{min}} \in [0, 1]$ GHz, and we computed the corresponding interpolation error. Figure 3.11 depicts this error as a function of $\omega_{\text{min}}$. This test shows that causality-constrained interpolation is 3 to 10 times more accurate than conventional algorithms. In the same plot, we also report the poor accuracy that one achieves if Kramers-Kröning relations (2.24) are blindly used, instead of employing the proposed generalized Hilbert transform operator.

The above results can be interpreted as follows. The causality-constrained interpolation guarantees a better performance since it resorts to interpolation for either the real or the imaginary part only, the other one being accurately computed with dispersion relations. Therefore, a careful choice for the part to be interpolated allows a reduction of the interpolation error. For the case of a missing interval located around the zero frequency (DC), it is always convenient to interpolate the imaginary part, since it vanishes for $f = 0$.
3.4 – Causality-constrained interpolation

Figure 3.11. Maximum error between the reconstructed $S_{11}$ parameter obtained from (3.41) with respect to the exact value, as a function of missing bandwidth. Results from different interpolation techniques are shown: proposed method (dashed line), proposed with standard Kramers-Kronig relations (solid line) and splines (dash-dot line).

because of basic spectrum symmetries. This condition provides an additional interpolation point that increases the accuracy. The proposed technique can be also applied to reconstruct data within any arbitrary bandwidth, not necessarily centered at DC. In this case, since there is no a priori information on which part should be preferred for interpolation, the only advantage of the proposed scheme is the guarantee of the causality of the result.

3.4.3 Application example

We consider here a package to package differential link, routed through the first package, a PCB, a connector, another PCB and then back through to the second package (courtesy of Dr. Karl Bois, HP). The scattering parameters of the interconnect were measured with a 4-port Vector Network Analyzer from 10 MHz up to 20 GHz. Due to the interconnect length, the parameters have very fast phase variations, as depicted in Figure 3.12 for the insertion loss $S_{31}$.

In order to recover the missing DC point, both standard spline interpolation and the proposed scheme were applied. The two results turn out to be quite different, as shown in Fig. 3.13. This difference is due to the interconnect length, that makes the samples spacing (10 MHz) quite coarse. Since a real measurement of the DC point was not available to validate the reconstruction accuracy, we devised the following alternative test. The differential link is connected to 50 Ω resistors on ports 2, 3, 4 and to a voltage source with 50 Ω internal impedance on port 1. The voltage source applies a pulse of unit amplitude, 30 ns wide, and with a 0.15 ns rise time. The voltage at the far end of the line is depicted in Figure 3.14 and shows how the large difference between the two reconstructed DC points affects the accuracy of the simulation result. Since the input pulse has a lower voltage level of 0 V, the output voltage is expected to have a vanishing DC baseline. However,
Figure 3.12. Insertion loss $S_{31}$ (real part) of the differential I/O link. The frequencies from 5 to 20 GHz are not shown for clarity.

Figure 3.13. Insertion loss $S_{31}$ for low frequencies, with the additional DC point estimated by the proposed method (dashed line) and standard spline interpolation (solid line).

The DC point obtained with spline interpolation leads to a transient response which is downshifted by more than 0.2 V. If the DC point is instead recovered with the proposed causality-controlled scheme, a much more realistic result is achieved. This example clearly points out the dramatic impact that simplistic data processing algorithms may have on the reliability of CAD simulations.
Figure 3.14. Far end response of the interconnect link to a periodic digital signal, obtained with inverse FFT. The solid line was obtained from the raw dataset using standard spline interpolation, the dashed line was obtained using the proposed causality-constrained interpolation algorithm.

3.5 Conclusions

We presented a numerical technique based on the generalized Hilbert transform which allows a precise characterization of the causality for tabulated frequency responses. Rigorous estimates for the numerical errors due to both finite sampling frequency and finite bandwidth have been derived and used in order to guarantee accuracy control and numerical robustness. The proposed algorithms allow to verify both causality and passivity of tabulated frequency responses coming from direct measurements or numerical field simulations. Thus, the results of this Chapter enable a data qualification process that can be inserted in the CAD workflow, in order to accept or reject frequency data based on physical consistency criteria. As argued in Chapter 2 this qualification process can prevent many typical modeling and simulation problems.
Part II

New macromodeling algorithms
Chapter 4

Parametric macromodels

In Part I the physical foundations of macromodeling have been deeply investigated, laying down the necessary basis to tackle some open macromodeling problems. The first problem, to be solved in this Chapter, arises in the design of any modern electronic system. Due to the very high complexity modern systems reached, the finalization of the design results from lengthy optimization processes, which select the best candidate among many possible choices. The optimization aims to maximize the system performance while meeting several constraints of reliability, immunity to interferers, electromagnetic compatibility, power consumption. Optimization is usually performed by repeated system-level simulations for various combinations of the design variables.

A fundamental enabling factor for the above process is the availability of models for each structure or component. These models should be available on-demand for any configuration of the design variables, typically geometrical parameters such as interconnect width and spacing or substrate height, or material parameters such as conductivity or permittivity. Unfortunately, this is not possible with standard macromodeling algorithms because they build macromodels valid only for one particular configuration of the design variables. Each time a parameter changes the entire macromodeling flow, including the full-wave characterization step, must thus be repeated, resulting in very large design times. In order to overcome this issue, parametric macromodels that preserve the explicit dependence on some design variables are needed.

The generation of such models from tabulated data is still an open problem, solved in this Chapter with two parametric modeling methodologies. After a general statement of the problem in Section 4.1, the first methodology will be introduced in Section 4.2, based on a polynomial parameterization of suitable model coefficients. The model formulation, identification, and uniform stability verification will be discussed together with some application examples. In Section 4.3 a different approach will be presented, making use of a piecewise linear parameterization. This characteristic allows a significant reduction of the computational cost in the construction of parametric macromodels, and the derivation of purely algebraic conditions for uniform stability. This in turn will enable the development of an efficient algorithm for uniform stability test of parametric macromodels. Some application examples will show the potential of these findings. The material in this Chapter
is taken from [3, 8, 9].

### 4.1 Problem statement

Parametric macromodels must reproduce the external ports behavior of the device under investigation as a function of both frequency and one or more design variables, such as geometry or material parameters. The ports behavior is usually known as a set of tabulated frequency responses for different parameters configurations coming from repeated full-wave simulations. In the following we will focus on this case, although our findings are applicable independently on the source of the raw structure characterization. For the sake of simplicity we will consider two design parameters only, denoted as \( \lambda \) and \( \mu \). All results are however valid in the most general case and for an arbitrary number of parameters, with obvious modifications.

The structure under modeling is assumed to have a linear input-output behavior, with \( N \) accessible electrical ports. Our starting point is the availability of the structure frequency responses, being known at several frequency points

\[
\omega \in \{ \omega_k \} \quad k = 1, \ldots, \bar{k}
\]

and for different parameters values

\[
\lambda \in \{ \lambda_l \} \quad l = 1, \ldots, \bar{l},
\]

\[
\mu \in \{ \mu_m \} \quad m = 1, \ldots, \bar{m}.
\]

Without any loss of generality we assume \( \omega_k, \lambda_l, \) and \( \mu_m \) being sorted in increasing order. We will denote the available data samples as

\[
H_{klm} \in \mathbb{C}^{N \times N},
\]

where the subscripts \( k, l \) and \( m \) refer, respectively, to the frequency sample \( \omega_k \), and to the two parameters values \( \lambda_l \) and \( \mu_m \). The grid of available parameters values \((\lambda_l, \mu_m)\) defines a hypercube in the parameters space

\[
[\bar{\lambda}, \tilde{\lambda}] \times [\bar{\mu}, \tilde{\mu}],
\]

where

\[
\bar{\lambda} = \lambda_1 \quad \tilde{\lambda} = \lambda_{\bar{l}},
\]

\[
\bar{\mu} = \mu_1 \quad \tilde{\mu} = \mu_{\bar{m}},
\]

which provides the range of validity of the parameterized models to be constructed. In summary, we want to derive a macromodel \( H(s; \lambda, \mu) \) which depends continuously on frequency and parameters, and which provides an accurate approximation to the input data \( H_{klm} \).

---

\(^1\)We adopt the following notation for the minimum and maximum values of a given variable \( x \): \( \underline{x} \) for the minimum and \( \bar{x} \) for the maximum.
4.2 Polynomial parameterization

4.2.1 Model formulation

We introduce in this Section the first parametric modeling algorithm, which employs a polynomial parameterization of the model coefficients. Our first step is to identify a good expression for the model, that can accurately follow the parameters variation with a small number of coefficients. Since macromodels are often written in poles-residues form, it would be tempting to parameterize these quantities. However, the variation of the poles induced by the parameters may be very odd and cumbersome to fit. We argue a better approach by recalling a general result on lumped networks [105].

**Theorem 4.1.** Let $\lambda$ and $\mu$ be the values of two components in a lumped network (each one being a resistor, inductor or capacitor). Then, any network transfer function, including $Z$, $Y$, and $S$ representations, can be written as

$$H(s; \lambda, \mu) = \sum_{n=0}^{\bar{n}} R'_n(\lambda, \mu) s^n,$$

where the coefficients $R'_n(\lambda, \mu)$ and $r'_n(\lambda, \mu)$ are multiaffine\(^2\) functions of $\lambda$ and $\mu$.

Therefore, if (4.8) is used instead of the poles-residues form, a large class of systems can be described, independently of the network order and complexity, even with a simple multiaffine parameterization. Although a similar result cannot be claimed for a generic parameter (e.g. geometrical dimensions, temperature), it is clear that the coefficients of the form (4.8) are preferred candidates for the parameterization scheme. We therefore adopt (4.8) as the expression for our parametric model, with $R'_n(\lambda, \mu)$ and $r'_n(\lambda, \mu)$ being polynomials of degree $\bar{p}$ in $\lambda$ and $\bar{q}$ in $\mu$

$$R'_n(\lambda, \mu) = \sum_{p=0}^{\bar{p}} \sum_{q=0}^{\bar{q}} R'_{npq} \lambda^p \mu^q,$$

$$r'_n(\lambda, \mu) = \sum_{p=0}^{\bar{p}} \sum_{q=0}^{\bar{q}} r'_{npq} \lambda^p \mu^q.$$

4.2.2 The parametric Sanathanan-Koerner iteration

We now develop a reliable algorithm to estimate the coefficients $R'_{npq}$ and $r'_{npq}$ of the model (4.8) in order to fit the given data (4.4), i.e., to numerically minimize the modeling

\(^2\)A function is multiaffine if linear in each one of its variables taken alone, with the other variables constant. Therefore multiaffine functions of two variables $\lambda$ and $\mu$ are always in the form

$$r_{00} + r_{10} \lambda + r_{01} \mu + r_{11} \lambda \mu.$$

Multiaffine functions are also called bilinear functions.
error $\mathcal{E}$ defined as

$$
\mathcal{E}^2 = \|H(j\omega_k; \lambda l, \mu m) - H_{klm}\|^2 = \sum_{k=1}^{\bar{k}} \sum_{l=1}^{\bar{l}} \sum_{m=1}^{\bar{m}} \|H(j\omega_k; \lambda l, \mu m) - H_{klm}\|^2, \tag{4.12}
$$

where $\|.\|_F$ denotes the Frobenius matrix norm.

This task presents two main difficulties:

- the strong ill-conditioning due to the high order powers of $s$ in (4.8);
- the nonlinear dependence of the error in the unknowns $r'_{npq}$ of the model denominator.

In the identification of standard, non-parametric models, Vector Fitting can be used to overcome these problems, as detailed in Appendix A. Unfortunately, it cannot be easily extended to the parametric case, because a parameterization of the model poles would be required (see Appendix A). However, VF has been recognized as a smart reformulation of the classical Sanathanan-Koerner iteration (SK) [106], which instead can be extended to the parametric case as follows.

We avoid the powers of $s$ by expressing the model numerator and denominator with respect to different frequency dependent basis functions $\phi_n(s)$

$$
H(s; \lambda, \mu) = \frac{H_1(s; \lambda, \mu)}{H_2(s; \lambda, \mu)} = \sum_{n=0}^{\bar{n}} r_n(\lambda, \mu) \phi_n(s) \tag{4.13}
$$

defined as

$$
\phi_n(s) = \begin{cases} 
1 & \text{for } n = 0 \\
\frac{1}{s-a_n} & \text{for } n = 1, \ldots, \bar{n}
\end{cases} \tag{4.14}
$$

In order to further improve the numerical conditioning of the fitting scheme we will develop, we also move from the canonical polynomial basis $\{\lambda^p \mu^q\}$ used in (4.11) to an orthogonal basis built using Chebychev polynomials

$$
R_n(\lambda, \mu) = \sum_{p=0}^{\bar{p}} \sum_{q=0}^{\bar{q}} R_{npq} T'_p(\lambda) T'_q(\mu), \tag{4.15}
$$

$$
r_n(\lambda, \mu) = \sum_{p=0}^{\bar{p}} \sum_{q=0}^{\bar{q}} r_{npq} T'_p(\lambda) T'_q(\mu). \tag{4.16}
$$

where $T'_p(\lambda)$ and $T'_q(\mu)$ are Chebychev polynomials [99] suitably rescaled to the range of the parameters $\lambda$ and $\mu$

$$
T'_p(\lambda) = T_p \left( \frac{2\lambda - \lambda_k}{\lambda_k - \lambda} - 1 \right), \tag{4.17}
$$

$$
T'_q(\mu) = T_q \left( \frac{2\mu - \mu_k}{\mu_k - \mu} - 1 \right). \tag{4.18}
$$

3The Frobenius norm of a matrix $A$ is given by $\|A\|_F = \sqrt{\text{Tr}(AA^H)}$ where Tr denotes the matrix trace and the superscript $^H$ the conjugate transpose.
The obtained expression is equivalent to (4.8), but much more suitable for numerical computations. Since (4.13) is a rational function of $s$, it can be converted for each value of the parameters to an equivalent circuit model compatible with any circuit simulator. The new basis $\phi_n(s)$ is the same basis used in Vector Fitting, with the quantities $a_n$ being constant and fixed a priori. Henceforth referred as basis poles, the $\{a_n\}$ can be chosen in two ways:

- linearly displaced over the available data bandwidth $[\omega_1, \omega_k]$ as in standard Vector Fitting [33], with real part much smaller than the imaginary part;
- set close to the real system poles, estimated by applying Vector Fitting to one of the available system responses $H_{klm}$ with $l$ and $m$ fixed.

The second choice will be used since it enhances the convergence of the Sanathanan-Koerner iteration.

The nonlinearity of the optimization problem is tackled using the SK iteration to minimize the modeling error (4.12), here written as a function of the model numerator and denominator

$$ E^2 = \left\| H_2^{-1}(j\omega_k; \lambda_l,\mu_m)[H_1(j\omega_k; \lambda_l,\mu_m) - H_2(j\omega_k; \lambda_l,\mu_m)H_{klm}] \right\|^2. \tag{4.19} $$

The SK method circumvents the nonlinearity due to the unknowns in $H_2^{-1}(j\omega_k; \lambda_l,\mu_m)$ by minimizing, in an iterative fashion, a linearized error function

$$ E_i^2 = \left\| w_{klm}^{(i)} \left[ H_1^{(i)}(j\omega_k; \lambda_l,\mu_m) - H_2^{(i)}(j\omega_k; \lambda_l,\mu_m)H_{klm} \right] \right\|^2 \tag{4.20} $$

where the weight

$$ w_{klm}^{(i)} = [H_2^{(i-1)}(j\omega_k; \lambda_l,\mu_m)]^{-1} \tag{4.21} $$

plays the role of the factor $H_2^{-1}(j\omega_k; \lambda_l,\mu_m)$ in (4.19), being obtained from the previous estimate of the denominator, and where

$$ H_1^{(i)}(s; \lambda,\mu) = \sum_{n=0}^{\bar{n}} \sum_{p=0}^{\bar{p}} \sum_{q=0}^{\bar{q}} R_{npq}^{(i)} T_p'(\lambda) T_q''(\mu) \phi_n(s), \tag{4.22} $$

$$ H_2^{(i)}(s; \lambda,\mu) = \sum_{n=0}^{\bar{n}} \sum_{p=0}^{\bar{p}} \sum_{q=0}^{\bar{q}} r_{npq}^{(i)} T_p'(\lambda) T_q''(\mu) \phi_n(s). \tag{4.23} $$

are the estimates at iteration $i$ for the numerator and denominator of (4.13). As the iterations go, the weight makes (4.20) progressively close to the original error (4.19). Since this new minimization problem is linear in the coefficients of $H_1^{(i)}(j\omega_k; \lambda_l,\mu_m)$ and $H_2^{(i)}(j\omega_k; \lambda_l,\mu_m)$, it can be solved with the efficient algorithms for linear least-squares equations. At the first iteration ($i = 1$) the weight $w_{klm}^{(1)}$ is set to unity for all $k, l$ and $m$ values.

A close inspection of (4.20) reveals that the SK iteration admits the trivial solution $R_{npq} = 0$, $r_{npq} = 0$, $\forall n,p,q$. Therefore, some non-triviality constraint must be added. It
turns out that the specific choice of this constraint has a significant influence on the quality of the results. The simplistic solution of fixing one of the model coefficients, e.g. \( r_{000} = 1 \) may slow down or even prevent the convergence of the SK iteration, as will be shown in Section 4.2.5. We propose therefore a better normalization scheme that guarantees improved performance both in terms of accuracy and convergence speed. This normalization is inspired to the relaxed normalization developed for VF [41]. It is established by adding to the least squares problem for the minimization of (4.20) the constraint

\[
\frac{1}{klm} \sum_{k,l,m} \Re \left\{ \sum_{n=0}^{\bar{n}} \sum_{p=0}^{\bar{p}} \sum_{q=0}^{\bar{q}} (r_{npq}^{(i)} - 1) \phi_n(j\omega_k) T_p^{(i)}(\lambda_l) T_q^{(i)}(\mu_m) \right\} = 0. \tag{4.24}
\]

This constraint is enforced in least squares sense and guarantees a non-vanishing weighted average of the coefficients \( r_{npq}^{(i)} \), thus making (4.13) well-defined at all iterations.

### 4.2.3 Uniform stability

The parametric SK iteration developed in the previous Section does not guarantee the model stability. The main difficulty towards this goal is that, for parametric models, stability must be uniformly ensured over a continuous set, the hypercube (4.5). Computationally tractable conditions must therefore be devised. Without them, fitting techniques with stability constraints cannot be derived, and only a rough check of the model stability can be performed with a sweep of the parameters.

Let us consider the model (4.13): its poles are given by the zeros of \( H_2(s; \lambda, \mu) \), since the basis poles \( a_n \) are common between the numerator and the denominator, and so cancel out. The denominator \( H_2(s; \lambda, \mu) \) can be written as

\[
H_2(s; \lambda, \mu) = r_0(\lambda, \mu) + c(\lambda, \mu) (sI - \Delta)^{-1} b \tag{4.25}
\]

where

\[
b = [1, \ldots, 1]^T \text{ (size } \bar{n} \times 1), \tag{4.26}
\]

\[
c(\lambda, \mu) = [r_1(\lambda, \mu), \ldots, r_{\bar{n}}(\lambda, \mu)], \tag{4.27}
\]

\[
\Delta = \text{diag}(a_n). \tag{4.28}
\]

The zeros of the denominator (4.25) are given by the eigenvalues of the matrix [41]

\[
A(\lambda, \mu) = \frac{r_0(\lambda, \mu) \Delta - bc(\lambda, \mu)}{r_0(\lambda, \mu)} \tag{4.29}
\]

Therefore, (4.13) is BIBO stable if and only if \( A(\lambda, \mu) \) is a stable matrix, having all eigenvalues in the strict half plane \( \Re \{s\} < 0 \). The stability of parametric matrices like (4.29) has been studied in the robust control literature. For the case of degrees \( \bar{p} \leq 1 \) and \( \bar{q} \leq 1 \) the following stability condition exists [107].
Theorem 4.2. If \( r_0(\lambda,\mu) \) is not vanishing\(^4\) in (4.5), \( A(\lambda,\mu) \) is stable for all \((\lambda,\mu) \in [\bar{\lambda},\bar{\lambda}] \times [\bar{\mu},\bar{\mu}]\) if a symmetric and positive definite matrix \( P \) exists such that

\[
\begin{align*}
A^T(\bar{\lambda},\bar{\mu})P + PA(\bar{\lambda},\bar{\mu}) &< 0 \\
A^T(\bar{\lambda},\bar{\mu})P + PA(\bar{\lambda},\bar{\mu}) &< 0 \\
A^T(\bar{\lambda},\bar{\mu})P + PA(\bar{\lambda},\bar{\mu}) &< 0 \\
A^T(\bar{\lambda},\bar{\mu})P + PA(\bar{\lambda},\bar{\mu}) &< 0
\end{align*}
\]

The linear matrix inequalities (LMIs) in (4.30) are basically the Lyapunov equation for \( A(\lambda,\mu) \) at the four corners of (4.5). Since LMIs can be efficiently solved with convex optimization techniques [109], this theorem directly allows to numerically ascertain the uniform stability of (4.13). The scope of this result is however much broader, since it reduces the complexity of the parametric stability condition to a computationally tractable level. Here Theorem 4.2 has been reported for the two parameters case; the general case can be found in [107].

For models with degree higher than one (\( \bar{p} > 1 \) and/or \( \bar{q} > 1 \)) a similar result involving only the extremes of the parameters range cannot exist [108]. Several conditions for stability have been proposed, like those derived in [107], that however involve a higher degree of conservativeness than Theorem 4.2. For this reason, we will not further investigate this possibility, but adopt in Section 4.3 a different macromodel formulation, for which uniform stability conditions of general validity exist.

4.2.4 Illustrative example

A two-port circuit composed by two microstrip lines and a resistor-capacitor pair is considered, depicted in Figure 4.1. This configuration idealizes a 5 cm interconnect link loaded by a device, here represented by the resistor-capacitor group. The nominal features of the components are: \( R = 4 \, \text{k}\Omega, \, C = 0.2 \, \text{pF}, \) microstrip width \( w = 80 \, \mu\text{m}, \) lengths \( l_1 = 3 \, \text{cm} \) and \( l_2 = 2 \, \text{cm}, \) dielectric height \( h = 0.3 \, \text{mm}, \) dielectric permittivity \( \epsilon_r = 4. \) The proposed technique was applied to derive two parameterized models.

The microstrips width \( w \) was first taken as design parameter, variable between 60 and 130 \( \mu\text{m}. \) The \( 2 \times 2 \) scattering matrix of the circuit was computed for 100 points from 10 MHz to 10 GHz, and for 15 different values of \( w \) between 60 and 130 \( \mu\text{m}, \) at steps of 5 \( \mu\text{m}. \) For the sake of simplicity the per-unit-length parameters of the lines were computed with standard Wheeler's formulas, while in a real design scenario a 2D electromagnetic simulator would have been used. Among the 15 responses, those for \( w = 60, 70, \ldots, 130 \, \mu\text{m} \) were used to construct the parametric model, and the others to verify, a posteriori, the approximation quality for intermediate values of \( w. \) In the top panel of Figure 4.2, the magnitude of \( S_{21} \) for a parametric model of order 16 and degree \( \bar{m} = 3 \) is depicted, together with the exact system response. A very good agreement can be observed: in

\(^4\)This hypothesis is verified if and only if \( r_0(\lambda,\mu) \) has the same sign at the four vertices of the hypercube (4.5), see [108].
fact, the maximum modeling error among the S parameters was found to be $1.3 \times 10^{-3}$ (absolute) and 0.8% (relative). Two SK iterations were required to identify the model, each one taking just 2s. Since the model degree is higher than one, the model stability could not be ascertained with Theorem 4.2, so a parameter sweep was used. The model turned out to be stable.

The second test case considered $R$ and $C$ as parameters. Since they are values of lumped components, a model of degree $\tilde{p} = \tilde{q} = 1$ was expected to be sufficient, even for large parameters variations. The ranges for $R$ and $C$ were chosen as [0.1,10]kΩ and [0.1,0.9]pF, respectively. The circuit S matrix was computed for nine values of $R$ and nine of $C$. Out of the 81 obtained datasets, only 9 corresponding to $R = 0.1,1,10kΩ$ and $C = 0.1,0.5,0.9pF$ were used for the model identification. The bottom panel of Figure 4.2 shows the very good match of the model response with the data, for the real part of the $S_{11}$ coefficient. The maximum absolute and relative errors between the model and the true system response are $8 \times 10^{-5}$ and 0.1% respectively. The estimated model, of order $\bar{n} = 18$ and degrees $\bar{p} = \bar{q} = 1$, was computed in only 11 s with two SK iterations. The application of the uniform stability test devised in Section 4.2.3 ascertained that the computed model is stable for all parameters values.

**4.2.5 Application example: backdrilled via**

The first application example concerns a via with stub, depicted in Fig. 4.3, connecting a microstrip line and a stripline in a multilayer PCB. When the via is created, the metalization runs from top to bottom. The bottom part is unnecessary to electrically connect the two lines and can be a potential source of Signal Integrity problems. To mitigate these problems, the stub length $h$ can be adjusted through backdrilling. Our objective is to generate a macromodel for the 2-port via structure preserving the dependence on the free parameter $h$.

The scattering parameters of the via were obtained using an EM solver. Several stub heights $h = 0, 200, 400, 478, 606, 716 \mu m$ were considered, including both the full length ($h = 716 \mu m$) and the completely backdrilled ($h = 0 \mu m$) stub. The parametric SK iteration was applied to compute the macromodel. First, the model was normalized by setting $r_{000} = 1$ in (4.18). Unfortunately, this compromised the model identification, because of convergence failure or low accuracy depending on the model order. Instead, using the proposed algorithm with normalization (4.24), we obtained an accurate macromodel with order $\bar{n} = 14$ and degree $\bar{p} = 3$. The model was computed in 2.5 minutes on a laptop with a 1.83 GHz processor.
To validate the model quality in the whole parameter range, additional frequency responses were computed for the new parameter values \( h = 100, 300, 418, 496 \) µm. The model responses at these values were computed by evaluating the closed-form expression (4.13). Figure 4.4 shows a model vs data comparison for all (reference and validation) values of \( h \), showing a very good agreement with a maximum deviation of \( 2.2 \times 10^{-2} \). Although stability is not guaranteed by the SK algorithm, the computed model is stable. Figure 4.5 displays the drift of the model poles caused by the parameter \( h \).
Figure 4.3. Backdrilled via cross section.
4.2 – Polynomial parameterization

Figure 4.4. Magnitude and phase of $S_{21}$ of the via macromodel. Raw frequency data (solid line) and model responses (dash dot line) are depicted for $h = 0, 100, 200, 300, 400, 418, 478, 496, 606, 716 \, \mu m$. 
Figure 4.5. Macromodel poles as a function of the parameter $h$. 

4 - Parametric macromodels
4.2 - Polynomial parameterization

4.2.6 Application example: integrated inductor

As a second application case, we consider a 2-turns integrated spiral inductor (courtesy of MSDT Consortium, Georgia Institute of Technology). The trace width $w$ and the substrate dielectric constant $\varepsilon_r$ were considered as first and second parameter, respectively. The $S_{11}$ of the inductor was computed with a full-wave solver from 0.1 up to 15 GHz for $w = 3, 4, 5, 6, 7$ mils and $\varepsilon_r = 30, 35, 40$. All available data were used to fit the parametric model, except for the responses corresponding to $w = 4, 6$ mils and $\varepsilon_r = 35$, reserved for validation purposes. In Fig. 4.6, the response of a model (order $n = 6$, degrees $\bar{p} = 2$ with respect to $w$ and $\bar{q} = 1$ with respect to $\varepsilon_r$) is compared to the raw data. A very good match can be observed, since the maximum modeling error is only $4 \times 10^{-3}$ (0.4%). Also in this example the relaxed normalization (4.24) was used, leading to a significantly higher model accuracy. The required computation time for this example was only 2 s. Also in this case the model poles are stable, as shown in Fig. 4.7.

![Fig. 4.6. Macromodel of a spiral inductor. Model responses (dash-dot lines) and raw data (solid lines) are compared for different combinations of $w$ (3, 4, 6, 7 mils) and $\varepsilon_r$ (40, 35, 35, 30).](image-url)
Figure 4.7. Poles of the inductor macromodel for \( w \in [3,7] \) mils and \( \varepsilon_r \in [30,40] \).
4.3 Piecewise linear parameterization

The technique developed in the previous Section proved to be a valid and general purpose algorithm to create parametric models from tabulated frequency data. Its general applicability was demonstrated with three examples involving electrical components of different nature: a short interconnect line, a discontinuity, and a passive device for RF. Although the computational cost of this algorithm is moderate, it is still too high for components with a medium to large ports count. Moreover, the uniform stability check algorithm we proposed is limited to models with multiaffine parameterization. In this Section we address both issues with a different macromodeling algorithm based on a piecewise linear parameterization.

4.3.1 Formulation

Our starting point is expression (4.13), here repeated for the convenience of the Reader

\[
H(s; \lambda, \mu) = \frac{H_1(s; \lambda, \mu)}{H_2(s; \lambda, \mu)} = \sum_{n=0}^{\bar{n}} R_n(\lambda, \mu) \phi_n(s) + \sum_{n=0}^{\bar{n}} r_n(\lambda, \mu) \phi_n(s).
\] (4.31)

This functional form was identified in Section 4.2.1 as a very convenient formulation for parametric macromodels, and thanks to the Vector Fitting basis \( \phi_n(s) \) defined by (4.14) it leads to a well-conditioned fitting algorithm. For the expansion coefficients of the numerator and denominator we introduce now the representation\(^5\)

\[
R_n(\lambda, \mu) = \sum_{p,q} R_{npq} \psi_p(\lambda) \xi_q(\mu), \tag{4.32a}
\]

\[
r_n(\lambda, \mu) = \sum_{p,q} r_{npq} \psi_p(\lambda) \xi_q(\mu), \tag{4.32b}
\]

with the parameter-dependent basis functions \( \{ \psi_p(\lambda) \} \) being piecewise linear

\[
\psi_p(\lambda) = \begin{cases} 
\frac{\lambda - \lambda_2}{\lambda_1 - \lambda_2} I_{[\lambda_1, \lambda_2]}(\lambda) & \text{for } p = 1 \\
\frac{\lambda - \lambda_{p-1}}{\lambda_p - \lambda_{p-1}} I_{[\lambda_{p-1}, \lambda_p]}(\lambda) + \frac{\lambda - \lambda_{p+1}}{\lambda_p - \lambda_{p+1}} I_{[\lambda_p, \lambda_{p+1}]}(\lambda) & \text{for } p = 2, \ldots, \bar{p} - 1 \\
\frac{\lambda - \lambda_{\bar{p}-1}}{\lambda_{\bar{p}} - \lambda_{\bar{p}-1}} I_{[\lambda_{\bar{p}-1}, \lambda_{\bar{p}}]}(\lambda) & \text{for } p = \bar{p}
\end{cases}
\] (4.33)

where \( \bar{p} = \bar{l} \). With \( I_A(\cdot) \) we denote the indicator function of set \( A \)

\[
I_A(x) = \begin{cases} 
1 & \text{if } x \in A \\
0 & \text{otherwise}
\end{cases}
\] (4.34)

\(^5\)In order to keep mathematical formulas reasonably compact, the lower and upper values of summation indexes will be often omitted in this Chapter. When not specified the lower value will always be 1 for all indexes except for \( n \), that will start from 0. The upper limit instead will always be the maximum value associated to the index (i.e., \( \bar{n} \) for \( n \), \( \bar{p} \) for \( p \), ...).
The basis functions (4.33) are linear in each interval \([\lambda_l, \lambda_{l+1}]\) between the available parameter points. A similar definition is used for \(\xi_q(\mu)\). The complete model expression including explicit dependence on all free variables reads

\[
H(s; \lambda, \mu) = \frac{\sum_{n,p,q} R_{npq} \phi_n(s) \psi_p(\lambda) \xi_q(\mu)}{\sum_{n,p,q} r_{npq} \phi_n(s) \psi_p(\lambda) \xi_q(\mu)}. \tag{4.35}
\]

We remark that the order of numerator and denominator is the same, since the same basis poles are used. This is indeed the main enabling factor for the developments of Section 4.3.5, which will show how the basis poles cancel out in the final model expression. Note however that a different order for numerator and denominator polynomials is indeed possible. For example, one may want a numerator degree larger than denominator degree by one, in order to build macromodels with linear growth for large frequency (e.g., reproducing purely inductive or capacitive asymptotic behavior for impedance or admittance forms). This is easily achieved by adding \(\phi_{-1}(s) = s\) to the set of numerator basis functions only. Therefore, without loss of generality, we will employ throughout the rest of this Chapter the same set of basis functions (4.14) for both numerator and denominator.

### 4.3.2 Identification technique

We present here an advanced identification procedure that allows estimation of the coefficients \(R_{npq}\) and \(r_{npq}\) in (4.35) from the raw data (4.4), while minimizing the modeling error (4.12). The numerical minimization of (4.12) for models in the form (4.35) could be performed using the parametric Sanathanan-Koerner iteration of Section 4.2.2 with minor modifications. However, the number of unknowns in the linear least squares problem (4.20) may grow very large. This number, which can be explicitly estimated and for the two-parameters case results \(\bar{l}\bar{m}(\bar{n} + 1) \times (N^2 + 1)\), grows exponentially with the number of parameters. The direct solution of (4.20) in a single step may therefore be unfeasible due to obvious complexity arguments. We therefore devise a smarter fitting algorithm that, using the properties of the piecewise linear model structure, leads to a great reduction of the overall computational cost for identification, both in terms of CPU and memory requirements.

Thanks to the piecewise-linear parameterization (4.32), the following interpolation property holds for the model coefficients

\[
R_n(\lambda_l, \mu_m) = R_{n\lambda m}, \tag{4.36}
\]

\[
r_n(\lambda_l, \mu_m) = r_{n\lambda m}. \tag{4.37}
\]

Therefore, the model response \(H(s; \lambda_l, \mu_m)\) evaluated at the grid point \((\lambda_l, \mu_m)\) is only function of the few coefficients

\[
\{R_{n\lambda m}\}, \{r_{n\lambda m}\} \quad n = 0, \ldots, \bar{n}, \tag{4.38}
\]

with \(l\) and \(m\) fixed. Indeed, the modeling error can be written as sum of independent contributions \(\mathcal{E}_{lm}^2\)

\[
\mathcal{E}^2 = \sum_{l=1}^{\bar{l}} \sum_{m=1}^{\bar{m}} \mathcal{E}_{lm}^2. \tag{4.39}
\]
where
\[ E^2_{lm} = \sum_{k=1}^{\bar{k}} \left\| \sum_n R_{nlm}(j\omega_k) \phi_n(j\omega_k) - H_{klm} \right\|_F^2. \] (4.40)

Each factor \( E^2_{lm} \) represents the local modeling error at the grid point \((\lambda_l, \mu_m)\). Since these local errors are independent, i.e., they are functions of separate subsets of model coefficients, they can be independently minimized. If the minimum of (4.40) is achieved for each grid point \((\lambda_l, \mu_m)\), the global error (4.12) will be also minimized. This observation shows that the minimization of the fitting error (4.12) can be reformulated as \( l\bar{m} \) independent local minimum problems.

Minimization of (4.40) is a standard, non-parameterized fitting problem, that tries to match the frequency response \( H_{klm} \) (with \( l \) and \( m \) fixed)
\[ H_{lm}(j\omega_k) \approx H_{klm}, \] (4.41)
with the transfer function
\[ H_{lm}(s) = \frac{\sum_n R_{nlm}(s)}{\sum_n r_{nlm}(s)}. \] (4.42)
of a local, non-parameterized model. This reformulation of the original problem allows to use standard modeling algorithms to identify parameterized models in the form (4.31).

The independent local fitting problems (4.41)-(4.42) are not critical, and highly robust and efficient methods exist for their solution, such as the VF algorithm. When applied to the raw data \( H_{klm} \) with \( l \) and \( m \) fixed, Vector Fitting returns a model in poles-residues form (see Appendix A)
\[ H_{lm}(s) = Q_{0lm} + \sum_{n=1}^{\bar{n}} \frac{Q_{nlm}}{s - p_{nlm}}, \] (4.43)
where \( Q_{nlm} \) and \( p_{nlm} \) are respectively the model residues and poles, with \( Q_{0lm} \) being the direct coupling constant.

If the above local identification process is applied to each independent grid point \((\lambda_l, \mu_m)\), the resulting sets of macromodel poles
\[ P_{lm} = \{p_{nlm}, n = 1, \ldots, \bar{n}\} \] (4.44)
have no relation with each other. It is well-known that model poles are very sensitive quantities to even small variations of the parameters. Bifurcation effects may occur, resulting in quite irregular variations of each pole in the parameters space. See Fig. 4.8 for a graphical illustration on a simple test case. In addition, the poles/residues form (4.43) is not fully compatible with the model representation (4.31), which is based on a set of frequency-domain expansion functions \( \{\phi_n(s)\} \), which are defined on a different but common set of poles \( \{a_n\} \).

Fortunately, there is a full compatibility between (4.43) and (4.42), since a direct conversion can be performed from one format to the other, and vice-versa. Here, we are interested in converting (4.43) into (4.42). The following theorem provides the conversion rules for the computation of global coefficients \( R_{nlm} \) and \( r_{nlm} \) from local coefficients \( Q_{nlm} \) and \( p_{nlm} \).
Figure 4.8. Sketch of poles location of a 4th order circuit with varying components. Bifurcation effects show that a direct parameterization of the model poles is non-smooth and should be avoided.

**Theorem 4.3.** Under the hypothesis that the poles \( \{a_n\} \) are distinct, the transfer functions

\[
F(s) = Q_0 + \sum_{n=1}^{\bar{n}} \frac{Q_n}{s - p_n}, \tag{4.45}
\]

and

\[
G(s) = \frac{R_0 + \sum_{n=1}^{\bar{n}} \frac{R_n}{s - a_n}}{r_0 + \sum_{n=1}^{\bar{n}} \frac{r_n}{s - a_n}} \tag{4.46}
\]

are equal if

\[
\sum_{n=1}^{\bar{n}} r_n \prod_{n' \neq n}^{\bar{n}} (p_l - a_{n'}) = -r_0 \prod_{n' \neq n}^{\bar{n}} (p_l - a_{n'}) \quad \text{for } l = 1, \ldots, \bar{n} \tag{4.47}
\]

\[
R_n = r_n F(a_n) \quad \text{for } n = 1, \ldots, \bar{n} \tag{4.48}
\]

\[
R_0 = r_0 Q_0. \tag{4.49}
\]

**Proof.** Let us rewrite (4.46) as ratio of two polynomials by multiplying numerator and denominator by \( \prod_{n' = 1}^{\bar{n}} (s - a_{n'}) \)

\[
G(s) = \frac{R_0 \prod_{n' = 1}^{\bar{n}} (s - a_{n'}) + \sum_{n=1}^{\bar{n}} R_n \prod_{n' = 1}^{\bar{n}} (s - a_{n'})}{r_0 \prod_{n' = 1}^{\bar{n}} (s - a_{n'}) + \sum_{n=1}^{\bar{n}} r_n \prod_{n' = 1}^{\bar{n}} (s - a_{n'})}. \tag{4.50}
\]

To prove the first condition (4.47) we force the poles of (4.50) to equal the poles of (4.45) by setting to zero the denominator of (4.50) for \( s = p_l \), with \( l = 1, \ldots, \bar{n} \), obtaining

\[
r_0 \prod_{n' = 1}^{\bar{n}} (p_l - a_{n'}) + \sum_{n=1}^{\bar{n}} r_n \prod_{n' = 1}^{\bar{n}} (p_l - a_{n'}) = 0. \tag{4.51}
\]
This condition allows to compute the denominator coefficients \( \{ r_n \} \) of (4.46) with the solution of a linear system. Since there is a degree of freedom in the coefficients of (4.46), the value of \( r_0 \) can be fixed at will. The second condition (4.48) is obtained by evaluating \( G(s) = F(s) \) for \( s = a_n, \ n = 1, \ldots, \bar{n} \), and from the fact that \( G(a_n) = R_n/r_n \), as evident from (4.50). Finally, condition (4.49) follows by imposing \( F(s) = G(s) \) for \( s \to \infty \).

The solution of linear system (4.47), followed by a direct evaluation of (4.48) and (4.49) for each grid point \( (\lambda_l, \mu_m) \) provides the full set of coefficients \( R_{nlm} \) and \( r_{nlm} \) defining the proposed parameterized macromodel (4.31), starting from the collective set of independent poles and residues obtained by local fitting (4.42) at individual grid points.

### 4.3.3 Choice of basis poles

We discuss now the choice of basis poles \( \{ a_n \} \). We start noting that these poles are not used in the first stage of the fitting algorithm, the identification of the local models (4.43), task performed using standard Vector Fitting. Their choice has therefore no effects on this task. Basis poles come into play later, when all local models (4.43) are expressed with respect to the common basis \( \phi_n(s) \) adopted for the final parametric model. The choice of basis poles \( a_n \) should be done in order to optimize the numerical conditioning of basis functions \( \phi_n(s) \), in order to ease the conversion from the VF form (4.43) to (4.42). It is well known that the proposed rational basis (4.14) is well conditioned if the basis poles \( a_n \) are equally spaced over the available bandwidth close to the imaginary axis [33]. We therefore adopt this rule to displace the basis poles \( a_n \).

To further support this conclusion we devised the following numerical test on the application example of Sec. 4.3.8. We computed the condition number of the linear system to be solved in the model conversion (4.47) for several displacements of the basis poles \( a_n \):

- Linear: linear displacement over the available bandwidth suggested in [33];
- Response poles: the basis poles \( \{ a_n \} \) are chosen as the poles of one of the available responses, estimated with standard VF;
- Linear, 10% bandwidth: linear displacement over one tenth of the available bandwidth;
- Linear, 10x bandwidth: linear displacement over ten times the available bandwidth;
- Linear with close poles: linear displacement for all poles except for the first one, placed very close to the second pole ( \( a_1 = 0.99 a_2 \)).

Table 4.1 reports the numerical results of this test, and shows that the adopted rule (first case) provides the best conditioning of the model conversion formulas.

### 4.3.4 Stability enforcement at grid points

Since Vector Fitting guarantees the stability of the identified model, the final parameterized model (4.31) will be stable by construction at all grid points \( (\lambda_l, \mu_m) \). It is clear that
this enforcement process does not guarantee uniform stability of (4.31) at each point in the parameters space (4.5). However, it strongly reduces the occurrence of violations, and can be used to guarantee stability in the whole parameters range. In fact, if the model turns out to be unstable in a certain region, the interpolation grid can be there refined by adding more interpolation points where stability is preserved by construction, until uniform stability is achieved. In case of multiple parameters, those that produce the most relevant variations of the system response should be refined first, since stability violations are unlikely where variations due to parameters are small. By adding more samples with respect to these most critical parameters, the cause of stability loss is removed and the probability of obtaining uniform stability is maximized.

Such a procedure clearly needs for a reliable algorithm to test uniform stability and localize possible violations. This need motivates the developments of Section 4.3.6, where we present a robust numerical algorithm for checking the uniform stability of parameterized models in the form (4.31), without resorting to brute-force sampling methods, which are slow and possibly misleading. Before presenting such testing procedures, we need to further elaborate on the model realization.

4.3.5 Model realization

In this Section, a state-space realization of the parameterized model transfer function (4.31) is first derived, and then converted to a descriptor form. This form is very convenient, since it allows the construction of a uniform stability test for the macromodel (4.31) with piecewise linear parameterization (4.32). All results from (4.53) to (4.65) hold for any model in the form (4.31), including the polynomially parameterized models of Section 4.2. A brief review of the definition and properties of descriptor systems is available in Appendix D.

In the following derivations, we will adopt the standard notation for state-space realizations

\[
H(s) = D + C(sI - A)^{-1}B \leftrightarrow \begin{bmatrix} A & B \\ C & D \end{bmatrix}
\]  

(4.52)
A state-space realization for (4.31) can be obtained if the model transfer function is interpreted as the ratio of two transfer functions $H_1(s; \lambda, \mu)$ and $H_2(s; \lambda, \mu)$

$$H(s; \lambda, \mu) = \frac{H_1(s; \lambda, \mu)}{H_2(s; \lambda, \mu)}, \quad (4.53)$$

where

$$H_1(s; \lambda, \mu) = R_0(\lambda, \mu) + \sum_{n=1}^{\bar{n}} \frac{R_n(\lambda, \mu)}{s - a_n}, \quad (4.54)$$

$$H_2(s; \lambda, \mu) = r_0(\lambda, \mu) + \sum_{n=1}^{\bar{n}} \frac{r_n(\lambda, \mu)}{s - a_n}, \quad (4.55)$$

First, we construct two separate state-space realizations for the numerator and denominator. We have

$$H_1(s; \lambda, \mu) \leftrightarrow \begin{bmatrix} A_1 & B_1 \\ C_1(\lambda, \mu) & D_1(\lambda, \mu) \end{bmatrix}, \quad (4.56)$$

$$H_2(s; \lambda, \mu) I_N \leftrightarrow \begin{bmatrix} A_2 & B_2 \\ C_2(\lambda, \mu) & D_2(\lambda, \mu) \end{bmatrix}, \quad (4.57)$$

where

- $A_1 = A_2 = \text{blkdiag}\{a_n I_N\}$ with $n = 1, \ldots, \bar{n}$;
- $B_1 = B_2 = [I_N, \ldots, I_N]^T$ is a block-column matrix obtained by stacking $\bar{n}$ identity matrices;
- $C_1(\lambda, \mu) = [R_1(\lambda, \mu), \ldots, R_{\bar{n}}(\lambda, \mu)];$
- $C_2(\lambda, \mu) = [r_1(\lambda, \mu) I_N, \ldots, r_{\bar{n}}(\lambda, \mu) I_N];$
- $D_1(\lambda, \mu) = R_0(\lambda, \mu)$
- $D_2(\lambda, \mu) = r_0(\lambda, \mu) I_N;$

where $I_N$ is the identity matrix of size $N \times N$. We remark that, in case of complex poles $\{a_n\}$, the above state-space matrices are complex. However, standard coordinate changes can be applied in the state space such that the realization is real [110]. Henceforth, we will sometimes omit the dependence of the matrices $C_1, C_2, D_1,$ and $D_2$ from $\lambda$ and $\mu$, in order to avoid formulas of excessive length.

Although the denominator function is scalar, its realization has been chosen to have $N$ ports, in order to be compatible in size with the realization of the numerator. Then, a realization for the inverse of $H_2(s; \lambda, \mu)$ is derived from (4.57)

$$H_2^{-1}(s; \lambda, \mu) I_N \leftrightarrow \begin{bmatrix} A_2 - B_2 D_2^{-1} C_2 & -B_2 D_2^{-1} \\ D_2^{-1} C_2 & D_2^{-1} \end{bmatrix}, \quad (4.58)$$
using standard manipulations, see [111] for details. Finally, $H(s; \lambda, \mu)$ is realized by cascading (4.56) and (4.58) as

$$H(s; \lambda, \mu) = H_1(s; \lambda, \mu)H_2^{-1}(s; \lambda, \mu)I_N \leftrightarrow \begin{bmatrix} A_1 & B_1D_2^{-1}C_2 & B_1D_2^{-1} \\ 0 & A_2 - B_2D_2^{-1}C_2 & -B_2D_2^{-1} \\ C_1 & D_1D_2^{-1}C_2 & D_1D_2^{-1} \end{bmatrix}$$  \hspace{1cm} (4.59)

The number of state variables in (4.59) is $2N\tilde{n}$, at least twice as large than necessary. In fact, it can be easily seen that $N\tilde{n}$ states are not controllable and do not contribute to the input-output transfer function. This is evident if one considers that the poles of (4.59) are equal to

$$\text{eig}\{A_1\} \cup \text{eig}\{A_2 - B_2D_2^{-1}(\lambda, \mu)C_2(\lambda, \mu)\} = \{a_n\} \cup \text{eig}\{A_2 - B_2D_2^{-1}(\lambda, \mu)C_2(\lambda, \mu)\}, \hspace{1cm} (4.60)$$

while the poles of the original transfer function (4.31) are only $\text{eig}\{A_2 - B_2D_2^{-1}(\lambda, \mu)C_2(\lambda, \mu)\}$, i.e., the zeros of $H_2(s; \lambda, \mu)$. Indeed, the poles $\{a_n\}$ cancel out since they are common to the numerator and the denominator of (4.31).

The redundant states can be removed if (4.59) is cast to the Kalman controllable canonical form [111] with the similarity transformation

$$x(t; \lambda, \mu) = Tw(t; \lambda, \mu), \hspace{1cm} (4.61)$$

where $x(t; \lambda, \mu)$ denotes the state vector of (4.59), $w(t; \lambda, \mu)$ is the new state vector in a different coordinate system, and the corresponding transformation matrix reads

$$T = \frac{1}{\sqrt{2}} \begin{bmatrix} I & I \\ -I & I \end{bmatrix}. \hspace{1cm} (4.62)$$

with $I$ being the identity matrix of size $N\tilde{n} \times N\tilde{n}$. Application of (4.61) to (4.59) leads, after removal of the uncontrollable states, to the more compact realization

$$H(s; \lambda, \mu) \leftrightarrow \begin{bmatrix} A_2 - B_2D_2^{-1}C_2 & B_2D_2^{-1} \\ C_1 - D_1D_2^{-1}C_2 & D_1D_2^{-1} \end{bmatrix}. \hspace{1cm} (4.63)$$

The realization (4.63) does not depend linearly on the matrices $C_1(\lambda, \mu)$, $C_2(\lambda, \mu)$, $D_1(\lambda, \mu)$, $D_2(\lambda, \mu)$ which are functions of the parameters $\lambda$ and $\mu$. We now show that a much simpler realization can be constructed such that linearity is preserved. This constraint will require to adopt a slightly more general form of realization, namely a so-called descriptor form, also known as a differential-algebraic system of equations (DAE) or, in circuit notation, as a modified nodal analysis (MNA) system. See Appendix D for details. A few straightforward manipulations show that the following descriptor system

$$\begin{align} E\dot{x}(t; \lambda, \mu) &= A(\lambda, \mu)x(t; \lambda, \mu) + Bu(t) \hspace{1cm} (4.64a) \\
y(t; \lambda, \mu) &= C(\lambda, \mu)x(t; \lambda, \mu) \hspace{1cm} (4.64b) \end{align}$$

\(^6\)Note that the poles $\{a_n\}$ of $A_1$ are multiple.
with
\[
\begin{align*}
E &= \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \\
B &= \begin{bmatrix} 0 \\ -I_N \end{bmatrix}, \\
A(\lambda, \mu) &= \begin{bmatrix} A_2 \\ C_2(\lambda, \mu) \\ D_2(\lambda, \mu) \end{bmatrix}, \\
C(\lambda, \mu) &= \begin{bmatrix} C_1(\lambda, \mu) \\ D_1(\lambda, \mu) \end{bmatrix}.
\end{align*}
\]

is equivalent to the realization (4.63). It is evident that all blocks of this representation depend linearly on the parameter-dependent matrices \(C_1(\lambda, \mu), C_2(\lambda, \mu), D_1(\lambda, \mu), D_2(\lambda, \mu)\). Therefore, this formulation inherits the same parameterization introduced in the model coefficients (4.32). Such property is essential for the development of the uniform stability check, to be introduced in next Section. It is also remarkable that a large part of this system (the first \(N\bar{n}\) equations) does not depend on the parameters.

### 4.3.6 Uniform stability assessment

In this Section we present a purely algebraic method for checking the stability of parameterized models uniformly with respect to the parameters \(\lambda\) and \(\mu\). The devised procedure is based on the descriptor form realization (4.64), since its matrices have a very simple dependence on the parameters.

Starting from (4.65a) and (4.65b), it can be easily seen that when the model coefficients are piecewise linear as in (4.32), both matrices \(A(\lambda, \mu)\) and \(C(\lambda, \mu)\) can be expressed as

\[
A(\lambda, \mu) = \sum_{p=1}^{\bar{p}} \sum_{q=1}^{\bar{q}} A_{pq} \psi_p(\lambda) \xi_q(\mu),
\]

with
\[
A_{pq} = \begin{bmatrix} \bar{r}_{1pq}I_N & \cdots & \bar{r}_{npq}I_N \end{bmatrix},
\]

and

\[
C(\lambda, \mu) = \sum_{p=1}^{\bar{p}} \sum_{q=1}^{\bar{q}} C_{pq} \psi_p(\lambda) \xi_q(\mu),
\]

with
\[
C_{pq} = \begin{bmatrix} \bar{R}_{1pq} & \cdots & \bar{R}_{\bar{q}pq} \\ \bar{R}_{\bar{q}pq} & \cdots & \bar{R}_{\bar{q}pq} \end{bmatrix}.
\]

Therefore, for each rectangular subdomain
\[
(\lambda, \mu) \in [\lambda_l, \lambda_{l+1}] \times [\mu_m, \mu_{m+1}]
\]

of the parameters space, the system matrices (4.65a) and (4.65b) are either multi-affine in the parameters or constant. In fact, \(A(\lambda, \mu)\) and \(C(\lambda, \mu)\) can be written in the subdomain (4.70) as

\[
A(\lambda, \mu) = A_{lm}(1-\lambda')(1-\mu') + A_{l+1,m}(1-\lambda')\mu' + A_{l,m+1}(1-\lambda')\mu' + A_{l+1,m+1}(1-\lambda)\mu',
\]

(4.71)
\[ C(\lambda, \mu) = C_{lm}(1-\lambda')(1-\mu') + C_{l+1,m}\lambda'(1-\mu') + C_{l,m+1}(1-\lambda')\mu' + C_{l+1,m+1}\lambda'\mu', \quad (4.72) \]

where

\[ \lambda' = \frac{\lambda - \lambda_l}{\lambda_{l+1} - \lambda_l}, \quad \mu' = \frac{\mu - \mu_m}{\mu_{m+1} - \mu_m}. \quad (4.73) \]

Expressions (4.71) and (4.72) show that, within each rectangle (4.70), \( A(\lambda, \mu) \) and \( C(\lambda, \mu) \) take values in the convex hull identified by the “corner” matrices

\[
\begin{bmatrix}
A(\lambda, \mu) \\
C(\lambda, \mu)
\end{bmatrix} \in \mathrm{Co}\left\{ \begin{bmatrix} A_{lm} \\ C_{lm} \end{bmatrix}, \begin{bmatrix} A_{l+1,m} \\ C_{l+1,m} \end{bmatrix}, \begin{bmatrix} A_{l,m+1} \\ C_{l,m+1} \end{bmatrix}, \begin{bmatrix} A_{l+1,m+1} \\ C_{l+1,m+1} \end{bmatrix} \right\}. \quad (4.74)
\]

Therefore, the system matrices belong to a so-called convex polytopic domain. For this kind of systems, conditions for uniform stability can be found in the robust control literature [112]. These conditions, which involve the feasibility of certain Linear Matrix Inequalities (LMI), are reviewed in the following. The reader is referred to [109, 113] for general concepts about LMI’s and their numerical solution.

We now consider testing the uniform stability of proposed macromodels. We have the following result [112], which we report without proof.

**Theorem 4.4.** The system (4.64) is admissible (i.e. stable, regular and impulse-free, see Appendix D) for all parameter values in the subdomain (4.70) if there exist two matrices \( P = P^T > 0 \) and \( Q \) such that the following set of LMIs holds

\[
(PE^T + SQ)^T A_{pq}^T + A_{pq}(PE^T + SQ) < 0 \quad (4.75)
\]

with \( p = l, l+1, q = m, m+1, \) and where \( S \in \mathbb{R}^{N(N+1) \times N} \) is any matrix with full column rank that satisfies \( ES = 0 \).

In our case, we can choose the explicit form \( S = \begin{bmatrix} 0 & I_N \end{bmatrix}^T \), which provides the simplest expression fulfilling the hypothesis of Theorem 4.4.

A few comments are in order. It is easily recognized that the LMI condition (4.75) collects a set of Lyapunov stability equations, each corresponding to a “corner” model, extracted from (4.64), for the subdomain (4.70). The theorem states that when these Lyapunov equations hold simultaneously with common matrices \( P \) and \( Q \), then the model is uniformly stable in the entire subdomain (4.70). This condition is very powerful, since only a feasibility check of LMI condition (4.75) is required. This operation can be performed with standard convex programming methods [109] in a finite number of steps. In this work, we use the well-known SeDuMi solver [114]. We remark that this procedure avoids possibly unreliable and time consuming brute-force dense sampling in the parameters space, that have to scan the whole space to ascertain model stability.

The uniform stability conditions set by Theorem 4.4 are sufficient but not necessary. Therefore, if these conditions do not hold in a certain subdomain (4.70), one cannot conclude that there is a stability violation for sure. In this case, one can adopt the following strategy to find stability violations. First, the subdomain (4.70) is further divided into
4.3 – Piecewise linear parameterization

Figure 4.9. Cross section of the transmission line example. Signal lines are numbered; reference conductors are grayed.

smaller rectangles, where Theorem 4.4 is applied in order to detect areas where (4.31) is surely stable. Then, if stability is still unknown in some of these small areas, a local sampling can be there performed to localize possible stability violations. We remark that this resort to sampling does not significantly affect the computational cost of the proposed uniform stability test. If fact, sampling is only applied in some small zones, where a few check points are sufficient. Moreover, since model is likely to be unstable in these areas, only a few check points must be scanned to find a stability violation and stop the sweep since the model instability has been proved.

4.3.7 Application example: coupled wires

We consider a multi-conductor transmission line composed by six parallel wires; the conductors are 2 cm long and have a diameter of 1 mm. The cross section is depicted in Figure 4.9 and depends on the two free variables $x$ and $y$, which represent the horizontal and vertical spacing between the conductors. The two inner conductors are the signal lines, while the other four act as reference. This example reproduces the simplified geometry of a high-speed connector, with varying horizontal and vertical pitch.

The purpose of this example is twofold. First, we want to test the algorithm accuracy on this 4-port structure that exhibits both large and small $S$ parameters, being the transmission coefficients and the lines crosstalk. Second, this structure will show on a practical example the low computational complexity of the proposed approach, which allows the quick identification of a two-parameters model for the whole 4-port structure.

The scattering parameters of the line were computed up to 20 GHz for several values of the two design parameters. The horizontal spacing $x$ was swept between 3.5 and 4.5 mm at steps of 0.1 mm. The vertical spacing from 2.5 up to 3.5 mm with the same 0.1 mm step. Out of these responses, those for $x \in \{3.5,3.7,3.9,4.1,4.3,4.5\}$ mm and $y \in \{2.5,2.7,2.9,3.1,3.3,3.5\}$ mm were used to identify a parameterized macromodel, as a multivariate function of frequency, $x$ and $y$. The other responses were instead used as a validation set, in order to assess the model quality in the whole parameters range once the identification was completed.

A parameterized model of order 16 was identified for the entire 4-port structure. The maximum modeling error turned out to be $9.3 \times 10^{-3}$ on the fitting responses and $9.7 \times 10^{-3}$.
on the validation responses. The modeling error for all available configurations of $x$ and $y$ is shown in Fig. 4.11. Figure 4.10 further demonstrates the excellent accuracy of the computed macromodel, which is able to precisely reproduce the weak far-end crosstalk $S_{14}$ between the two signal lines. Thanks to the low computational complexity of the proposed algorithm, the identification of the 4 port multivariate model took only 19 s on a 1.8 GHz laptop with 1.5 GB of memory. This is a very remarkable result compared to the technique presented in Section 4.2 that in this case would have required about 2 GB of memory just to allocate the matrix associated with the fitting equations (4.20).

The uniform stability of the model in the parameters range was assessed with the LMI based algorithm proposed in Sec. 4.3.6. In all subdomains (4.70) the model turned out to be uniformly stable, as further confirmed by Figure 4.12, where the model poles are depicted for several values of the parameters. A brute-force stability test was also performed by sweeping the two parameters in (4.5) at steps of 0.02 mm. Model stability
was confirmed also by this test, that however is less accurate than the LMI-based test and more time consuming. For this example, the algorithm devised in Sec. 4.3.6 took only 63 s, while the brute force check took more than 5 minutes.
4.3.8 Application example: RF circuit block

As a second example we consider a seven-ports RF circuit block located in several multimode GSM and EDGE transceivers for wireless applications, built in 130 nm CMOS technology (courtesy of Dr. P. Brenner, Infineon Technologies AG). A brief description of the circuitry ports is provided in Table 4.2. The small-signal scattering matrix depends on the bias level $V_B$, which is a free parameter ranging from 0.15 V up to 1 V. The device exhibits a strong nonlinear behavior with respect to the $V_B$ parameter and is thus a good benchmark for the presented technique.

In order to construct a bias-dependent parameterized macromodel, the scattering parameters of the linearized device were computed from 0 up to 40 GHz for increasing bias values $V_B$ in 25 mV steps. An additional set of responses for intermediate values of $V_B$ was then used to validate the model quality after identification.

<table>
<thead>
<tr>
<th>Port</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>1,2</td>
<td>Differential input</td>
</tr>
<tr>
<td>3,4</td>
<td>Differential output</td>
</tr>
<tr>
<td>5</td>
<td>Block enable/disable</td>
</tr>
<tr>
<td>6</td>
<td>Power supply (high)</td>
</tr>
<tr>
<td>7</td>
<td>Power supply (low)</td>
</tr>
</tbody>
</table>

Table 4.2. Ports description for the RF device of Sec. 4.3.8.

Figure 4.13 compares the transmission coefficient $S_{31}$ of a parameterized model of order 6 with the original data, for the validation values of $V_B$. An excellent match between the model behavior and the reference data can be observed, even if the model response changes quite rapidly, as can be observed in Fig. 4.14, where the magnitude of the macromodel $S_{31}$ parameter is depicted versus frequency and $V_B$. The computed model turned out to be very accurate on all 49 elements of the S matrix, with a maximum error between the model and the validation data of $3.2 \times 10^{-3}$. In Fig. 4.15 the reflection coefficient at port 4 of the model is shown versus frequency and parameter $V_B$. With the proposed technique, the model was identified in only 6 s, since standard Vector Fitting was applied sequentially to the different responses, minimizing both computational cost and memory consumption, that was negligible. This result shows again the potential of this algorithm in dealing with devices with large ports count. Using the technique of Section 4.2 for this modeling problem would require the solution of some fully coupled least squares problems with about 230000 constraints and more than 700 unknowns, for a memory consumption larger than 1.3 GB.

Finally, application of the uniform stability check proposed in Section 4.3.6 showed that the computed model is stable uniformly over the whole parameter range $V_B \in [0.15,1]V$. 

96
4.4 Conclusions

This Chapter introduced two different strategies for the construction of parametric macromodels from tabulated frequency data, able to include in symbolic form the dependence of the original system behavior from one or more design parameters. The first algorithm generalizes the well known Sanathanan-Koerner iteration, while the second reduces the parametric identification problem to a series of standard macromodeling tasks, allowing
for a significant reduction of the computational cost of the process. For both methods, we discussed the problem of stability, giving purely algebraic conditions for uniform stability in the whole parameters space. Several examples demonstrated the accuracy and robustness of the proposed algorithm for both passive components of interconnect networks as well as active linear devices.
4.4 – Conclusions

Figure 4.15. Real (top panel) and imaginary (bottom panel) part of the $S_{44}$ parameter for the RF device model.
4 - Parametric macromodels
Chapter 5

Macromodels with delays

The design of new generation electronic systems often implies several system-level simulations, with a lot of components and interconnects involved. As optimization procedures, these simulations may be very time consuming and significantly affect the product design time, time-to-market, and competitiveness in the fast-evolving global market. The main simulation bottleneck are long interconnect chains, whose electrical length may be very large. In this case, standard purely rational macromodeling techniques such as Vector Fitting lead to very inefficient models with an unacceptable number of poles. Moreover, a pure rational approximation of a structure that is characterized by a significant propagation delay inevitably leads to a non-zero response before the time-of-flight has elapsed. This effect, which is intrinsic in the structure of the model, may be the source of serious accuracy degradation in system-level simulations [72]. The relevance of these issues grows with the interconnect length and the maximum operating frequency.

The above difficulties are easily overcome for interconnects that may be represented as transmission lines, governed by the telegraphers’ equations. In such case, there exist several techniques that explicitly extract the propagation delays [25–29, 32, 115], leading to macromodels that structurally take into account the correct physics of signal transmission. Unfortunately, not all electrically long interconnects may be represented with pure transmission line models. Examples can be buses made of a chain of various blocks, as typically found in all systems for chip-to-chip, chip-to-memory or even larger scale links, or equivalently, interconnects including discontinuities along their path.

In this Chapter, we extend standard purely rational macromodeling techniques to the case of electrically long interconnects. We consider a model structure that explicitly includes propagation delay terms, mixed with suitable rational coefficients [70, 71, 116]. The resulting delayed rational approximation is computed from raw tabulated frequency data using an iterative weighted linear least squares process analogous to the parametric Sanathanan-Koerner iteration developed in the previous chapter. In addition to the standard implementation, denoted as Delayed Sanathanan-Koerner (DSK) iteration, we will also propose a modified version with a poles relocation step, denoted as Delayed Vector Fitting (DVF) that generalizes the purely rational VF algorithm. This second algorithm guarantees by construction the stability and causality of the computed model.
The material in this Chapter is taken from [4,10] and is organized as follows. Section 5.1 introduces the model structure. Section 5.2 describes the main model identification algorithm. Section 5.3 presents the SPICE netlist synthesis process. Finally, Section 5.4 applies the proposed scheme to several test cases and real application examples.

5.1 Macromodels with delays

We consider an arbitrary electrically long interconnect with \( N \) input/output ports and represented by an unknown transfer function \( H(s) \). Our aim is to identify an approximation of \( H(s) \) from the sampled frequency response of the system, available from numerical simulation or direct measurement. Let us denote the available response samples as

\[
H_k \in \mathbb{C}^{N \times N},
\]

and the available frequency points as

\[
\omega = \omega_1, \ldots, \omega_k.
\]

Without loss of generality, we restrict our attention to a single element \( H_{ij}(s) \) of the transfer function. Results will be then generalized to the most general multiport case by applying the proposed algorithm to each element of the transfer function independently, and then combining the results in a multiport model. In order to simplify the notation, we will drop the subscripts \( ij \).

We assume that the interconnect is structured as a chain of cascaded blocks [71]. Each of these basic elements can be a transmission-line structure, a lumped block, or another (simpler) electrically-long 3D interconnect. For this class of structures, it can be shown (see Appendix E) that each element of the transfer function can be written as

\[
H(s) = \sum_m Q_m(s)e^{-sT_m},
\]

where \( T_m \) represent the physical delays due to the propagation of the electromagnetic field inside the structure. These delays are properly defined in time domain, and measure the time taken by an electromagnetic wave to travel from one port to another port through the interconnect [117]. In addition, since the interconnect is not assumed to be homogeneous, but composed by several different blocks, these delays take also into account the multiple reflections a wave may experience inside the structure. The terms \( Q_m(s) \) are instead suitable frequency-dependent coefficients representing other effects such as attenuation and dispersion.

The above consideration naturally leads to the macromodel structure that we adopt. Essentially, two approximations are applied to (5.3). First, the number of delays is truncated to a (small) finite number \( \bar{m} \). Second, a rational approximation is applied to each coefficient \( Q_m(s) \) which, in general, is not a rational function. The resulting delayed rational model is further represented as

\[
H(s) \simeq \sum_{m=1}^{\bar{m}} \frac{R_{mn} \phi_m(s) \chi_m(s)}{\sum_{n=0}^{\bar{n}} r_n \phi_n(s)},
\]

where
where $\phi_n(s)$ are partial fractions (4.14) associated to a prescribed set of basis poles $a_n$, and $\chi_m(s)$ is a basis of delay elements

$$\chi_m(s) = e^{-s\tau_m} \quad m = 1, \ldots, \tilde{m}.$$  \hfill (5.5)

The delays in (5.5)

$$\tau_1, \ldots, \tau_{\tilde{m}},$$  \hfill (5.6)

are approximations to the dominant delays in (5.3). This model formulation is similar to (4.35), with the parameter dependent basis factors replaced by the delay elements. We remark that a delayed-rational form can also be obtained by replacing the partial fraction functions $\phi_n(s)$ in (4.14) with polynomials $\{1, s, s^2, \ldots\}$ or other systems of rational basis functions (such as orthogonal rational functions [38]). Polynomials are ruled out here due to possible ill-conditioning of the resulting model fitting equations. Among different sets of rational functions, the partial fractions basis (4.14) is adopted here due to its simple form and its excellent approximation and numerical stability properties. The basis poles $a_n$ are chosen to optimize the numerical conditioning of the model identification process. For this purpose, we resort to the widely adopted solution proposed in [33], with initial poles being linearly distributed over the data bandwidth $[\omega_1, \omega_{\tilde{k}}]$ and close to the imaginary axis.

### 5.2 Model identification

Model identification requires to estimate the coefficients $R_{mn}, r_n$ and $\tau_m$ of the model (5.4) from the raw tabulated data (5.1), such that the deviation between model response and raw data is minimized at the available frequency points. We can define the approximation error at a single frequency point $\omega_k$ as

$$E_k = H_k - \frac{\sum_{n=0}^{\bar{n}} \sum_{m=1}^{\bar{m}} R_{mn} \phi_n(j\omega_k) e^{-j\omega_k \tau_m}}{\sum_n r_n \phi_n(j\omega_k)}. \quad (5.7)$$

The cumulative (RMS) error for the entire response reads

$$E = \sqrt{\frac{1}{k} \sum_{k=1}^{k} |E_k|^2}. \quad (5.8)$$

It is clear that the cumulative error $E$ is a complex nonlinear function of the unknown model coefficients $R_{mn}, r_n$ and $\tau_m$. Main difficulty is the presence of the coefficients $\tau_m$ in the exponentials and the coefficients $r_n$ at the denominator, which are responsible for the representation of the model poles. In order to avoid a direct minimization of (5.8) with a nonlinear optimization routine, we devise an alternative fitting algorithm with lower computational cost and higher robustness. It employs a time-frequency decomposition to estimate first the dominant delays $\tau_m$, and then a generalized Sanathanan-Koerner iteration for the identification of $R_{mn}$ and $r_n$. 

103
5.2.1 Estimation of propagation delays

The first stage for the identification of a delayed macromodel (5.4) is the estimation of the dominant delay terms (5.6) from the raw data. For this task, we adopt the algorithm described in [71], based on the so-called Gabor transform [118]. In this work, we define the Gabor transform starting from frequency domain instead of the more standard time-domain representation, as

\[ G(\omega, \tau) = \int_{-\infty}^{+\infty} H(j\xi)W^*_{\omega, \tau}(\xi)d\xi. \]  

(5.9)

The “basis” functions

\[ W_{\omega, \tau}(\xi) = W(\xi - \omega)e^{-j\xi\tau} \]  

(5.10)

are amplitude-modulated (parameter \( \tau \) is proportional to the number of oscillations) and frequency-shifted (parameter \( \omega \) is the center of the translation) versions of a normalized Gaussian window

\[ W(\xi) = \pi^{-1/4}e^{-\xi^2/2}. \]  

(5.11)

If \( W(\xi) \) were taken to be identically one, the definition in (5.9) would become (up to a normalization constant) exactly the inverse Fourier transform of \( H(j\xi) \), which is the system impulse response \( h(\tau) \). Hence, the variable \( \tau \) has the physical meaning of time or time-delay.

The Gaussian window \( W(\xi) \) in (5.9) plays the role of a sharp bandpass filter. Therefore, \( G(\omega, \tau) \) can be regarded as the inverse Fourier transform of \( H(j\xi) \), but retaining only those frequency components located in a frequency band centered around \( \omega \). For this reason, \( G(\omega, \tau) \) belongs to the class of the so-called time-frequency transforms, since it provides a localization of the various components of \( H \) both in frequency \( \omega \) and time \( \tau \).

Local maxima \((\tilde{\omega}_m, \tilde{\tau}_m)\) of \( |G(\omega, \tau)|^2 \) pinpoint the location in time (delay) and frequency of the dominant energy contributions of \( H(j\omega) \). It turns out that typical interconnect responses are characterized by well-separated single-delay components, see Fig. 5.1 for a graphical illustration. Therefore, the time (delay) coordinates of the local maxima \( \tilde{\tau}_m \) provide good estimates for the individual propagation delays in (5.4).

The number of significant delays can be automatically determined as follows. First, we average the spectrogram \( |G(\omega, \tau)|^2 \) over the available bandwidth \( \Omega \) according to

\[ G^2(\tau) = \frac{1}{2\pi} \int_{\omega} |G(\omega, \tau)|^2d\omega. \]  

(5.12)

Then, starting from any local maximum \( \tilde{\tau}_m \), we determine the closest local minimum \( \tau_m^- \) of \( G_\omega(\tau) \) such that \( \tau_m^- < \tilde{\tau}_m \). The energy content of the \( m \)-th individual delay term is thus estimated as

\[ G_m^2 = \frac{1}{2\pi} \int_{\tau_m^-}^{\tilde{\tau}_m} G_\omega^2(\tau)d\tau. \]  

(5.13)

All delay terms such that their relative contribution exceeds a predefined threshold \( \gamma \)

\[ \frac{G_m^2}{\sum_m G_m^2} > \gamma \]  

(5.14)
Figure 5.1. Magnitude of Gabor coefficients for the return loss $S_{2,2}$ of a measured PCB interconnect. The same example will be analyzed in Section 5.4.2.

are retained in the model. Since the neglected energy contributions are small, this procedure does not significantly affect the accuracy of the final model. However, model efficiency is optimized, since the number of terms in (5.4) is minimal. More details on the actual implementation can be found in [71].

5.2.2 Model coefficients identification

Once the set of dominant delays (5.6) is known, the fundamental task is the estimation of the coefficients $R_{mn}$ and $r_n$ in (5.4). This problem is solved here in a way similar to the identification of parametric macromodels of Section 4.2, i.e. by an iterative weighting process similar to the Sanathanan-Koerner (SK) iteration [35]. An outer iteration loop is devised. The $i$-th pass of this loop minimizes a modified error metric obtained by multiplying (5.7) by a weighting factor

$$\mathcal{W}_k^{(i)} = \frac{\sum_{n=0}^{\bar{n}} r_n^{(i)} \phi_n(j \omega_k)}{\sum_{n=0}^{\bar{n}} r_n^{(i-1)} \phi_n(j \omega_k)},$$

which is the ratio between the (unknown) denominator at current iteration and the known denominator at previous iteration $i - 1$. At the first iteration we set

$$r_n^{(0)} = \begin{cases} 1 & \text{for } n = 0 \\ 0 & \text{for } n = 1, \ldots, \bar{n} \end{cases}$$

(5.16)
as an initialization. The following single-frequency weighted error

\[ E_k^{(i)} = \frac{H_k \sum_n r_n^{(i)} \phi_n(j\omega_k) - \sum_n \sum_m R_m^{(i)} \phi_n(j\omega_k)e^{-j\omega_k \tau_m}}{\sum_n r_n^{(i-1)} \phi_n(j\omega_k)} \]  

(5.17)

is obtained. The above error is linear in the unknowns \( R_m^{(i)} \) and \( r_n^{(i)} \), therefore its minimization is readily achieved by the solution of a standard linear least squares system, whose \( k \)-th row is the right-hand-side of (5.17). As an implementation detail, we add the following non-triviality constraint

\[ \frac{1}{k} \sum_k \Re \left\{ \frac{\sum_n (r_n^{(i)} - 1) \phi_n(j\omega_k)}{\sum_n r_n^{(i-1)} \phi_n(j\omega_k)} \right\} = 0 \]  

(5.18)

as a last row in the least squares system, in order to rule out the all-vanishing solution. This constraint is analogous to (4.24).

Iterations are stopped when all coefficients of the model representation are stabilized. Note that, upon convergence of the coefficients \( r_n^{(i)} \), the weighting factor \( W_k \) tends to one uniformly, and the weighted least squares problem (5.17) becomes equivalent to the original formulation (5.7).

### 5.2.3 Stability enforcement: the DSK and DVF schemes

Let us take a closer look at the rational approximation of a single-delay element

\[ Q_m(s) \simeq R_m0 + \sum_{n=1}^{\bar{n}} \frac{R_m}{s - a_n} \]  

(5.19)

As we saw in Chapter 2, stability require that the poles \( p_n \) of this rational function are constrained to the \( \Re \{s\} < 0 \) region of the complex plane. This condition is readily checked by explicitly computing the poles, i.e., zeros of the denominator, which are the eigenvalues of matrix

\[ \begin{pmatrix} 
    a_1 - \rho_1 & -\rho_2 & \cdots & -\rho_{\bar{n}} \\
    -\rho_1 & a_2 - \rho_2 & \cdots & -\rho_{\bar{n}} \\
    \vdots & \vdots & \ddots & \vdots \\
    -\rho_1 & -\rho_2 & \cdots & a_{\bar{n}} - \rho_{\bar{n}} 
\end{pmatrix}, \]  

(5.20)

where \( \rho_n = r_0^{-1} r_n \). In case of unstable poles, they are flipped into the left hand complex plane, and the corresponding new set of coefficients \( r_n \) in (5.19) is derived. This procedure forces the denominator to be a minimum phase rational function [33, 119, 120].

Two alternatives are possible for stepping through iterations. The first choice preserves the basis poles \( a_n \) through the iterations. The resulting scheme is a direct generalization of the SK iteration, which is therefore denoted as DSK (Delayed Sanathanan-Koerner) algorithm. The DSK scheme may be more robust, but unstable poles may appear at each iteration. The second choice updates also the “basis” poles \( a_n \) and, consequently, the
5.3 – Model synthesis

partial fractions $\phi(s)$ of (4.14) at each iteration. This update is performed as follows. Starting from the poles $a_{n(i-1)}$, the numerator of the weighting function (5.15) is formed as

$$\sigma(s) = r_0 + \sum_{n=1}^{\hat{n}} \frac{r_{n(i)}}{s - a_{n(i-1)}}.$$  (5.21)

The zeros of this function define the set of poles $a_{n(i)}$ to be used at next iteration. It is easily recognized that this scheme is a generalization of VF, which is thus denoted as DVF (Delayed Vector Fitting). The DVF produces guaranteed stable models by construction, provided that the poles $a_{n(i)}$ are constrained to the left hand plane by a suitable flipping process. Both DSK and DVF schemes will be applied to the various examples of Section 5.4 and compared.

5.2.4 Causality and passivity

In Chapter 2 we remarked the strong importance of passivity in macromodeling. For the class of structures under investigation, passivity conditions require both the causality of each response and the non-expansivity (no energy gain) of the overall transfer matrix. Causality conditions are satisfied by construction by the proposed algorithm, since each term in the model (5.3) is causal, because of the positivity of the delays $\tau_n$ and the stability of the model poles. If we assume that the raw data are passive, any possible passivity violation of the model will be small and of the same order of the approximation error that is achieved in the identification process. Therefore, passivity enforcement can be obtained using a perturbation approach, which is a standard practice for lumped macromodels [43–48, 119, 120]. Recent developments [30, 31, 76, 77, 121] extended such techniques to the class of delay-based macromodels, with specific attention to transmission-line macromodels based on the Method of Characteristics [30, 31]. Since the proposed model structure (5.4) falls in this class, it is argued that the perturbation approach of [30, 31] can be applied here after a suitable modification, that will be subject of future studies.

5.3 Model synthesis

In this Section, we derive a compact SPICE-compatible circuit stamp for the delayed macromodel form (5.4). The first step is to rewrite the model expression as a summation
of delayed partial fraction expansions

\[ Y(s) = \sum_{m=1}^{\bar{m}} \left( K_{m0} + \sum_{n=1}^{\bar{n}} \frac{K_{mn}}{s - p_n} \right) e^{-s\tau_m} U(s), \]  

(5.22)

where \( U(s) \) and \( Y(s) \) are the Laplace-domain input excitation and output response of the model. Due to the adopted model representation, the poles \( p_n \) are common to all delay terms. This allows us to decouple the synthesis of the rational part from the synthesis of the delays. In the following, we only provide details for real poles \( p_n \), the case of complex poles being a trivial extension. We rewrite (5.22) as

\[ Y(s) = \sum_{m=1}^{\bar{m}} \left( K_{m0} U(s) + \sum_{n=1}^{\bar{n}} K'_{mn} U_n(s) \right) e^{-s\tau_m}, \]  

(5.23)

with \( K'_{mn} = -K_{mn} p_n^{-1} \) and where

\[ U_n(s) = \frac{1}{1 - s/p_n} U(s) \]  

(5.24)

represents the signal at the output of one-pole lowpass filter, which can be synthesized by the single RC cell depicted in Fig 5.2.

Depending on the adopted SPICE platform, the delay element can be synthesized in various ways. If delayed controlled sources are available, the synthesis of (5.23) is direct. If instead such elements are not available, delays can be synthesized using ideal transmission lines elements. We explore this second option in the following.

We need \( \bar{m} \) different transmission line segments, since there are \( \bar{m} \) distinct delays in (5.23). Application of a single delay \( \tau_m \) is achieved by the circuit depicted in Fig. 5.3, representing a transmission line with unitary characteristic impedance, which is matched...
at both ends in order to avoid spurious signal reflections. The line is excited by a set of controlled sources which reproduce the summation over $n$ in (5.23).

The outputs $Y_m(s)$ of the $\bar{m}$ delay lines are finally collected and reported to the output port via another set of controlled sources. The schematic of Fig. 5.4 represents the circuit that is directly connected to the output port, valid for scattering representations. In such case the model input is the impinging port wave, defined as $U(s) = R_0^{-1}V(s) + I(s)$, which feeds the single RC cells of Fig. 5.2. Since the matched ideal transmission line in Fig. 5.3 produces a current division factor equal to 0.5, the gain of the controlled sources in Fig. 5.4 is defined with a correction factor 2 to obtain a full equivalence to (5.23).

In case of multiport networks, the same synthesis process is performed for each pair of input-output couplings. This procedure is straightforward and not further commented here.

5.4 Examples

5.4.1 Synthetic lumped-distributed network

This first example is intended to validate the delayed model identification approaches presented in Section 5.2.2. We consider the structure depicted in Fig. 5.5, made of a chain of two transmission line segments with a capacitive discontinuity inbetween. The frequency-dependent per-unit-length parameters of the two transmission lines have been computed from DC up to 10 GHz using a surface-based MoM$^1$ solver based on [122]. Conductor skin and proximity effects are explicitly taken into account. From this computation, we also obtain the asymptotic values of capacitance and inductance,

$$C_\infty = 4.47 \times 10^{-11} \text{ F}, \quad L_\infty = 2.49 \times 10^{-7} \text{ H}, \quad (5.25)$$

which are used to derive the nominal propagation delays of the two line segments

$$T_1 = \ell_1 \sqrt{C_\infty L_\infty} = 1.67 \times 10^{-8} \text{ s},$$

$$T_2 = \ell_2 \sqrt{C_\infty L_\infty} = 2.34 \times 10^{-8} \text{ s}. \quad (5.26)$$

---

$^1$MoM: Method of Moments.
We can derive analytically the complete set of delays to be used in the macromodel expression,

\[
\tau^{1,1} = \tau^{2,2} = \left\{ \sum_{i=1}^{2} 2 m_i T_i : m_i \geq 0 \right\},
\]

\[
\tau^{1,2} = \tau^{2,1} = \left\{ \sum_{i=1}^{2} (2 m_i + 1) T_i : m_i > 0 \right\},
\]  

where the superscripts indicate the respective scattering response. It is to be noted that, for \(\tau^{1,1}\) not all the combinations of \(m_i\) are possible, in particular \(m_2 > 0\) only if \(m_1 > 0\). The same applies to \(\tau^{2,2}\), for which \(m_1 > 0\) only if \(m_2 > 0\).

A total number \(\bar{k} = 5001\) frequency samples with \(\bar{m} = 6\) delay terms were used in the DSK and DVF model identification. We report in Table 5.1 the number of model poles used in the rational approximation, and the corresponding RMS approximation errors obtained with the DSK and DVF algorithms. The table includes also the results of the standard VF scheme for comparison.

<table>
<thead>
<tr>
<th>Order</th>
<th>RMS Error ×10^{-3}</th>
</tr>
</thead>
<tbody>
<tr>
<td>VF</td>
<td>DSK</td>
</tr>
<tr>
<td>(S_{1,1})</td>
<td>1046</td>
</tr>
<tr>
<td>(S_{1,2})</td>
<td>1124</td>
</tr>
<tr>
<td>(S_{2,2})</td>
<td>1208</td>
</tr>
</tbody>
</table>

Table 5.1. Macromodeling results for the structure depicted in Fig. 5.5

The approximation errors are well below engineering accuracy for all three algorithms. However, application of standard VF algorithm requires a very large number of poles for obtaining a purely rational approximation of the terminal scattering responses with a level of accuracy that is comparable with the DSK and DVF results. Conversely, due to the explicit extraction of the propagation delays, both DSK and DVF achieve excellent accuracy with a very small number of poles.

Figures 5.6–5.8 compare the frequency-responses of the delayed macromodel to the raw data used for the model identification. We only report the results of DVF algorithm, since both VF and DSK results appear identical on this scale. Also, only one tenth of the modeling bandwidth is displayed for readability of the plots. Similar results are obtained over the full bandwidth up to 10 GHz. These statements are confirmed by Table 5.1, which reports error metrics computed for all models over the entire bandwidth.
Figure 5.6. Comparison between DVF model and data for the scattering element \( S_{11} \) of structure depicted in Fig. 5.5. Only a reduced bandwidth of 1 GHz is depicted in the Figure for readability.
Figure 5.7. As in Figure 5.6, but for $S_{12}$. 
Figure 5.8. As in Figure 5.6, but for $S_{22}$. 
5.4.2 Measured PCB interconnect

The second example is a 10 cm long PCB interconnect. The structure, which includes signal launches and discontinuities, is characterized via measured scattering responses up to 40 GHz (courtesy of Prof. C. Schuster, formerly IBM). A total number of \( k = 801 \) frequency samples are available. For this case, the delays are not known a priori and must be inferred from the data. The estimation procedure of Section 5.2.1 leads to the results of Table 5.2. Figure 5.1 reports an illustration of the time-frequency energy localization for \( S_{2,2} \), which leads to the delay estimates reported in the table.

<table>
<thead>
<tr>
<th></th>
<th>( \tau_1 )</th>
<th>( \tau_2 )</th>
<th>( \tau_3 )</th>
<th>( \tau_4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( S_{1,1} )</td>
<td>0.00 ns</td>
<td>1.3 ns</td>
<td>2.64 ns</td>
<td>4.08 ns</td>
</tr>
<tr>
<td>( S_{1,2} )</td>
<td>0.66 ns</td>
<td>1.99 ns</td>
<td>3.37 ns</td>
<td>–</td>
</tr>
<tr>
<td>( S_{2,2} )</td>
<td>0.00 ns</td>
<td>1.3 ns</td>
<td>2.64 ns</td>
<td>4.08 ns</td>
</tr>
</tbody>
</table>

Table 5.2. Delays estimates for the PCB interconnect of Sec. 5.4.2.

Table 5.3 reports the modeling parameters and results of the VF, DSK, and DVF algorithms. The level of approximation is excellent, taking into account that modeling accuracy cannot be as low as desired due to the measurement noise floor. Figures 5.9-5.11 compare the DVF model responses to the corresponding raw measured data. Also for this case, no visible difference is evident from the plots. This applies also to the DSK and VF results, not reported. A confirmation is provided by Fig. 5.12, which reports the frequency-dependent model vs data error obtained by all three algorithms for the \( S_{1,2} \) response.

<table>
<thead>
<tr>
<th></th>
<th>Order</th>
<th>RMS Error ( \times 10^{-2} )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>VF</td>
<td>DSK</td>
</tr>
<tr>
<td>( S_{1,1} )</td>
<td>72</td>
<td>16</td>
</tr>
<tr>
<td>( S_{1,2} )</td>
<td>72</td>
<td>16</td>
</tr>
<tr>
<td>( S_{2,2} )</td>
<td>76</td>
<td>16</td>
</tr>
</tbody>
</table>

Table 5.3. Macromodeling results for the PCB interconnect of Sec. 5.4.2
Figure 5.9. Comparison between DVF model and data for the return loss $S_{11}$ of the PCB interconnect of Sec. 5.4.2.
Figure 5.10. As in Fig. 5.9, but for insertion loss $S_{12}$. 
5.4 – Examples

Figure 5.11. As in Fig. 5.9, but for $S_{22}$.

Figure 5.12. Model vs data deviation for the scattering element $S_{12}$ of the PCB interconnect of Sec. 5.4.2.
5.4.3 Complex bus

We now consider a complex bus structure, namely the IBM GX bus. The raw specification is a set of frequency-dependent scattering parameters (courtesy of IBM), obtained by cascading several different simpler models of lumped blocks and frequency-dependent transmission lines, and performing a simple frequency-domain solution of the interconnected system. Main task is to compute a global model from the terminal responses of the entire bus, without using any information on the internal structure.

<table>
<thead>
<tr>
<th></th>
<th>$\tau_1$</th>
<th>$\tau_2$</th>
<th>$\tau_3$</th>
<th>$\tau_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_{1,1}$</td>
<td>0.00 ns</td>
<td>0.56 ns</td>
<td>5.44 ns</td>
<td>6.71 ns</td>
</tr>
<tr>
<td>$S_{2,2}$</td>
<td>0.00 ns</td>
<td>1.29 ns</td>
<td>5.87 ns</td>
<td>–</td>
</tr>
<tr>
<td>$S_{1,2}$</td>
<td>2.73 ns</td>
<td>–</td>
<td>–</td>
<td>–</td>
</tr>
</tbody>
</table>

Table 5.4. Delays estimates for the complex bus of Sec. 5.4.3.

For this structure, the number of dominant delays varies depending on the considered response. Table 5.4 summarizes the various delays estimates that are used to extract the model.

Both delayed macromodeling schemes DSK and DVF and the classical strictly rational VF were applied, in order to compare model accuracy and complexity. The model identification results are summarized in Table 5.5 for the modeled scattering responses. The results show that a significant saving in terms of number of poles is achieved by DSK and DVF with respect to standard VF. The DSK produces a model that is not stable due to the presence of poles with positive real part. Conversely, the DVF model is stable and thus represents the best compromise between model accuracy and complexity.

<table>
<thead>
<tr>
<th></th>
<th>Order</th>
<th>RMS Error $\times 10^{-3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>VF</td>
<td>DSK</td>
</tr>
<tr>
<td>$S_{1,1}$</td>
<td>80</td>
<td>20</td>
</tr>
<tr>
<td>$S_{1,2}$</td>
<td>48</td>
<td>15</td>
</tr>
<tr>
<td>$S_{2,2}$</td>
<td>66</td>
<td>25</td>
</tr>
</tbody>
</table>

Table 5.5. Macromodeling results for the complex bus of Sec. 5.4.3

Figures 5.13-5.15 report a comparison between the global DVF model and the raw data, showing excellent correlation. Figure 5.16 reports the magnitude of the $S_{11}$ model vs data errors, which are uniformly bounded over frequency for all VF, DSK, and SVF algorithms.
Figure 5.13. Comparison between model and data for the return loss $S_{11}$ of the complex bus of Sec. 5.4.3.
Figure 5.14. As in Fig. 5.13, but for insertion loss $S_{12}$. 

---

5 – Macromodels with delays
Figure 5.15. As in Fig. 5.13, but for return loss $S_{22}$.

Figure 5.16. Model vs data deviation for the scattering element $S_{11}$ of the complex bus of Sec. 5.4.3.
5.4.4 Transient analysis

We consider the three interconnect structures discussed in Sections 5.4.1, 5.4.2, and 5.4.3, and we compare the performance in terms of accuracy and simulation time of the SPICE netlists corresponding to the VF and DVF macromodels. In all cases, we adopt the same model termination scheme, in order to draw meaningful conclusions. In particular, all model ports are matched into their reference resistance $R_0 = 50 \, \Omega$. Each model is excited at one port using a single pulse, and the received voltage is monitored at the other interconnect port.

The results of the SPICE transient simulations are reported in Figures 5.17, 5.18, 5.19 and 5.20. We also include an additional structure, which is the same PCB interconnect of Sec. 5.4.2, but with a length of 50 cm. This structure is included with the aim of relating the macromodeling efficiency with the electrical length of the interconnect. In all cases, a quite good match is observed between the VF and DVF model.

However, a closer look at the results reveals that VF models produce spurious oscillations before the expected propagation delay has elapsed, since this delay is only approximated by VF via a finite-order rational function. Figure 5.21 illustrates this for the 50 cm long PCB interconnect. Conversely, the DVF models are exempt from this inconsistency, since the delays are explicitly extracted and accounted for in the model.

We conclude by presenting in Table 5.6 a summary of the CPU time required for the various transient simulations. It can be observed that the speedup factor of DVF with respect to standard VF macromodels scales almost linearly with the electrical size of the interconnect at the highest frequency of interest.

<table>
<thead>
<tr>
<th>Structure</th>
<th>VF</th>
<th>DVF</th>
<th>Speedup</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sec. 5.4.1</td>
<td>113 s</td>
<td>4.47 s</td>
<td>25X</td>
</tr>
<tr>
<td>Sec. 5.4.2 (10 cm)</td>
<td>2.77 s</td>
<td>1.30 s</td>
<td>2.1X</td>
</tr>
<tr>
<td>Sec. 5.4.2 (50 cm)</td>
<td>7.49 s</td>
<td>0.81 s</td>
<td>9.3X</td>
</tr>
<tr>
<td>Sec. 5.4.3</td>
<td>1.84 s</td>
<td>0.80 s</td>
<td>2.3X</td>
</tr>
</tbody>
</table>

Table 5.6. Transient SPICE simulation time

5.5 Conclusions

This Chapter tackled the open problem of the identification of distributed macromodels for long interconnect systems. The modeling strategy is based on a combination of propagation delay extraction and rational approximation in the frequency domain. The explicit inclusion of delay terms allows for a direct representation of rapid phase variations, whereas slowly varying variations are captured by low-order rational coefficients. Model identification is performed directly on terminal scattering responses via iterative solution of suitably weighted linear least-squares problems. This process leads to particularly compact models that can be readily synthesized into SPICE-compatible netlists, allowing for
5.5 – Conclusions

Figure 5.17. Transient SPICE results for the distributed circuit of Sec. 5.4.1

Figure 5.18. Transient SPICE results for the measured PCB interconnect of Sec. 5.4.2

significant reduction of simulation times with respect to more standard purely rational macromodeling techniques.
Figure 5.19. As in Fig. 5.18 but with an interconnect length of 50 cm

Figure 5.20. Transient SPICE results for the bus structure of Sec. 5.4.3.
Figure 5.21. Zoom of Fig. 5.19.
Conclusions

This thesis led to several improvements of the state of the art in the macromodeling and simulation of high-speed interconnects. The detailed study on the causality, stability and passivity of macromodels showed the significant impact that physical consistency has on Computer Aided Design (CAD) techniques, providing a clear explanation to several pitfalls commonly encountered with macromodeling algorithms. This study received the 2007 Best Paper Award of the IEEE Transactions on Advanced Packaging.

As a complement to the theoretical analysis, we developed new algorithms for physical consistency validation of tabulated frequency data, with excellent accuracy and robustness compared to previously published techniques. They are suitable to harden Computer Aided Design workflows as well as to improve the reliability of measurement and simulation processes through the detection of causality and passivity inconsistencies. In order to devise these algorithms we developed a new numerical method for the computation of generalized dispersion relations, currently the best in the state of the art. Its application to the interpolation with causality constraints received the Best Student Paper Award at the IEEE 15th Topical Meeting on Electrical Performance of Electronic Packaging.

Capitalizing on this work on physical consistency, we proposed new macromodeling algorithms for the efficient simulation and design of complex interconnect networks. This aim was pursued along two directions: the parameterization of macromodels and the inclusion of delay elements. The first characteristic is crucial to perform fast optimizations of complex interconnect links. We proposed two numerical algorithms to create parametric models from tabulated data, that provide excellent performance in terms of accuracy and computational complexity. We also identified algebraic conditions for uniform stability, leading to a fast and accurate algorithm for stability verification of parametric macromodels.

The inclusion of delay elements in macromodels, an open problem of great industrial relevance, was solved with the invention of Delayed Vector Fitting, one of the first algorithms able to produce distributed macromodels for long interconnects with possibly non-uniform structure. This technique was shown to reduce simulation times for long links by at least one order of magnitude. This contribution received the Best Paper Award at the IEEE 17th Topical Meeting on Electrical Performance of Electronic Packaging.

Finally, all proposed techniques have been implemented in commercial CAD softwares, including the circuit simulator Nexxim® of the high-frequency CAD market leader Ansoft, and have become part of the design workflow of several electronic industries.
Appendix A

The Vector Fitting algorithm

Vector Fitting is currently the most popular algorithm for the identification of rational macromodels from tabulated frequency data. From a set of samples of the system frequency response of an $N$-ports linear device

$$H_k \in \mathbb{C}^{N \times N}, \quad (A.1)$$

and the corresponding frequency points

$$\{\omega_k\} \quad k = 1, \ldots, \bar{k}, \quad (A.2)$$

Vector Fitting identifies a rational macromodel in poles and residues form

$$H(s) = R_0 + \sum_{n=1}^{\bar{n}} \frac{R_n}{s - p_n}, \quad (A.3)$$

such that the model response approximates the given samples (A.1)

$$H(j\omega_k) = R_0 + \sum_{n=1}^{\bar{n}} \frac{R_n}{j\omega_k - p_n} \simeq H_k. \quad (A.4)$$

The approximation condition (A.4) is to be intended in least squares sense. Since both the residues $R_n$ and poles $p_n$ are unknown, (A.4) is a nonlinear problem. For a direct solution of (A.4) one must resort to a method such as Newton’s iteration. This choice unfortunately has several drawbacks, including high computational cost and poor robustness. Vector Fitting instead solves (A.4) via the iterative solution of linear least squares problems only.

First, we present the Vector Fitting algorithm outline, and then we will comment each step in details.

1) We set $i = 1$ and define two rational functions $\sigma H^{(i)}(s)$ and $\sigma^{(i)}(s)$ with common poles as

$$\sigma H^{(i)}(s) = Q_0^{(i)} + \sum_{n=1}^{\bar{n}} \frac{Q_n^{(i)}}{s - a_n^{(i)}} \quad (A.5)$$
\[ \sigma^{(i)}(s) = 1 + \sum_{n=1}^{\hat{n}} \frac{q^{(i)}_n}{s - a^{(i)}_n} \]  
(A.6)

These functions represent respectively the numerator and denominator of the model estimate at iteration \(i\)

\[ H^{(i)}(s) = \frac{\sigma H^{(i)}(s)}{\sigma^{(i)}(s)} . \]  
(A.7)

The poles \(a^{(1)}_n\) are linearly distributed over the bandwidth spanned by the input frequency points (A.2), with real part much smaller than the imaginary part [33].

2) Equation

\[ \sigma H^{(i)}(j\omega_k) \simeq H_k \sigma^{(i)}(j\omega_k) \]  
(A.8)

is solved in least squares sense for the unknowns \(Q^{(i)}_n\) and \(q^{(i)}_n\).

3) The poles of (A.5) and (A.6) are relocated using the zeros of \(\sigma^{(i)}(s)\)

\[ a^{(i+1)}_n = \text{zeros}\{\sigma^{(i)}(s)\} , \]  
(A.9)

and constrained to have negative real part by

\[ a^{(i+1)}_n = -\left| \text{Re}\left\{a^{(i+1)}_n\right\}\right| + j\text{Im}\left\{a^{(i+1)}_n\right\} \]  
(A.10)

4) \(i = i + 1\), and steps 2 and 3 are repeated until

\[ \sigma^{(i)}(s) \rightarrow 1 . \]  
(A.11)

5) The poles \(p_n\) of (A.3) are obtained as

\[ p_n = a^{(i)}_n ; \]  
(A.12)

6) Equation

\[ \mathbf{R}_0 + \sum_{n=1}^{\hat{n}} \frac{\mathbf{R}_n}{j\omega_k - p_n} \simeq \mathbf{H}_k \]  
(A.13)

is solved in least squares sense for the residues \(\mathbf{R}_n\) of the final macromodel (A.3).

Step 1 defines the two rational functions with common and known poles used to represent the model numerator and denominator. Function \(\sigma^{(i)}(s)\) plays a particularly important role, since it is used in steps 2 and 3 to estimate the unknown system poles. The equation (A.8) solved at step 2 represents a linearized version of the approximation condition (A.4). Since \(\sigma H^{(i)}(s)\) and \(\sigma^{(i)}(s)\) have the same poles, condition (A.8) will force the zeros of \(\sigma^{(i)}(s)\) to approximate the unknown system poles. This fact is then used in step 3 to relocate the poles of (A.5) and (A.6) closer to the exact poles, and the whole process is repeated until convergence. Usually, only 4-5 iterations are enough. In order to
enforce the stability and causality of the final macromodel, possible poles \( a^{(i)}_n \) with positive real part are flipped in step 3 in the stable half place by equation (A.10). In step 5 the estimated poles are set as the poles of the final macromodel. Once the poles \( p_n \) are known, estimation of the residues \( R_n \) is an easy task since it amounts to the solution of a linear least squares problem (in step 6).

The main advantages of Vector Fitting are the reduced computational cost and the high robustness. Only the solution of a few linear least squares problems is in fact necessary, task that can be performed with well established and robust algorithms based on QR or SVD decomposition [123]. While a mathematical proof of convergence has not been found yet, Vector Fitting has been shown to be a very efficient and reliable algorithm for rational macromodels identification.
A – The Vector Fitting algorithm
Appendix B

Dispersion relations: the bandpass data case

In Section 3.1 we developed a robust numerical method for the computation of dispersion relations with subtractions for bandlimited data, applied then in Sections 3.2, 3.3 for the causality and passivity check of tabulated frequency data. Section 3.1 focused on the baseband data case (3.5), where the samples are known from \( \omega = 0 \) up to a maximum frequency \( \omega_{\text{max}} \). The algorithms proposed in Chapter 3 are also valid in the most general case of bandpass data, with the minor modifications reported in this Appendix.

We consider the case (3.6), for which the frequency response is available in
\[
\Omega = [-\omega_{\text{max}}, -\omega_{\text{min}}] \cup [\omega_{\text{min}}, \omega_{\text{max}}].
\]

If \( L(x) \) of (3.15) is redefined as
\[
L(x) = \ln \left| \frac{(\omega_{\text{max}} + x)(\omega_{\text{min}} - x)}{(\omega_{\text{max}} - x)(\omega_{\text{min}} + x)} \right|, \tag{B.1}
\]
all formulas in Sections 3.1.3 and 3.1.5 remain valid except for the bound (3.19), which includes now an additional term accounting for the missing low frequency data

\[
T_n(\omega) = \frac{M}{\pi} \sum_{q=1}^{n} (\bar{\omega}_q)^\alpha \left( \prod_{p=1, p \neq q}^{n} \frac{\omega - \bar{\omega}_p}{\bar{\omega}_q - \bar{\omega}_p} \right) \times \left\{ \ln \left| \frac{\omega_{\text{max}} - \bar{\omega}_q}{\omega_{\text{max}} - \omega} \right| + (-1)^{\alpha+n+1} \ln \left| \frac{\omega_{\text{min}} + \bar{\omega}_q}{\omega_{\text{max}} + \omega} \right| + (-1)^n \frac{\ln \left( (\omega_{\text{min}} - \bar{\omega}_q) \omega \right)}{(\omega_{\text{min}} - \omega) \bar{\omega}_q} + \left( -1 \right)^n \frac{\ln \left( (\omega_{\text{min}} + \bar{\omega}_q) \omega \right)}{(\omega_{\text{min}} + \omega) \bar{\omega}_q} \right\}, \tag{B.2}
\]
where \( n^+ \) is the number of positive subtractions points \( \bar{\omega}_q > 0 \). This expression can be derived following the same guidelines presented in Appendix C for the baseband case. A detailed proof is therefore omitted to avoid duplications.

We discuss now the displacement of subtraction points that should be adopted in the bandpass case in order to minimize the truncation error. For simplicity, we consider an
even number of subtractions \( n \), symmetrically placed around \( \omega = 0 \). Only the placement of the \( n/2 \) subtraction points \( \bar{\omega}_{\frac{n}{2}+1}, \ldots, \bar{\omega}_n \) laying in the positive frequencies axis is discussed, since the other \( n/2 \) points \( \bar{\omega}_1, \ldots, \bar{\omega}_{n/2} \) can be easily obtained by symmetry with respect to \( \omega = 0 \). First, we place the two edge subtractions \( \bar{\omega}_{\frac{n}{2}+1} \) and \( \bar{\omega}_n \) close to \( \omega_{\min} \) and \( \omega_{\max} \) respectively (\( \epsilon \ll 1 \))

\[
\bar{\omega}_{\frac{n}{2}+1} = \omega_{\min} + (\omega_{\max} - \omega_{\min}) \frac{\epsilon}{2}, \quad (B.3)
\]

\[
\bar{\omega}_n = \omega_{\max} - (\omega_{\max} - \omega_{\min}) \frac{\epsilon}{2}. \quad (B.4)
\]

The position of the other subtraction points depends on the ratio \( \omega_{\min}/\omega_{\max} \). We start by considering the two limiting cases \( \omega_{\min}/\omega_{\max} \simeq 0 \) and \( \omega_{\min}/\omega_{\max} \simeq 1 \). In the first case, since the available data cover the entire bandwidth \([ -\omega_{\max}, \omega_{\max} ]\) except for a very small interval \([ -\omega_{\min}, \omega_{\min} ]\), subtractions must be dense near \( \omega = \omega_{\max} \) and rare at low frequency, where the missing bandwidth \([ -\omega_{\min}, \omega_{\min} ]\) is very small. A displacement similar to (3.20) is therefore optimal, provided that subtractions are not placed in \([ -\omega_{\min}, \omega_{\min} ]\). So we adopt the following rule

\[
\bar{\omega}'_q = -\left( \bar{\omega}_n - \bar{\omega}_{\frac{n}{2}+1} \right) \cos \left( \pi \frac{q - \frac{2}{n} - \frac{1}{2}}{n - \frac{1}{2}} \right) + \bar{\omega}_{\frac{n}{2}+1} \quad (B.5)
\]

for \( q = \frac{n}{2} + 1, \ldots, n \), which leads to a Chebyshev-like distribution of subtractions in \( \Omega \). This distribution is depicted in Figure B.1 (top). In the second case, when \( \omega_{\min}/\omega_{\max} \simeq 1 \), subtractions must be concentrated near both \( \omega = \omega_{\min} \) and \( \omega = \omega_{\max} \). A Chebyshev distribution in the interval \([ \omega_{\min}, \omega_{\max} ]\) is therefore adopted

\[
\bar{\omega}''_q = -\frac{\bar{\omega}_n - \bar{\omega}_{\frac{n}{2}+1}}{2} \cos \left( \pi \frac{q - \frac{n}{2} - \frac{1}{2}}{n - \frac{1}{2}} \right) + \frac{\bar{\omega}_{\frac{n}{2}+1} + \bar{\omega}_n}{2} \quad (B.6)
\]

with \( q = \frac{n}{2} + 1, \ldots, n \). This distribution is depicted in Figure B.1 (bottom). Based on the two displacements \( \{ \bar{\omega}'_q \} \) and \( \{ \bar{\omega}''_q \} \), a nearly-optimal grid for any value of \( \omega_{\min}/\omega_{\max} \) can be obtained as a convex combination of the two

\[
\bar{\omega}_q = \bar{\omega}'_q \left( 1 - \frac{\omega_{\min}}{\omega_{\max}} \right) + \bar{\omega}''_q \left( \frac{\omega_{\min}}{\omega_{\max}} \right) \gamma. \quad (B.7)
\]

We experimentally verified that (B.7) with \( \gamma = 0.3 \) approximately minimizes the truncation error bound (B.2) for any value of \( \omega_{\min}/\omega_{\max} \). Figures B.2 and B.3 show how the proposed displacement rule approximately minimizes the truncation error bound for very different ratios of \( \omega_{\min}/\omega_{\max} \), ranging from 0.01 up to 0.99 and for different numbers of subtraction points (\( n = 6 \) and \( n = 16 \) respectively). We remark that this empirical rule, although being not optimal in mathematical sense, is sufficient for practical applications.
Figure B.1. Qualitative illustration of the optimal subtractions displacements for the limiting cases $\omega_{\min}/\omega_{\max} \simeq 0$ and $\omega_{\min}/\omega_{\max} \simeq 1$.

Figure B.2. Magnitude of the truncation error bound for $n = 6$ subtractions, for different $\omega_{\min}/\omega_{\max}$ ratios (solid line: $\omega_{\min}/\omega_{\max} = 0.01$, dash-dot line: $\omega_{\min}/\omega_{\max} = 0.5$, dashed line: $\omega_{\min}/\omega_{\max} = 0.99$).

Figure B.3. As in Fig. B.2 but for 16 subtraction points.
Appendix C

Dispersion relations: mathematical proofs

We report here a detailed proof for the bound (3.19) on the truncation error (3.16) related to the algorithm for generalized dispersion relations computation presented in Chapter 3. We consider the baseband case (3.5). Starting from (3.16) and taking the magnitude, we can write

\[ |E_n(j\omega)| = \left| \frac{\prod_{q=1}^{n} (\omega - \bar{\omega}_q)}{j\pi} \int_{\Omega C} \frac{-H(j\omega')}{|\omega' - \bar{\omega}_q| \omega - \omega'} \frac{d\omega'}{\prod_{q=1}^{n} (\omega' - \bar{\omega}_q)} \right| \leq \prod_{q=1}^{n} |\omega - \bar{\omega}_q| \frac{|H(j\omega')|}{\pi} \int_{\Omega C} \frac{d\omega'}{\prod_{q=1}^{n} |\omega' - \bar{\omega}_q| |\omega - \omega'|}. \]  

(C.1)

Let us denote with \( I_- \) and \( I_+ \) the individual contributions to the integral in (C.1) due to the negative and positive frequencies, respectively,

\[ I_- = \int_{-\infty}^{-\omega_{\text{max}}} \frac{|H(j\omega')|}{\prod_{q=1}^{n} |\omega' - \bar{\omega}_q| |\omega - \omega'|} d\omega', \]  

(C.2)

\[ I_+ = \int_{\omega_{\text{max}}}^{+\infty} \frac{|H(j\omega')|}{\prod_{q=1}^{n} |\omega' - \bar{\omega}_q| |\omega - \omega'|} d\omega'. \]  

(C.3)

Under the assumption (3.17), integrals (C.2) and (C.3) can be bounded with a closed form expression. For \( I_+ \), we have the following chain of inequalities

\[ I_+ = \int_{\omega_{\text{max}}}^{+\infty} \frac{|H(j\omega')|}{\prod_{q=1}^{n} |\omega' - \bar{\omega}_q| |\omega - \omega'|} d\omega' \leq M \int_{\omega_{\text{max}}}^{+\infty} \frac{(\omega')^\alpha}{\prod_{q=1}^{n} (\omega' - \bar{\omega}_q)} d\omega' = \frac{M}{\prod_{q=1}^{n} (\omega_{\text{max}} - \bar{\omega}_q)} \frac{1}{\omega - \omega'} \sum_{p=1}^{n} \frac{(\bar{\omega}_p)^\alpha}{\prod_{q=1}^{n} (\bar{\omega}_p - \bar{\omega}_q)} \omega' - \omega = M \sum_{p=1}^{n} \frac{(\bar{\omega}_p)^\alpha}{\prod_{q=1}^{n} (\bar{\omega}_p - \bar{\omega}_q)} \frac{1}{|\omega_{\text{max}} - \bar{\omega}_p|} \ln \frac{|\omega_{\text{max}} - \bar{\omega}_p|}{|\omega - \omega'|}. \]  

(C.4)
The key step of this derivation is the partial fraction expansion of the boxed quantity. An analogous calculation shows that $I_-$ is bounded by

$$I_- \leq M \sum_{p=1}^{n} \frac{(-\bar{\omega}_p)^{\alpha}}{\prod_{q=1}^{n} (\bar{\omega}_q - \bar{\omega}_p)} \frac{1}{\omega_{\max} \omega_p - \omega_p} \ln \left| \frac{\omega_{\max} + \bar{\omega}_p}{\omega_{\max} + \omega} \right|. \quad (C.5)$$

A direct substitution of (C.4) and (C.5) into (C.1) leads to (3.19), concluding the proof. This bound is the tightest possible for the considered class of functions, defined by (3.17). In fact, there exists a particular frequency response for which the magnitude of the truncation error (3.16) equals the bound (3.19). This response reads

$$H(j\omega) = jM |\omega|^\alpha \left[ \text{sign}(\omega) \right]^{\alpha+1} \prod_{q=1}^{n} \frac{\omega - \bar{\omega}_q}{|\omega - \bar{\omega}_q|} \quad (C.6)$$

as can be verified by direct substitution. This proves that a tighter bounds does not exist, hence the proposed treatment of truncation errors is indeed optimal.

Finally, we remark that the above proofs are easily adapted to the bandpass case (3.6) with minor modifications, leading to the bound (B.2), which can be shown to be tight, as for the baseband case.
Appendix D

Descriptor systems

This Appendix collects some important concepts on the descriptor representation of linear time-invariant systems used in the thesis. The Reader is referred to [112,124,125] for more details and for a comprehensive set of bibliographic references.

The descriptor form is a mathematical representation of singular systems, that generalizes the very popular state-space representation with the introduction of a matrix $E$

\[
E \dot{x}(t) = A x(t) + Bu(t), \quad (D.1a)
\]
\[
y(t) = C x(t), \quad (D.1b)
\]

with $x(t) \in \mathbb{R}^n$ and $\text{rank}\{E\} = r$. In general $r < n$, so this representation supports singular systems governed by mixed dynamic and non-dynamic (algebraic) equations, that do not fit in the state-space formalism. The transfer function $G(s)$ of (D.1) is given by [112]

\[
G(s) = C(sE - A)^{-1}B, \quad (D.2)
\]

with the poles being the generalized eigenvalues [126] of the matrix pencil $(E,A)$

\[
\rho_i = \text{zeros}\{\det(sE - A)\}, \quad (D.3)
\]

located both at finite and infinite frequency. The behavior of descriptor systems is therefore much richer than state-space systems, and is classified according to the following properties [112]:

- **regular** if $\det(sE - A)$ is not identically vanishing;
- **impulse-free** if $\deg(\det(sE - A)) = r$;
- **stable** if all poles $\rho_i$ have negative real part;
- **admissible** if regular, impulse-free and stable.

The regularity property ensures the existence and uniqueness of the solution of (D.1) for any initial condition, while the impulse-free property the absence of impulsive modes [112].
Appendix E

Mathematical representation of a cascade of lumped blocks and transmission lines

We prove in this appendix that composite structures obtained by cascading lumped multiprot elements and transmission line segments may be represented as in (5.3). This class of structures, exemplified in Fig. E.1, can represent a lot of interconnects of practical interest, with line segments being any kind of uniform transmission line (microstrips, striplines, coaxial cables,...) and the blocks being any discontinuity (connectors, bends, vias, ferrites, junctions, variations of line cross section,...). We denote with \( R \) the set of the pseudorational transfer functions in the form

\[
H(s) = \sum_m Q_m(s)e^{-sT_m} \tag{E.1}
\]

where \( Q_m(s) \) are proper rational transfer functions and the summation may include infinite terms. Also, we denote with \( R^{n \times k} \) the matrices \( n \times k \) with the entries in \( R \). The proof is conducted for \( k = n = 2 \), since the generalization to larger port counts follows the same scheme.

We first prove that the cascade connection of two scattering matrices \( S'(s) \in R^{2 \times 2} \) and \( S''(s) \in R^{2 \times 2} \) leads to a scattering matrix \( S(s) \) that also belongs to \( R^{2 \times 2} \). Simple calculations show that connection of port 2 of \( S'(s) \) with port 1 of \( S''(s) \) leads to the following expressions

\[
S_{11} = S'_{11} + S'_{12}S''_{11}(1 - S''_{22}S''_{11})^{-1}S'_{21},
S_{12} = S'_{12}(1 - S''_{11}S''_{22})^{-1}S''_{12},
S_{21} = S''_{21}(1 - S''_{22}S''_{11})^{-1}S'_{21},
S_{22} = S''_{22} + S''_{21}(1 - S''_{22}S''_{11})^{-1}S'_{22}S''_{12}. \tag{E.2}
\]

Clearly, the set \( R \) is closed under the sum and product operations. Using the result

\[
(1 - A)^{-1} = \sum_{m=0}^{\infty} A^m,
\]

141
valid for $|A| < 1$, we can conclude that $A \in \mathbb{R}$ implies $(1 - A)^{-1} \in \mathbb{R}$. Therefore, all elements of $S(s)$ in (E.2) belong to $\mathbb{R}$. The above result can be applied recursively to show that the cascade connection of any number of multiport elements in $\mathbb{R}^{2 \times 2}$ also belongs to $\mathbb{R}^{2 \times 2}$.

We conclude the proof by showing that the scattering matrix elements for lumped circuit blocks and transmission-line structures belong to $\mathbb{R}$. The case of lumped circuits is trivial, since their responses are purely rational. Conversely, the case of transmission-lines, including the lossy and frequency-dependent cases, requires some care and some approximation. We consider in the following a scalar transmission line of length $L$ with frequency dependent per-unit-length parameters $R(s)$, $L(s)$, $G(s)$ and $C(s)$. We define the propagation factor and characteristic admittance, respectively, as

$$\Gamma(s) = \sqrt{(R(s) + sL(s))(G(s) + sC(s))}$$
$$Y_c(s) = \sqrt{\frac{G(s) + sC(s)}{R(s) + sL(s)}}$$

A straightforward calculation leads to the following expressions for the scattering matrix elements,

$$S_{11} = S_{22} = \frac{-\alpha_+ \alpha_-(1 - Q^2)}{\alpha_+^2 - \alpha_-^2 Q^2}$$
$$S_{12} = S_{21} = \frac{Q(\alpha_+^2 - \alpha_-^2)}{\alpha_+^2 - \alpha_-^2 Q^2}$$  \hspace{1cm} (E.3)

where

$$\alpha_{\pm}(s) = Y_c(s) \pm Y_R, \quad Q(s) = e^{-\tau \Gamma(s)}$$

with $Y_R$ representing the port reference admittance. Following the well-known Method of Characteristics approach [25,27], we extract the propagation delay $T$ from the propagation operator,

$$Q(s) = e^{-sT} P(s)$$
where $P(s)$ includes no delay and represents mainly frequency-dependent attenuation and dispersion effects. Then, we compute rational approximations for characteristic admittance $\tilde{Y}_c(s) \simeq Y_c(s)$ and delayless propagation operator $\tilde{P}(s) \simeq P(s)$. Note that the corresponding approximation errors can be reduced below any prescribed threshold due to the universal approximation properties of rational functions [127]. Setting

$$\tilde{\alpha}_\pm(s) = \tilde{Y}_c(s) \pm Y_R, \quad \tilde{Q}(s) = e^{-sT\tilde{P}(s)},$$

we can approximate the denominators of (E.3) with the expansion

$$\begin{align*}
(\tilde{\alpha}_+^2 - \tilde{Q}_-^2 \tilde{\alpha}_-^2)^{-1} &= \tilde{\alpha}_+^{-2} \left(1 - \frac{(\tilde{Q}_- \tilde{\alpha}_- / \tilde{\alpha}_+)^2}{2}\right)^{-1} \\
&= \tilde{\alpha}_+^{-2} \left(1 - \frac{(e^{-sT\tilde{P}_-} \tilde{\alpha}_- / \tilde{\alpha}_+)^2}{2}\right)^{-1} \\
&= \tilde{\alpha}_+^{-2} \sum_{m=0}^{\infty} \left(\frac{\tilde{P}_- \tilde{\alpha}_- / \tilde{\alpha}_+}{2}\right)^{2m} e^{-2smT}.
\end{align*}$$

It is easily recognized that this expression belongs to $R$. This proves that the scattering matrix of frequency-dependent transmission lines belongs to $R^{2 \times 2}$, provided that suitable rational approximations are used for both characteristic admittance and delayless propagation operators.
E – Mathematical representation of a cascade of lumped blocks and transmission lines
Bibliography


Bibliography


