REFERENCES

[1] V. L. Blake, "Reflection of radio waves from a rough sea," PROC. IRE, vol. 38, pp. 301–304; March, 1950.
[2] S. O. Rice, "Reflection of electromagnetic waves from slightly rough surfaces," *Comm. Pure and Appl. Math.*, vol. 4, pp. 351–378; August, 1951.
[3] W. S. Ament, "Toward a theory of reflection from a rough surface," PROC. IRE, vol. 41, pp. 142–146; January, 1953.
[4] H. Davies, "The reflection of electromagnetic waves from a rough surface," *J. IEE*, vol. 101, pt. IV, pp. 209–214; August, 1954.
[5] J. Feinstein, "Some stochastic problems in wave propagation," IRE TRANS. ON ANTENNAS AND PROPAGATION, vol. AP-2, pp. 23–30; January, 1954.
[6] W. C. Hoffman, "Scatter of electromagnetic waves from a random surface," *Quart. Appl. Math.*, vol. 13, pp. 291–304; October, 1955.
[7] L. M. Brekhovskikh and M. A. Isakovich, "Three papers on diffraction of waves from a rough surface," The Journal of Experimental and Theoretical Physics, U. S. Navy Electronics Lab., San Diego, Calif., vol. 23; 1952. (Trans. by R. N. Goss.)
[8] C. I. Beard, I. Katz, and L. M. Spetner, "Phenomenological vector model of microwave reflection from the ocean," IRE TRANS. ON ANTENNAS AND PROPAGATION, vol. AP-4, pp. 162–167; April, 1956.
[9] M. Katzin, "On the mechanism of radar sea clutter," PROC. IRE (*Correspondence*), vol. 45, p. 50; January, 1957.
[10] D. E. Kerr, "Propagation of Short Radio Waves," Radiation Laboratory Series, McGraw-Hill Book Co., Inc., New York, N. Y. vol. 13, 1951.
[11] G. H. Hagn, D. L. Nielson and F. H. Smith, "Backscatter Literature Survey," prepared for Office of Secretary of Defense, by Stanford Research Institute, Menlo Park, Calif., Contract No. SD-66; 1961.
[12] J. A. Saxton and J. A. Lane, "Electrical properties of sea water," *Wireless Engr.*, pp. 269–275; October, 1952.
[13] L. J. Cote, *et al.*, "The directional spectrum of a wind generated sea as determined from data obtained by the stereo wave observation project," *Meteorological Papers*, vol. 2, p. 57; June, 1960.
[14] W. J. Pierson, G. Neumann and R. W. James, "Observing and Forecasting Ocean Waves," Hydrographic Office Publication, Washington, D. C., No. 603; 1958.
[15] C. Cox and W. Munk, "Statistics of the sea surface derived from sun glitter," *J. Marine Res.*, vol. 13, pp. 198–227; November, 1954.
[16] C. S. Williams, *et al.*, "Radar Return From the Vertical for Ground and Water Surfaces," Sandia Corporation, Albuquerque, N. M., Monograph No. SCR-107; April, 1960.
[17] H. Davies and G. G. Macfarlane, "Radar echoes from the sea surface at centimetre wavelengths," *Proc. Phys. Soc.* (*London*), vol. 58, pp. 717–729; May, 1946.
[18] L. N. Ridenour, "Radar System Engineering," Radiation Laboratory Series, McGraw-Hill Book Co., Inc., New York, N. Y., vol. 1, p. 203; 1947.

The Impedance of a Short Dipole Antenna in a Magnetoplasma

K. G. BALMAIN, MEMBER, IEEE

Summary—**A formula for the impedance of a short cylindrical dipole in a magnetoplasma is derived using quasi-static electromagnetic theory. The formula is valid in a lossy plasma and for any dipole orientation with respect to the magnetic field. The dipole impedance is found to have a positive real part under lossless conditions when the quasi-static differential equation is hyperbolic; this indicates that the quasi-static theory predicts a form of radiation. It is shown that the quasi-static theory can be interpreted in terms of scaled coordinates and that a cylindrical dipole in a magnetoplasma has a free space equivalent with a distorted shape. A conductance correction term obtained from Langmuir probe theory is shown to be significant. Laboratory measurements of monopole impedance are compared with the theoretical calculations.**

I. INTRODUCTION

WHEN AN ANTENNA is immersed in some medium, knowledge of its impedance is important whether the antenna is regarded as part of a communications system or as a probe for studying the properties of the medium. For the former application, energy reflection from the antenna must be minimized and, for the latter, the relationship between impedance and medium properties must be well established. The foregoing statements apply especially to rocket and satellite exploration of the ionosphere and also to plasma diagnostics in the laboratory. For these reasons it was decided to study both theoretically and experimentally the small-signal impedance of a short, cylindrical dipole antenna immersed in a magnetoplasma.

The analysis is limited to short antennas (short compared to a wavelength) in order to avoid the problems of obtaining theoretically the antenna current distribution. If the antenna is short enough, the current may be assumed to vary linearly from a maximum at the center to zero at both ends. Furthermore, a short antenna may be analyzed conveniently using quasi-static electromagnetic theory, a method which (in free space at least) gives good impedance results but does not predict radiation. In this paper a derivation of the quasi-static theory is presented and it is shown that the quasi-static electric field is identical to the near field term in the expressions derived by Kogelnik [1] and by Mittra and Deschamps [2]. Furthermore, it is shown that the quasi-static electric field in a magnetoplasma induces a magnetic field, a phenomenon not present in isotropic media. The quasi-static field expressions thus derived

Manuscript received November 26, 1963; revised March 30, 1964. The work described here was supported by the Geophysics Research Directorate, Air Force Cambridge Research Laboratories, under Contract No. AF 19(604)-5565. Support was also provided by the National Aeronautics and Space Administration under Contract No. NSG-395. This paper is taken from a dissertation submitted in partial fulfillment of the requirements for the Ph.D. degree in electrical engineering, University of Illinois, Urbana, Ill.
The author is with the Dept. of Electrical Engineering, University of Illinois, Urbana, Ill.

are used to calculate the impedance of a short dipole for any orientation with respect to the steady magnetic field.

There are relatively few published papers dealing with the impedance of antennas in anisotropic media. Kononov, *et al.* [3], have applied quasi-static theory to the problem of an infinitesimal dipole but their field and impedance expressions differ with those in this paper due to their choice of an integration contour. Katzin and Katzin [4] have derived an impedance formula for longer dipoles but a great deal of numerical integration would be necessary to extract impedance values from their formula. Whale [5] has discussed some aspects of the problem, including the effect of plasma wave excitation on radiation resistance. Bramley [6] has obtained an impedance expression valid for low electron density or weak magnetic field. Kaiser [7] has applied quasi-static theory to the biconical dipole problem and has obtained impedance results similar to those in this paper.

Some papers on related topics should be mentioned for the sake of completeness. The impedance of antennas in conducting isotropic media has been studied by King and Harrison [8] and also by Deschamps [9] whose impedance relation is particularly simple and useful. Quasi-static theory has been applied to propagation problems in plasmas by Trivelpiece and Gould [10] and in ferrites by Trivelpiece, *et al.* [11] and several other authors [12], [13]. The paper by Bunkin [14] is one of the earliest on sources in anisotropic, cold plasma and, in addition, a quasi-static Green's function has been discussed by Tai [15]. A thorough discussion of source problems in isotropic warm plasma has been presented by Cohen [16] in a series of three articles.

II. Derivation of the Basic Equations

The impedance analysis of an antenna requires knowledge of its near field. If all the dimensions of the antenna are small compared to a wavelength, the use of an approximate near field theory is indicated in order to simplify the otherwise complicated calculations. Such an approximate theory can be obtained by first formulating general field expressions and then letting the antenna dimensions become very small in terms of wavelengths. It will be shown that the electric field derived in this manner is the quasi-static field (which can be obtained from a single scalar potential).

In a plasma with a z-directed dc magnetic field, Maxwell's equations are

$$\nabla \times \boldsymbol{H} = j\omega\epsilon_0 \boldsymbol{K}\boldsymbol{E} + \boldsymbol{J} \tag{1}$$

$$\nabla \times \boldsymbol{E} = -j\omega\mu_0 \boldsymbol{H}. \tag{2}$$

The relative permittivity matrix $\boldsymbol{K}$ is

$$\boldsymbol{K} = \begin{bmatrix} K' & jK'' & 0 \\ -jK'' & K' & 0 \\ 0 & 0 & K_0 \end{bmatrix} \tag{3}$$

in which

$$K_0 = 1 - \frac{X}{U}$$

$$K' = 1 - \frac{XU}{U^2 - Y^2}$$

$$K'' = \frac{-XY}{U^2 - Y^2}$$

$$X = \frac{\omega_N^2}{\omega^2}, \quad Y = \frac{\omega_H}{\omega}, \quad \omega_N^2 = \frac{Ne^2}{m\epsilon_0}, \quad \omega_H = \frac{eB_0}{m}.$$

$U = 1 - jZ = 1 - j(\nu/\omega)$, ν = collision frequency
N = electron density
B_0 = dc magnetic flux density
ω = angular frequency of signal source
$-e$ = electron charge
m = electron mass
ϵ_0 = permittivity of free space
μ_0 = permeability of free space
$k_0 = \omega\sqrt{\mu_0\epsilon_0} = 2\pi/\lambda_0$ = free-space propagation constant
λ_0 = free space wavelength.

Meter-kilogram-second units (rationalized) are used throughout. Ion motion may be included in the analysis by the use of appropriate expressions for K_0, K' and K''.

The first step is to obtain a general field formulation valid for electromagnetic problems in a magnetoplasma. It is desired to derive $\boldsymbol{E}$ and $\boldsymbol{H}$ from a pair of potentials chosen in such a manner as to display the quasi-static electric field as a distinct part of the total electric field. The electric and magnetic fields can be expressed in terms of a scalar potential ψ and a vector potential $\boldsymbol{A}$,

$$\boldsymbol{E} = -\nabla\psi - j\omega\boldsymbol{A} \tag{4}$$

$$\mu_0\boldsymbol{H} = \nabla \times \boldsymbol{A}. \tag{5}$$

The above two relations, together with (1) give

$$\nabla \times \nabla \times \boldsymbol{A} - k_0^2\boldsymbol{K}\boldsymbol{A} = -j\omega\mu_0\epsilon_0\boldsymbol{K}\nabla\psi + \mu_0\boldsymbol{J}. \tag{6}$$

Operation on (6) with the divergence operator gives

$$\nabla\cdot\boldsymbol{K}\nabla\psi + j\omega\nabla\cdot\boldsymbol{K}\boldsymbol{A} = \frac{\nabla\cdot\boldsymbol{J}}{j\omega\epsilon_0}. \tag{7}$$

This equation can be simplified by introducing the following restriction on $\boldsymbol{A}$:

$$\nabla\cdot\boldsymbol{K}\boldsymbol{A} = 0. \tag{8}$$

This is a modified Coulomb gauge condition and is discussed in the Appendix. Eq. (7) becomes

$$\nabla\cdot\boldsymbol{K}\nabla\psi = \frac{\nabla\cdot\boldsymbol{J}}{j\omega\epsilon_0}. \tag{9}$$

If q is the charge density, the equation of continuity $(\nabla\cdot\boldsymbol{J} + j\omega q = 0)$ puts (9) into the form

$$\nabla\cdot\boldsymbol{K}\nabla\psi = -\frac{q}{\epsilon_0} \tag{10}$$

which may be regarded as a modified Poisson's equation. This equation is widely used and is the quasi-static differential equation for the scalar potential ψ.

Solution of the above equations can be facilitated by the use of spatial Fourier transforms. A transform will be indicated with a tilde ($\sim$) and the transform variables will be represented by the vector k having components k_1, k_2, k_3,

$$\tilde{f}(k) = \iiint_{-\infty}^{\infty} f(r)e^{-jk\cdot r}dxdydz,$$

$$f(r) = \frac{1}{(2\pi)^3}\iiint_{-\infty}^{\infty} \tilde{f}(k)e^{jk\cdot r}dk_1dk_2dk_3.$$

Transformation of (6) gives

$$M\tilde{A} = \omega\mu_0\epsilon_0 Kk\tilde{\psi} + \mu_0\tilde{J} \tag{11}$$

where

$$-M\tilde{A} = k \times \tilde{A} + k_0^2 K\tilde{A}.$$

Transformation of (9) gives

$$\tilde{\psi} = -\frac{1}{\omega\epsilon_0}\frac{k\cdot\tilde{J}}{k\cdot Kk}\cdot \tag{12}$$

Substitution of (12) in (11) gives

$$\tilde{A} = \mu_0 M^{-1}\left(-\frac{Kk(k\cdot\tilde{J})}{k\cdot Kk} + \tilde{J}\right). \tag{13}$$

The field quantities can be expressed in terms of the pontentials by transforming (4) and (5),

$$\tilde{E} = -j(k\tilde{\psi} + \omega\tilde{A}) \tag{14}$$

$$\tilde{H} = \frac{j}{\mu_0} k \times \tilde{A}. \tag{15}$$

In addition, $\tilde{E}$ and $\tilde{H}$ may be expressed in terms of $\tilde{J}$ by noting that

$$-k = k_0^2 M^{-1}Kk. \tag{16}$$

This relation may be used in (14) and (15) to derive the following expressions:

$$\tilde{E} = -j\omega\mu_0 M^{-1}\tilde{J} \tag{17}$$

$$\tilde{H} = jk \times M^{-1}\tilde{J}. \tag{18}$$

An examination of the equations in the preceding paragraphs suggests that some simplification may result if $\tilde{E}$ and $\tilde{J}$ are each separated into two parts as follows:

$$\tilde{E} = \tilde{E}_0 + \tilde{E}_1 \qquad \tilde{J} = \tilde{J}_0 + \tilde{J}_1 \tag{19}$$

in which

$$\tilde{E}_0 = -jk\tilde{\psi} \qquad \tilde{J}_0 = \frac{Kk(k\cdot\tilde{J})}{k\cdot Kk}$$

$$\tilde{E}_1 = -j\omega\tilde{A} \qquad \tilde{J}_1 = \tilde{J} - \frac{Kk(k\cdot\tilde{J})}{k\cdot Kk}\cdot \tag{20}$$

The following relations may be deduced readily:

$$k \times K^{-1}\tilde{J}_0 = 0 \tag{21}$$

$$k\cdot\tilde{J}_1 = 0. \tag{22}$$

$\tilde{J}_1$ is clearly a transverse vector. However it is not the entire transverse part of the current density since the other part $\tilde{J}_0$ is not longitudinal; rather, $K^{-1}\tilde{J}_0$ is longitudinal. Eq. (13) for the vector potential becomes

$$\tilde{A} = \mu_0 M^{-1}\tilde{J}_1. \tag{23}$$

Eqs. (12), (14) and (23) permit the two parts of $\tilde{E}$ to be expressed as

$$\tilde{E}_0 = \frac{j}{\omega\epsilon_0}K^{-1}\tilde{J}_0 \tag{24}$$

$$\tilde{E}_1 = -j\omega\mu_0 M^{-1}\tilde{J}_1. \tag{25}$$

Eq. (21) shows that $\tilde{E}_0$ is a longitudinal vector. However it is not the entire longitudinal part of $\tilde{E}$ since in general $k\cdot\tilde{E}\neq 0$. Rather $K\tilde{E}_1$ is transverse, a fact which may be deduced from the gauge condition. An expression for the magnetic field follows from (15) and (23). It is

$$\tilde{H} = jk \times M^{-1}\tilde{J}_1. \tag{26}$$

The decomposition of the current density into two parts (a procedure suggested by Deschamps) simplifies the equations considerably. Furthermore it is clear that E_0 is derived entirely from J_0 and that both E_1 and H are derived from J_1. Similarly, ψ and A are derived from J_0 and J_1, respectively. Thus the entire field problem has been divided into two distinct halves, one with the source J_0 and the other with the source J_1. Although J may be confined to a finite region in space, J_0 and J_1 both exist outside that region.

The theory developed above does not use any approximations and is valid as long as the constant permittivity matrix K is a valid representation for the properties of the medium. Limiting the general analysis to short antennas requires that the antenna dimensions approach zero while the wavelength remains constant. One way to carry out this limiting process is to consider the antenna dimensions as being fixed and to let the wavelength become artitrarily large, that is, to let the frequency approach zero. Since k_0 is a parameter proportional to frequency, the LF approximation can be effected by letting k_0 approach zero. It should be noted that the LF approximation is not applied to the elements of the permittivity matrix K; that is, the elements of K are to be considered fixed as k_0 approaches zero. It will be shown that the first term of the approximation gives an electric field equal to E_0 (the quasi-static electric field) and that the LF approximation gives a magnetic field consisting of two parts. One part is the familiar magnetic field obtainable from the dc form of Ampere's law and the other part is an induced magnetic field which is nonzero only in an anisotropic medium.

Expansion of the field expression requires knowledge of the matrices M and M^{-1}

$$M = \begin{pmatrix} k_2^2 + k_3^2 - k_0^2K' & -k_1k_2 - jk_0^2K'' & -k_1k_3 \\ -k_1k_2 + jk_0^2K'' & k_1^2 + k_3^2 - k_0^2K' & -k_2k_3 \\ -k_1k_3 & -k_2k_3 & k_1^2 + k_2^2 - k_0^2K_0 \end{pmatrix} \tag{27}$$

$$M^{-1} = \frac{N}{d} = \frac{N_0 + k_0^2N_1 + k_0^4N_2}{k_0^2(d_0 + k_0^2d_1 + k_0^4d_2)} \tag{28}$$

in which d is the determinant of M. In order to consider the LF limit, it is necessary to know the scalars d_0, d_1, d_2 and the matrices N_0, N_1, N_2. They are

$$d_0 = -k^2k\cdot Kk \tag{29}$$

$$d_1 = (K'^2 - K''^2)(k_1^2 + k_2^2) + K'K_0(k_1^2 + k_2^2 + 2k_3^2) \tag{30}$$

$$d_2 = -\det K \tag{31}$$

$$N_0\tilde{J} = k^2k(k\cdot\tilde{J}) \tag{32}$$

$$N_1 = -\begin{pmatrix} K'(k_1^2 + k_2^2) + K_0(k_1^2 + k_3^2) & K_0k_1k_2 - jK''(k_1^2 + k_2^2) & K'k_1k_3 - jK''k_2k_3 \\ K_0k_1k_2 + jK''(k_1^2 + k_2^2) & K'(k_1^2 + k_2^2) + K_0(k_2^2 + k_3^2) & K'k_2k_3 + jK''k_1k_3 \\ K'k_1k_3 + jK''k_2k_3 & K'k_2k_3 - jK''k_1k_3 & K'(k_1^2 + k_2^2 + 2k_3^2) \end{pmatrix} \tag{33}$$

$$N_2 = K_0\begin{pmatrix} K' & -jK'' & 0 \\ jK'' & K' & 0 \\ 0 & 0 & \dfrac{K'^2 - K''^2}{K_0} \end{pmatrix}. \tag{34}$$

Application of the LF approximation is facilitated by carrying out two steps in the division operation indicated in (28). Thus (17) becomes

$$\tilde{E} = \frac{-j}{\omega\epsilon_0}\left[\frac{N_0}{d_0} + \frac{k_0^2}{d_0}\left\{N_1 - \frac{d_1N_0}{d_0}\right\} + k_0^4\left\{\frac{N_2 - d_1N_1 + (d_1^2 - d_2)\dfrac{N_0}{d_0} + k_0^2d_2\left(\dfrac{d_1N_0}{d_0} - N_1\right)}{d_0 + k_0^2d_1 + k_0^4d_2}\right\}\right]\tilde{J}. \tag{35}$$

In the analysis of an infinitesimal current element [1], [2], the first two terms have been interpreted as near field terms because the electric field terms derived from them are singular at the current element. Furthermore in the limit as $k_0^2\to 0$ the first term evidently predominates. However, it may be shown that the first term in the above expression is precisely $\tilde{E}_0$, which has been identified as the quasi-static electric field. Thus the quasi-static electric field should be a good approximation to the total electric field close to a short antenna in a magnetoplasma.

The transformed magnetic field may be treated similarly. If $\tilde{H}_0$ is designated as the LF limit of $\tilde{H}$, it can be shown that

$$\tilde{H}_0 = j\,\frac{k \times N_1\tilde{J}}{d_0}\,. \tag{36}$$

Further insight into the meaning of $\tilde{H}_0$ can be obtained by employing a different derivation. Taking the curl of (1) and setting $\nabla \cdot H = 0$ gives

$$\nabla^2 H = -j\omega\epsilon_0 \nabla \times KE - \nabla \times J. \tag{37}$$

In the LF or quasi-static limit, $E = -\nabla\psi$. Substitution of this in (37) gives an equation in H', the magnetic field resulting when the current and the quasi-static electric field are specified.

$$\nabla^2 H' = j\omega\epsilon_0 \nabla \times K\nabla\psi - \nabla \times J. \tag{38}$$

If K is a scalar the first term on the right-hand side is identically zero and H' and J are related only by the point form of Ampere's law for direct currents. A convenient expression for the magnetic field can be obtained by taking the Fourier transform of (38). This gives

$$\tilde{H}' = \frac{j}{k^2}\left(-\frac{k \times Kk(k\cdot\tilde{J})}{k\cdot Kk} + k \times \tilde{J}\right) \tag{39}$$

$$= \frac{j}{k^2} k \times \tilde{J}_1. \tag{40}$$

It can be shown that this LF expression for $\tilde{H}'$ is identical to (36) and thus $H' = H_0$. The advantage of this derivation is that it displays the LF magnetic field as the sum of two terms, the first term being identically zero in isotropic media and the second simply a statement of Ampere's law for direct currents. The meaning of the first term can be clarified by relating it to the induced current J_i which flows in the medium due to the quasi-static electric field. If $\tilde{\sigma}$ is the conductivity matrix, then

$$\begin{aligned}\tilde{J}_i &= \tilde{\sigma} E_0 \\ &= -\tilde{J}_0 - j\omega\epsilon_0\tilde{E}_0.\end{aligned} \tag{41}$$

The induced current is seen to consist of two parts; the first part is irrotational only when K is a scalar and the second is always irrotational. The magnetic field resulting from the quasi-static induced current is given by

$$\begin{aligned}\tilde{H}_i &= \frac{j}{k^2} k \times \tilde{J}_i \\ &= \frac{-j}{k^2}\frac{k \times Kk(k\cdot\tilde{J})}{k\cdot Kk}\end{aligned} \tag{42}$$

which is exactly the first term of (39). The existence of an induced magnetic field H_i in the LF limit suggests that unusual electromagnetic effects may be predicted by quasi-static theory when it is applied to problems in anisotropic media. Quasi-static propagation effects in magnetoplasmas and ferrites have been described in the literature in connection with source-free problems [10], [11]; in this paper source currents are included and it will be shown that the quasi-static theory predicts a form of radiation.

III. The Field of a Short Dipole

The quasi-static differential equation (10) may be written

$$\psi_{xx} + \psi_{yy} + a^{-2}\psi_{zz} = \frac{-q}{\epsilon_0 K'} \tag{43}$$

in which $a^2 = K'/K_0$. For the case of a lossless plasma, (43) is elliptic [17] when a^2 is positive and hyperbolic when a^2 is negative (see Fig. 1). Under hyperbolic conditions the characteristic surfaces of the differential equation are real cones in space with axes parallel to the z axis. Under elliptic conditions the characteristic surfaces are complex and thus have no physical significance.

The solution of (43) can be expressed as

$$\psi(r) = \frac{1}{\epsilon_0 K'(2\pi)^3}\iiint_{-\infty}^{\infty} \frac{\tilde{q}(k)e^{jk\cdot r}}{k_1^2 + k_2^2 + \frac{k_3^2}{a^2}} dk_1 dk_2 dk_3. \tag{44}$$

The cylindrical dipole and its coordinate system are sketched in Fig. 2. The dipole field will be derived assuming the filamentary, triangular current distribution shown in Fig. 3. The corresponding charge distribution is obtained from the equation of continuity.

$$\begin{aligned}q(r) &= -\frac{1}{j\omega}\frac{\partial J_u}{\partial u}\delta(y)\delta(v) \\ &= \frac{1}{j\omega L} T(u)\delta(y)\delta(v).\end{aligned} \tag{45}$$

The function $T(u)$ is also shown in Fig. 3. Substitution of $\tilde{q}(k)$ into (44) and subsequent integration will result in an expression for the potential ψ. However, for impedance calculation, the electric field parallel to the current (E_u) is required.

$$\begin{aligned}E_u(r) &= -\frac{\partial\psi}{\partial u} \\ &= \frac{1}{j\omega\epsilon_0 K' L}[I(L) + I(-L) - 2I(0)].\end{aligned} \tag{46}$$

The integral $I(L)$ can be expressed as

$$I(L) = \frac{1}{(2\pi)^3}\iiint_{-\infty}^{\infty} \frac{e^{j[k_1(x-L\sin\theta)+k_2y+k_3(z-L\cos\theta)]}}{k_1^2 + k_2^2 + \frac{k_3^2}{a^2}} dk_1 dk_2 dk_3. \tag{47}$$

The following transformation to cylindrical coordinates

$$\begin{aligned} x - L\sin\theta &= \rho_1\cos\phi_1 \qquad & k_1 &= \gamma\cos\eta \\ y &= \rho_1\sin\phi_1 & k_2 &= \gamma\sin\eta \\ z - L\cos\theta &= z_1 \end{aligned} \tag{48}$$

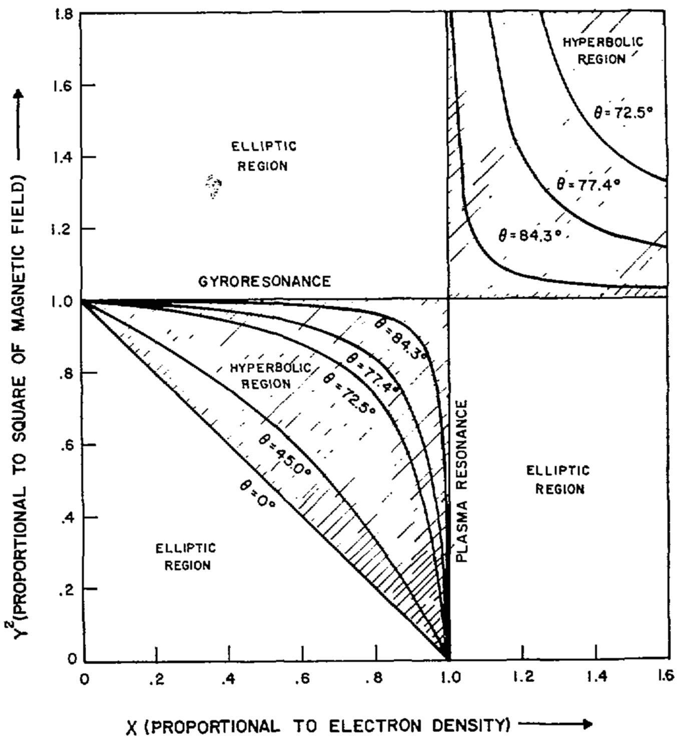

Fig. 1—The elliptic and hyperbolic regions. Note: θ is the angle (with respect to the z axis) of the characteristic cone when the differential equation is hyperbolic.

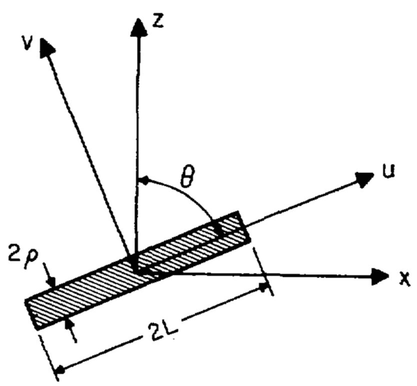

Fig. 2—The cylindrical dipole and the coordinate system (the y axis is directed into the paper).

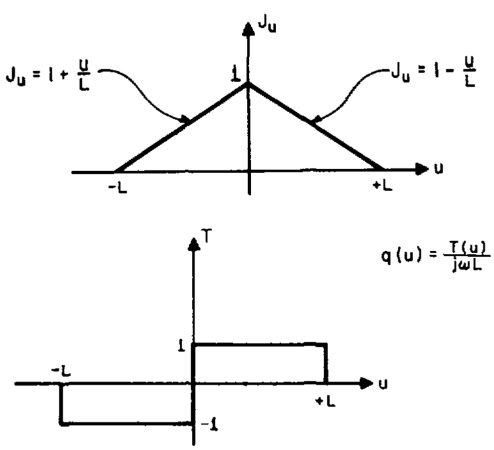

Fig. 3—The assumed current and charge distributions.

permits the integral to be expressed as

$$I(L) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_0^{\infty} \frac{e^{jk_3z_1} J_0(\gamma\rho_1)\gamma}{\gamma^2 + \frac{k_3^2}{a^2}} \, d\gamma dk_3 \qquad (49)$$

since

$$J_0(\gamma\rho_1) = \frac{1}{2\pi} \int_0^{2\pi} e^{j\gamma\rho_1 \cos(\eta - \phi_1)} d\eta. \qquad (50)$$

The next step involves contour integration with respect to k_3. It is convenient to designate by "a" the square root of K'/K_0 which has a positive real part, however small. Under lossless hyperbolic conditions (a^2 negative) the correct choice for "a" must be made by taking the limit as the collision frequency (ν) approaches zero. The integration contours are shown in Fig. 4; note that C_1 is used for z_1 positive and C_2 for z_1 negative. The coutour integration gives the following expression and integration with respect to γ completes the evaluation of $I(L)$:

$$I(L) = \frac{a}{4\pi} \int_0^{\infty} e^{-a\gamma|z_1|} J_0(\gamma\rho_1) d\gamma \qquad (51)$$

$$= \frac{a}{4\pi} (\rho_1^2 + a^2z_1^2)^{-1/2}. \qquad (52)$$

Similar expressions for $I(-L)$ and $I(0)$ may be derived. If it is noted that (ρ_1, z_1), (ρ_2, z_2) and (ρ_0, z_0) are cylindrical coordinates originating at $u = L$, $u = -L$ and $u = 0$, respectively (the ends and center of the dipole), then the electric field parallel to the dipole may be expressed as

$$E_u = \frac{a}{j\omega 4\pi\epsilon_0 K' L} \left[(\rho_1^2 + a^2z_1^2)^{-1/2} + (\rho_2^2 + a^2z_2^2)^{-1/2} - 2(\rho_0^2 + a^2z_0^2)^{-1/2} \right]. \qquad (53)$$

Under lossless hyperbolic conditions (a^2 real and negative), E_u becomes infinite on three conical surfaces emanating from the ends and center of the dipole. These surfaces are characteristic surfaces of the quasistatic differential equation (43).

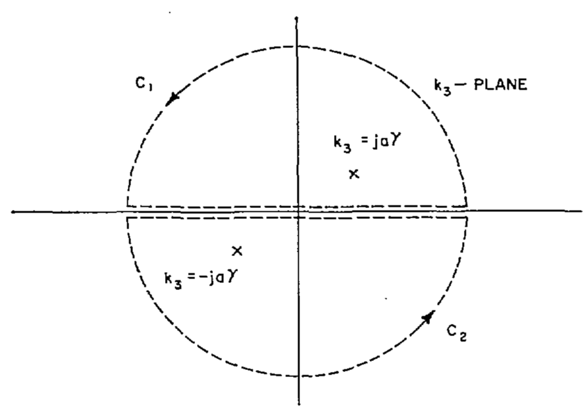

Fig. 4—The integration contours.

IV. The Impedance of a Short Dipole

For an input current of unit magnitude, the input impedance of an antenna with a conducting surface is given by

$$Z_{\text{in}} = -\int_S J \cdot E ds \tag{54}$$

where S is the antenna surface. In this formula J is the current density on the antenna surface and E is the electric field at the antenna surface when the conducting material in the antenna is removed. The impedance formula may be derived using the "reaction concept" and such derivations have been discussed frequently in textbooks (Harrington [18], for instance).

If the current is spread uniformly over the antenna surface, the current density (in the coordinate system of Fig. 5) is given by

$$\begin{aligned} J_u &= \frac{1 - \dfrac{u}{L}}{2\pi\rho} \quad \delta(r - \rho) \quad \text{for } u > 0 \\ &= \frac{1 + \dfrac{u}{L}}{2\pi\rho} \quad \delta(r - \rho) \quad \text{for } u < 0. \end{aligned} \tag{55}$$

If the dipole is very thin, the electric field E may be approximated by the field of a current filament lying along the dipole axis as given by (53). In order to simplify the calculations, one can obtain an expression for the impedance of a monopole length L; the impedance of a dipole of length $2L$ is just twice the monopole impedance. The monopole impedance is

$$Z_{\text{in}} = -\frac{1}{2\pi}\int_0^{2\pi}\int_0^L \left(1 - \frac{u}{L}\right) E_u(u, \rho, \phi,) du d\phi. \tag{56}$$

The expression for E_u may be put into the new coordinate system by the use of the following relations:

$$\rho_1^2 = [(u - L)\sin\theta - \rho\cos\theta\sin\phi]^2 + [\rho\cos\phi]^2$$

$$z_1 = (u - L)\cos\theta + \rho\sin\theta\sin\phi.$$

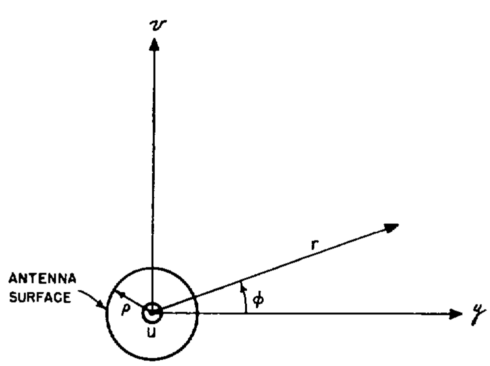

Fig. 5—The cylindrical coordinate system.

Similar relations for (ρ_2, z_2) and (ρ_0, z_0) can be obtained by replacing $-L$ by $+L$ and by zero, respectively. The algebra can be simplified by introducing

$$\begin{aligned} F &= 1 + (a^2 - 1)\cos^2\theta \\ G &= 2\rho(a^2 - 1)\sin\theta\cos\theta\sin\phi \\ H &= \rho^2[1 + (a^2 - 1)\sin^2\theta\sin^2\phi] \end{aligned} \tag{57}$$

$$\begin{aligned} N(\alpha) &= [F\alpha^2 + G\alpha + H]^{1/2} \\ M(\alpha) &= 2[F(F\alpha^2 + G\alpha + H)]^{1/2} + 2F\alpha + G. \end{aligned} \tag{58}$$

The monopole impedance now may be expressed as

$$Z_{\text{in}} = \frac{-a}{j\omega 4\pi\epsilon_0 K' L} \frac{1}{2\pi}\int_0^{2\pi} (I_1 + I_2 - 2I_3) d\phi \tag{59}$$

in which

$$I_1 = \int_0^L \frac{1 - \dfrac{u}{L}}{N(u - L)} du, \quad I_2 = \int_0^L \frac{1 - \dfrac{u}{L}}{N(u + L)} du,$$

$$I_3 = \int_0^L \frac{1 - \dfrac{u}{L}}{N(u)} du.$$

Integration with respect to u gives

$$\begin{aligned} &I_1 + I_2 - 2I_3 \\ &= \frac{-1}{FL}[3N(0) - 3N(L) + N(2L) - N(-L)] \\ &\quad + \frac{G}{2LF^{3/2}} \ln\frac{M^3(0)M(2L)}{M^3(L)M(-L)} + \frac{2}{F^{1/2}} \ln\frac{M(0)M(2L)}{M^2(L)} \cdot \end{aligned} \tag{60}$$

If it is assumed that $\rho \ll L$, then the above formula takes the form

$$\begin{aligned} I_1 + I_2 - 2I_3 = \frac{2}{F^{1/2}}\Big\{&1 - \ln\frac{FL}{\rho} \\ &+ \ln\{F^{1/2}[1 + (a^2 - 1)\sin^2\theta\sin^2\phi]^{1/2} \\ &+ (a^2 - 1)\sin\theta\cos\theta\sin\phi\}\Big\}. \end{aligned} \tag{61}$$

Substitution of the above in (59) and subsequent integration gives

$$Z_{\text{in}} = \frac{a}{j\omega 2\pi\epsilon_0 K' L F^{1/2}}\left[\ln\frac{L}{\rho} - 1 - \ln\frac{a + F^{1/2}}{2F}\right] \tag{62}$$

in which $F = \sin^2\theta + a^2\cos^2\theta$ and $a^2 = K'/K_0$).

It is important to recall that in computing $a = (K'/K_0)^{1/2}$ and $F^{1/2}$, the square root having a positive real part should be used. For the lossless hyperbolic case the correct sign of the square root must be determined by taking the limit as the collision frequency approaches zero; for instance, $a = |a| e^{j\pi/2}$ for $K' > 0$, $K_0 < 0$ and $a = |a| e^{-j\pi/2}$ for $K' < 0$, $K_0 > 0$.

Two special cases are of interest, $\theta=0$ (monopole parallel to B_0) and $\theta=\pi/2$ (monopole perpendicular to B_0).

Parallel case:

$$Z_{\text{in}} = \frac{1}{j\omega 2\pi\epsilon_0 K' L}\left[\ln\frac{L}{\rho} - 1 + \ln a\right]. \tag{63}$$

Perpendicular case:

$$Z_{\text{in}} = \frac{a}{j\omega 2\pi\epsilon_0 K' L}\left[\ln\frac{L}{\rho} - 1 - \ln\frac{a+1}{2}\right]. \tag{64}$$

It is interesting to observe that the impedance formula (62) can be rewritten in the same form as the free-space impedance [19] if the dimensions L and ρ are suitably scaled. That is,

$$Z_{\text{in}} = \frac{1}{j\omega 2\pi\epsilon_0 L'}\left[\ln\frac{L'}{\rho'} - 1\right] \tag{65}$$

where

$$L' = L\sqrt{K'}\sqrt{K_0\sin^2\theta + K'\cos^2\theta} \tag{66}$$

and

$$\rho' = \frac{\rho}{2}\left(\frac{K'\sqrt{K_0}}{\sqrt{K_2\sin^2\theta + K'\cos^2\theta}} + \sqrt{K'K_0}\right). \tag{67}$$

The impedance formula also may be derived by scaling the coordinates, the frequency ω and the source charge density q. The scale factors are chosen to bring the quasi-static differential equation and the equation of continuity into free-space form. There exists a family of such scalings, two of which are the most useful. If primes are used to designate the equivalent free-space coordinates, the two scalings are as follows.

1) Frequency-invariant scaling

$$\begin{aligned} x' &= \sqrt{K'K_0}\,x & \omega' &= \omega \\ y' &= \sqrt{K'K_0}\,y & q' &= \frac{q}{K'^2K_0} \\ z' &= K'\,z. \end{aligned} \tag{68}$$

2) Charge-invariant scaling

$$\begin{aligned} x' &= \frac{x}{\sqrt{K'}} & \omega' &= K'\sqrt{K_0}\,\omega \\ y' &= \frac{y}{\sqrt{K'}} & q' &= q \\ 2' &= \frac{2}{\sqrt{K_0}}. \end{aligned} \tag{69}$$

If scaling is applied to the problem of a thin cylindrical dipole in a magnetoplasma and if only positive, real scale factors are considered, then the equivalent free-space dipole has a different length and an elliptical cross section. The equivalent dipole with circular cross section has been shown by Lo [20] to have a radius equal to the arithmetic mean of the ellipse semi-axes. This leads to the scale factors (66), (67) already derived. Thus the scaled impedance formula is quite general although the derivation outlined above holds only for positive, real scale factors. Further references to scaling procedures in anisotropic media may be found in the papers by Clemmow [21] and by Arbel and Felsen [22].

It is worth noting that, for the special case of a monopole parallel to B_0, an impedance formula may be obtained without the approximation $\rho \ll L$ used to simplify (60). The impedance is

$$Z_{\text{in}} = \frac{1}{j\omega 2\pi\epsilon_0 K' L}\left\{\ln\frac{(aL+\sqrt{a^2L^2+\rho^2})^2}{\rho(2aL+\sqrt{4a^2L^2+\rho^2})} + \frac{1}{2aL}\left[3\rho + \sqrt{4a^2L^2+\rho^2} - 4\sqrt{a^2L^2+\rho^2}\right]\right\}. \tag{70}$$

Although the approximation $\rho \ll L$ was not used directly in the derivation of the above formula, such an approximation is implicit in it because the source current distribution is assumed to be filamentary in the derivation of the electric field formula (53).

V. Discussion

Under lossless hyperbolic conditions each of the above input impedance formulas has a positive real part indicating power flow into the plasma. It has been shown [23] that, between the characteristic cones extending from the ends of the dipole, the Poynting theorem also indicates power flow arising from the quasi-static electric field and the induced magnetic field H_i; this suggests that the real part of the antenna impedance is due to electromagnetic radiation. Kaiser [7] also has observed the real part but he asserts that it is associated with the infinity in the electric field. However, a field infinity occurs only if the assumed current distribution has a discontinuous slope. Fig. 6 shows graphically the continuous field produced by a third-order polynomial current distribution with zero slope at the ends and center of the dipole. The corresponding impedance expression for a thin monopole parallel to B_0 is [23]

$$Z_{\text{in}} = \frac{1.2}{j\omega 2\pi\epsilon_0 K' L}\left(\ln\frac{L}{\rho} - 1.375 + \ln a\right) \tag{71}$$

which is almost identical to (63). Evidently the real part of the impedance is not associated with the field infinity itself but rather with the hyperbolic plasma conditions under which field infinities can occur. Furthermore as long as the current distribution is continuous it apparently has little effect on impedance. However, the quasi-static theory predicts infinite power radiation from a uniform current distribution (which is not con-

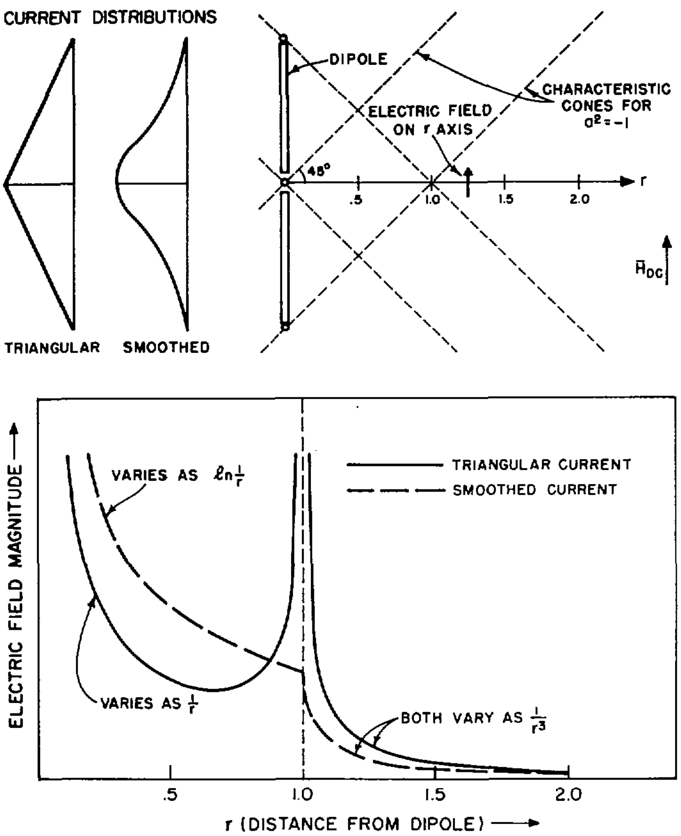

Fig. 6—Electric field discontinuities for two current distributions.

tinuous), an effect which has also been discussed by Arbel and Felsen [22].

The derivation of the quasi-static theory by the use of the limit $k_0 \rightarrow 0$ indicates that the impedance expressions obtained will be in error for dipoles of finite length. This error has been estimated [23] by evaluating the impedance correction arising from the second term in (35). The impedance correction has the same form as the impedance expressions already derived but it has in addition a $(L/\lambda_0)^2$ multiplier. This multiplier indicates that the impedance correction is small if $L \ll \lambda_0$; in the experimental work to be described, the correction term changes the impedance magnitude by 20 per cent at cyclotron resonance (at which the error is maximum).

The discussion of radiation effects holds for the lossless case only and this lossless condition is reached by taking the limit as the collision frequency approaches zero. However, a realistic evaluation of the theory can be made only after considering the case of a nonzero collision frequency. Consider the case of a z-directed dipole and define r to be the distance from the dipole. Under isotropic (or, more generally, elliptic) conditions and for large values of r, $E_z \propto r^{-3}$. Under lossless hyperbolic conditions, if r is not measured in the direction of a characteristic cone, then $E_z \propto r^{-3}$ (see Fig. 6). If r is measured in the direction of a characteristic cone, then $E_z \propto r^{-1/2}$, a variation with distance which permits outward power flow between conical surfaces. However, if a small but finite relative collision frequency Z is introduced, it may be shown that

$$
\begin{aligned}
E_z &\propto r^{-1/2} \qquad \text{for } r < \frac{L}{Z} \\
E_z &\propto r^{-3} \qquad \text{for } r > \frac{L}{Z}
\end{aligned}
\tag{72}
$$

in which the transition region $(r \approx L/Z)$ is an order-of-magnitude approximation. Thus if r is sufficiently large, the field decay with distance reverts to that of a static dipole and the high-level near fields extend outward in the characteristic directions to a distance of about L/Z. Evidently in a lossy medium "extended near field" describes the phenomenon better than "radiation field."

VI. An Ion Sheath Correction

Mlodnosky and Garriott [24] have obtained expressions for the conductance and capacitance of a cylindrical dipole in the ionosphere. The conductance term is the slope of a Langmuir probe voltage-current curve and the capacitance formula has the same form as that for a concentric, cylindrical capacitor (the outer cylinder is the sheath edge and the inner cylinder is the dipole surface). In the experiments to be described, the sheath is collapsed so that only the conductance term is used. It is given by

$$
G = \frac{ANe^2}{2(2\pi mkT)^{1/2}} \tag{73}
$$

in which A is the area of one-half of the dipole, T is the electron temperature and k is Boltzmann's constant. This conductance is in parallel with the impedance already derived. However, the conductance derivation is for isotropic plasma and thus it is useful only for low values of dc magnetic field.

VII. Impedance Measurements

The experimental apparatus is shown schematically in Fig. 7. The discharge is initiated by a two microsecond dc pulse and this is followed by the plasma decay period (afterglow) lasting several milliseconds. In the cathode region of the tube a small continuous discharge assures dependable starting of the pulsed discharge. The vacuum system is capable of pump-down to 10^{-6} mm Hg and the evacuated discharge tube can be back-filled with 1 to 10 mm Hg of neon or helium.

The resonance probe method [25] is used to measure the electron density as is shown in Fig. 8. Only the $X=1$ point is obtained since the measurement is made using the same oscillator frequency as in the impedance measurements. The resonance probe measurements are made only at zero dc magnetic field since the presence of the magnetic field tends to broaden and flatten the resonance peak.

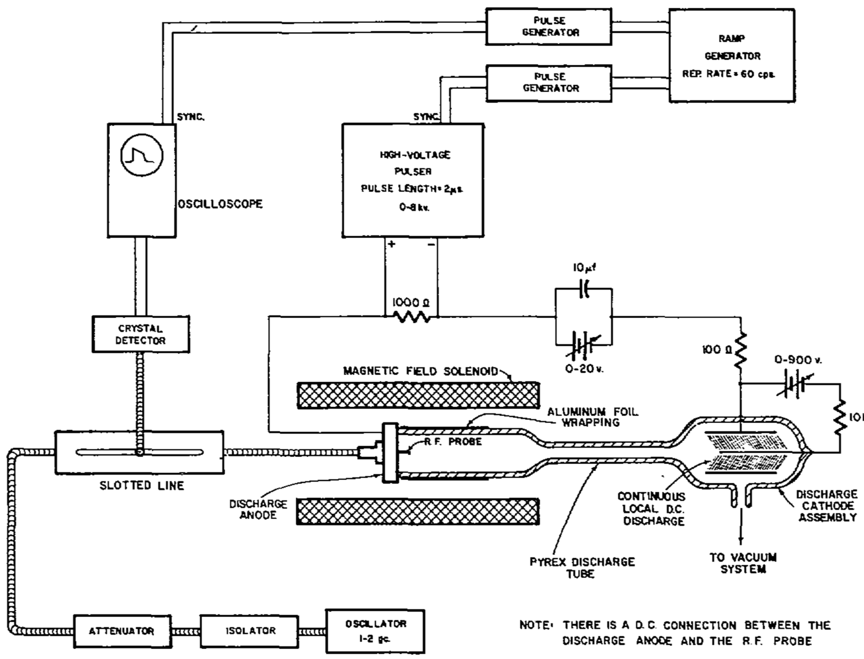

Fig. 7—The experimental apparatus

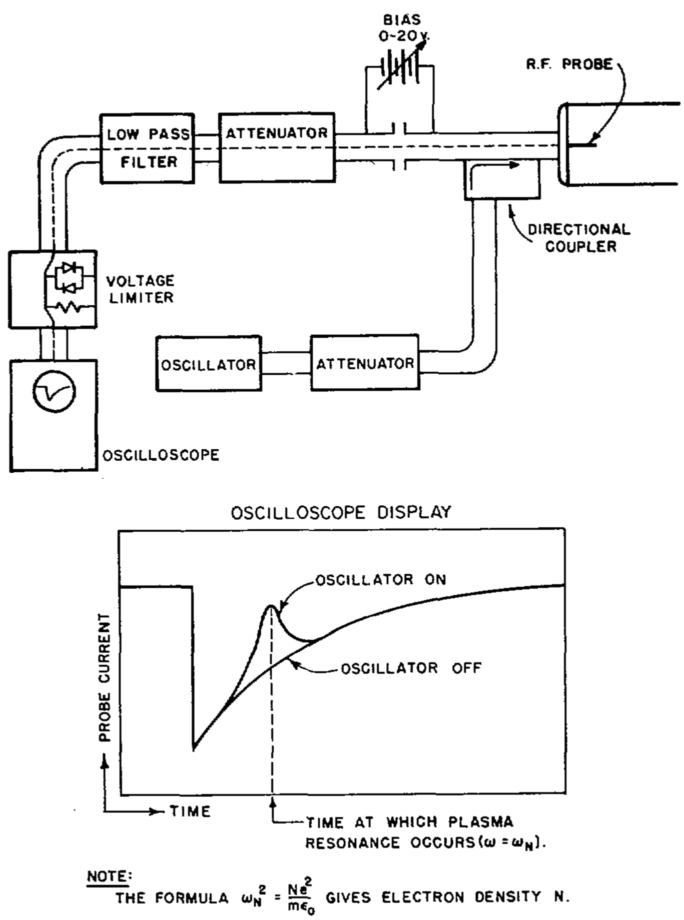

Fig. 8—Determination of electron density by the "Resonance Probe" technique.

The RF probe has the dimensions $L=8.0$ mm and $L/\rho=12.0$. Its impedance is measured at 1.6 Gc using the four-probe method [26] which involves photographic recording of the oscilloscope trace at four slotted line positions spaced $\frac{1}{8}$ wavelength apart. The impedance measurements are presented as loci plotted on Smith charts in Figs. 9 to 12). Each locus traces the impedance from shortly after the discharge pulse to complete deionization (going from left to right on the charts). For comparison, corresponding theoretical loci are shown in Figs. 13 and 14.

The theoretical graphs indicate that an increasing magnetic field sweeps the impedance locus from the top of the Smith chart nearly to the bottom; this effect is due primarily to the factor $1/K'$ appearing in the impedance expression (63). A prominent feature of each theoretical locus is the presence of a "kink" in the vicinity of $X=1$ (plasma resonance). This kink arises from the logarithm in the impedance formula and is thus related to the elliptic/hyperbolic feature of the quasi-static theory; the point $X=1$ is always on the boundary between an elliptic and a hyperbolic region (see Fig. 1). It should be noted that the line $X+Y^2=1$ is also an elliptic-hyperbolic boundary for $X<1$, $Y^2<1$; however, the Smith chart graphs reveal no unusual behavior at $X=1-Y^2$.

In general there is good qualitative agreement between experiment and theory. The movement of the impedance loci from the top of the Smith chart to the bottom with increasing magnetic field is evident in every experiment. In all cases (theoretical and experimental) the cyclotron resonance locus ($Y^2=1$) meets the rim of the Smith chart at right angles. It is particularly important to note that the ion sheath conductivity correction improves the agreement between theory and experiment at low values of dc magnetic field. For the case of high magnetic field ($Y^2>1$) the correction is not applicable and only one such locus is shown in Fig. 14.

In each experiment (as in the theory) the points $X=1$ follow an approximately circular path. Since these points were determined at zero magnetic field and since an increasing magnetic field tends to increase the time required for afterglow decay, the experimental points $X=1$ are in error for $Y^2>0$. Thus the true plasma resonance points are somewhat to the right of the indicated points.

The addition of a small quantity of argon apparently has little effect. This can be seen by comparing Fig. 9 with Fig. 10. In contrast to the case of argon, the addition of a very small amount of air has a pronounced effect on the impedance loci (see Figs. 11 and 12). The effect of the addition of air is to bring the experimental results into closer agreement with the theory, especially in the regions of the plasma resonance kinks. The air percentages indicated on the graphs are rough approximations obtained by extrapolating a low pressure leakage graph to 5 hours (0.03 per cent air at 4.3 mm) and to 25 hours (0.15 per cent air at 4.3 mm). The addition of air shortens the over-all decay period and most of this shortening is in the early part of the afterglow when the electron density is high. It is suggested that the addition of air tends to cause the predominance of volume processes (recombination, attachment) over surface processes (diffusion) in the afterglow decay. This should produce a more uniform plasma and hence better agreement between theory and experiment.

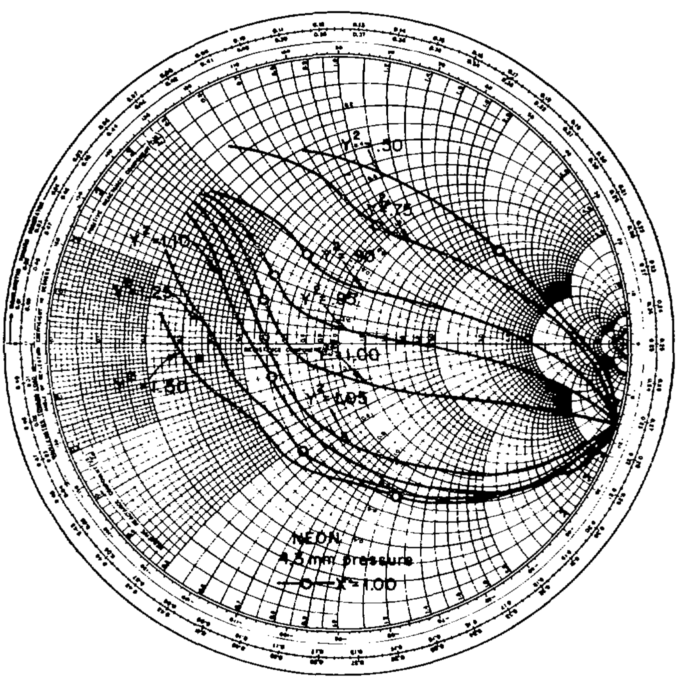

Fig. 9—Experimental impedance loci for neon.

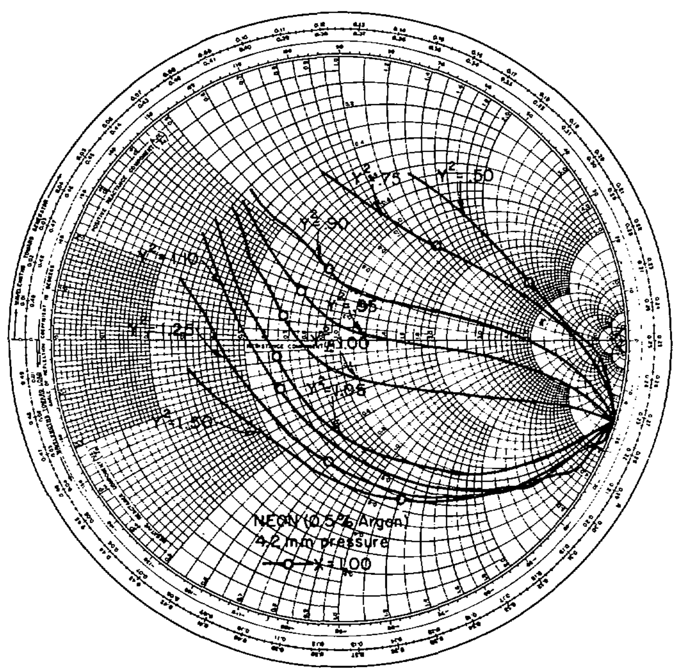

Fig. 10—Experimental impedance loci for neon (0.5 per cent argon).

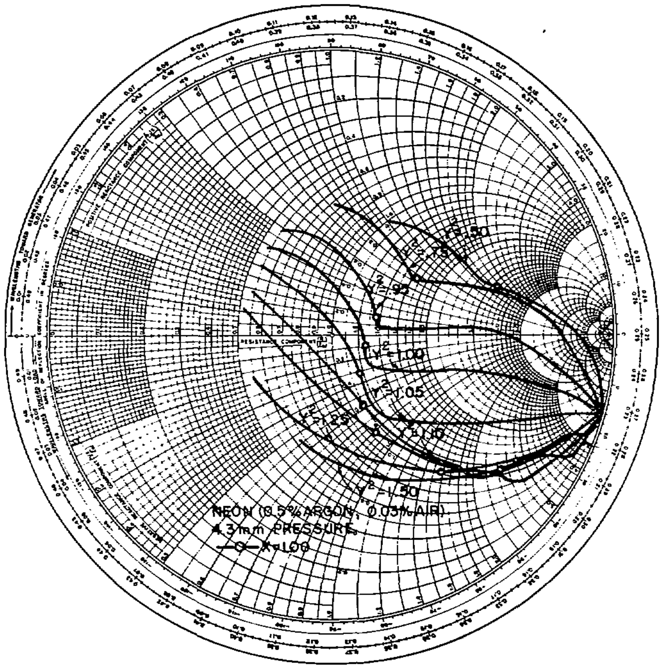

Fig. 11—Experimental impedance loci for neon (0.5 per cent argon, 0.03 per cent air).

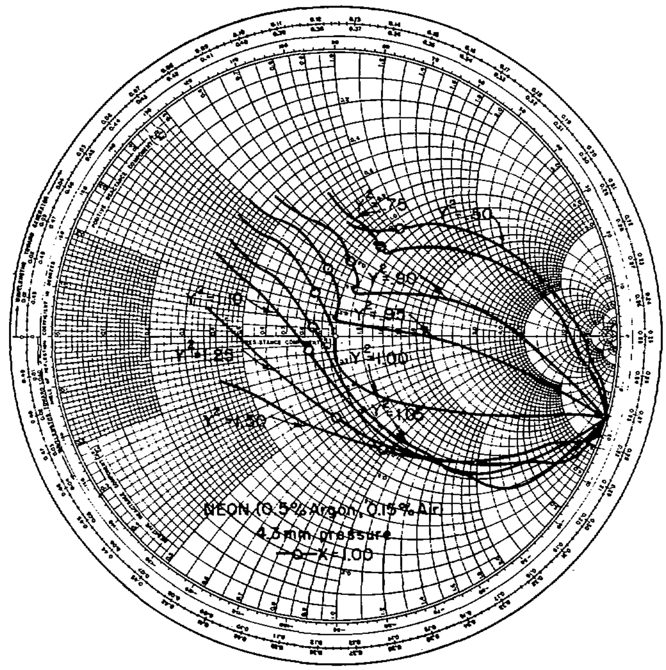

Fig. 12—Experimental impedance loci for neon (0.5 per cent argon, 0.15 per cent air).

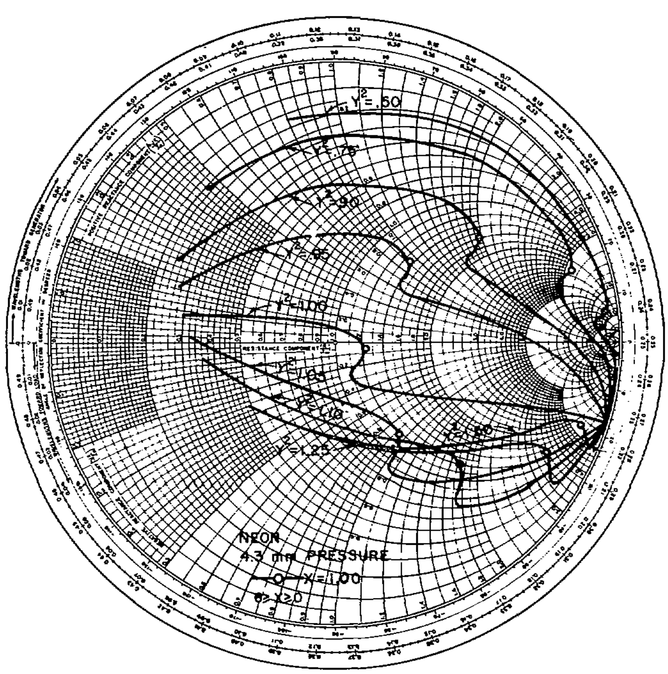

Fig. 13—Theoretical impedance loci for neon.

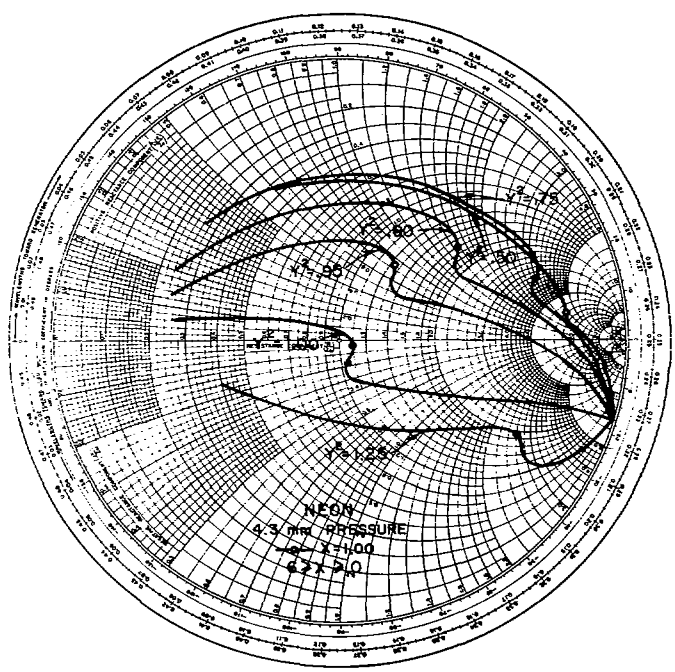

Fig. 14—Theoretical impedance loci with ion sheath correction.

Appendix

The Modified Coulomb Gauge Condition

The gauge condition is

$$\nabla \cdot \boldsymbol{KA} = 0. \tag{74}$$

This will be referred to as the modified Coulomb gauge condition because of its similarity to the Coulomb gauge condition

$$\nabla \cdot \boldsymbol{A} = 0 \tag{75}$$

which is mentioned in various texts.

In general, a particular gauge condition is chosen in order to simplify some aspect of electromagnetic theory. It is necessary to show that the choice of gauge condition has no effect on the field solution for $\boldsymbol{E}$ and $\boldsymbol{H}$ and that it is always possible to find potentials which satisfy the gauge condition. Suppose that $\boldsymbol{A}$ and ψ are potentials which satisfy Maxwell's equations through the relations

$$\boldsymbol{E} = -\nabla\psi - j\omega \boldsymbol{A} \tag{76}$$

$$\mu_0 \boldsymbol{H} = \nabla \times \boldsymbol{A}. \tag{77}$$

It is assumed that no restriction (such as a gauge condition) has been applied to $\boldsymbol{A}$ and ψ. It is known that Maxwell's equations are invariant under a gauge transformation of the type

$$\boldsymbol{A}' = \boldsymbol{A} + \nabla\beta \tag{78}$$

$$\psi' = \psi - j\omega\beta \tag{79}$$

in which $\boldsymbol{A}'$, ψ' are the new potentials and β is the gauge function. If it is required that the new potentials satisfy the modified Coulomb gauge condition, (74) becomes

$$\nabla \cdot \boldsymbol{K}\nabla\beta = -\nabla \cdot \boldsymbol{KA}. \tag{80}$$

Eq. (80) has the same form as the quasi-static equation for the scalar potential and solutions for this equation may be obtained easily. Thus a gauge function β can always be found such that the gauge condition is satisfied. Furthermore the invariance of Maxwell's equations under a gauge transformation assures that the field solutions are unaffected by the choice of gauge condition.

Acknowledgment

The author is indebted to Prof. G. A. Deschamps for his helpful advice and guidance. The author also wishes to thank J. C. Wissmiller and G. L. Duff for their contributions to the experimental apparatus and numerical computations.

References

[1] H. Kogelnik, "On electromagnetic radiation in magneto-ionic media," *J. Res. NBS*, vol. 64D, pp. 515–523; September–October, 1960.
[2] R. Mittra and G. A. Deschamps, "Field solution for a dipole in an anisotropic medium," E. C. Jordan, Ed., *Proc. Symp. on Electromagnetic Theory and Antennas*, Pergamon Press, Inc., New York, N. Y., pp. 495–512; 1963.
[3] B. P. Kononov, A. A. Rukhadze, and G. V. Solodukhov, "Electric field of a radiator in a plasma in an external magnetic field," *Soviet Phys. Tech. Phys.*, vol. 6, pp. 405–410; November, 1961.
[4] J. C. Katzin and M. Katzin, "The Impedance of a Cylindrical Dipole in a Homogeneous Anisotropic Ionosphere," Electromagnetic Research Corp., College Park, Md., Rept. No. NAS 585-2; September 26, 1961.
[5] H. A. Whale, "The excitation of electroacoustic waves by antennas in the ionosphere," *J. Geophys. Res.*, vol. 68, pp. 415–422; January 15, 1963.
[6] E. N. Bramley, "The admittance of a short cylindrical dipole in the ionosphere," *Planet. Space Sci.*, vol. 9, pp. 445–454; August, 1962.
[7] T. R. Kaiser, "The admittance of an electric dipole in a magneto-ionic environment," *Planet Space Sci.*, vol. 9, pp. 639–657; October, 1962.
[8] R. King and C. W. Harrison, "Half-wave cylindrical antenna in a dissipative medium: current and impedance," *J. Res. NBS*, vol. 64D, p. 365; July–August, 1960.
[9] G. A. Deschamps, "Impedance of an antenna in a conducting medium," IRE Trans. on Antennas and Propagation, vol. AP-10, pp. 648–650; September, 1962.
[10] A. W. Trivelpiece and R. W. Gould, "Space charge waves in cylindrical plasma columns," *J. Appl. Phys.*, vol. 30, p. 1784; November, 1959.
[11] A. W. Trivelpiece, A. Ignatius, and P. C. Holscher, "Backward waves in longitudinally magnetized ferrite rods," *J. Appl. Phys.*, vol. 32, pp. 259–267; February, 1961.
[12] R. W. Damon and J. R. Eshbach, "Magnetostatic modes of a ferromagnetic slab," *J. Appl. Phys.*, vol. 31, pp. 104S–105S; May, 1960.
[13] R. I. Joseph and E. Schlomann, "Theory of magnetostatic modes in long, axially magnetized cylinders," *J. Appl. Phys.*, vol. 32, pp. 1001–1005; June, 1961.
[14] F. V. Bunkin, "On radiation in anisotropic media," *Soviet Phys. JETP*, vol. 5, pp. 277–283; September, 1957.
[15] C. T. Tai, "Some wave functions and potential functions pertaining to spherically stratified media," *J. Res. NBS*, vol. 67D, pp. 199–202; March–April, 1963.
[16] M. H. Cohen, "Radiation in plasmas," *Phys. Rev.*, vol. 123, pt. I, pp. 711–721, August 1, 1961; vol. 126, pts. II and III, pp. 389–397, pp. 398–404, April 15, 1962.
[17] I. N. Sneddon, "Elements of Partial Differential Equations," McGraw-Hill Book Company, Inc., New York, N. Y.; 1957.
[18] R. F. Harrington, "Time-Harmonic Electromagnetic Fields," McGraw-Hill Book Company, Inc., New York, N. Y., p. 348; 1961.
[19] S. A. Schelkunoff and H. T. Friis, "Antennas; Theory and Practice," John Wiley and Sons, Inc., New York, N. Y. ch. 10; 1952.
[20] Y. T. Lo, "A note on the cylindrical antenna of non-circular cross section," *J. Appl. Phys.*, vol. 24, pp. 1338–1339; October, 1953.
[21] P. C. Clemmow, "The theory of electromagnetic waves in a simple anisotropic medium," *Proc. IEE (London)*, vol. 110, pp. 101–106; January, 1963.
[22] E. Arbel and L. B. Felsen, "Theory of radiation from sources in anisotropic media, part II: point source in infinite, homogeneous medium," E. C. Jordan, Ed., *Proc. Symp. on Electromagnetic Theory and Antennas*, Pergamon Press, Inc., New York, N. Y., pp. 421–459; 1963.
[23] K. G. Balmain, "The Impedance of a Short Dipole Antenna in a Magnetoplasma," Ph.D. dissertation, University of Illinois, Urbana, Ill.; 1963.
[24] R. F. Mlodnosky and O. K. Garriott, "The V.L.F. admittance of a dipole in the lower ionosphere," *Proc. Internat'l Conf. on the Ionosphere;* Institute of Physics and Physical Society, London, England, pp. 484–491; 1963.
[25] K. Takayama, H. Ikegami, and S. Miyazaki, "Plasma resonance in a radio frequency probe," *Phys. Rev. Lett.*, vol. 5, pp. 238–240; September 15, 1960.
[26] E. L. Ginzton, "Microwave Measurements," McGraw-Hill Book Company, Inc., New York, N. Y.; 1957.