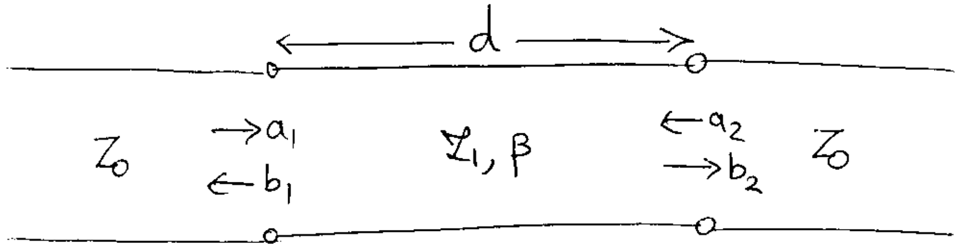
Problem ①



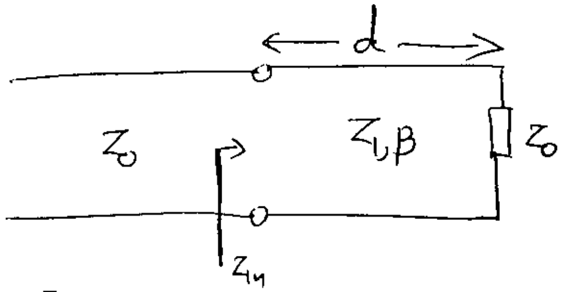
$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

- The circuit is symmetric $\Rightarrow S_{11} = S_{22}$, reciprocal $\Rightarrow S_{12} = S_{21}$ and Lossless $\Rightarrow SS^* = U$.

$$S = \begin{bmatrix} S_{11} & S_{12} \\ S_{12} & S_{11} \end{bmatrix}$$

- Computing S_{11} (and S_{22}):

$S_{11} = \left.\dfrac{b_1}{a_1}\right|_{a_2=0}$, i.e. S_{11} is equivalent to the reflection coefficient in the circuit:



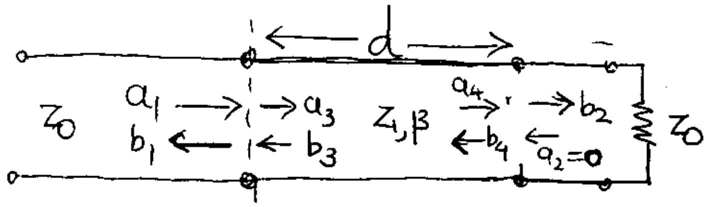
i.e. $S_{11} = \Gamma_{in} = \dfrac{Z_{in} - Z_0}{Z_{in} + Z_0}$ where $Z_{in} = Z_1 \dfrac{Z_0 + jZ_1 \tan\theta}{Z_1 + jZ_0 \tan\theta}$, $\boxed{\theta = \beta d}$

$$S_{11} = \frac{j(Z_1^2 - Z_0^2)\tan\theta}{2Z_1Z_0 + j(Z_1^2 + Z_0^2)\tan\theta} = \boxed{\frac{j(Z_1^2 - Z_0^2)\sin\theta}{2Z_0Z_1\cos\theta + j(Z_0^2 + Z_1^2)\sin\theta} = S_{11}}$$

Computing S_{21} (and S_{12}):

- One way to proceed is to exploit the fact that the circuit is lossless and use $SS^* = U$. However, we choose the definition for illustration purposes:

$S_{21} = \left.\frac{b_2}{a_1}\right|_{a_2=0}$. Therefore, we have to compute the transmission coefficient in the following circuit:



We can write:

$$V_1 = \sqrt{Z_0}\,(a_1+b_1) = \sqrt{Z_1}\,(a_3+b_3) \quad (1)$$
$$V_4 = \sqrt{Z_0}\,b_2 = \sqrt{Z_1}\,(a_4+b_4) \quad (2)$$
$$a_4 = a_3 e^{-j\theta} \quad (3)$$
$$b_3 = b_4 e^{-j\theta} \quad (4)$$
$$\frac{b_4}{a_4} = \Gamma_L = \frac{Z_0-Z_1}{Z_0+Z_1} \quad (5)$$
$$b_1 = S_{11} a_1 \quad (6)$$

From (1) and (6): $V_1 = \sqrt{Z_0}\,a_1(1+S_{11}) = \sqrt{Z_1}\,(a_3+b_3)$ (7)

From (2) and (5): $\sqrt{Z_1}\,a_4(1+\Gamma_L) = \sqrt{Z_0}\,b_2$ (8)

From (7) and (3),(4): $\sqrt{Z_0}\,a_1(1+S_{11}) = \sqrt{Z_1}\,(a_4 e^{j\theta} + b_4 e^{-j\theta})$

$$\overset{(5)}{=} \sqrt{Z_1}\,a_4\,(e^{j\theta} + \Gamma_L e^{-j\theta}) \quad \ldots \quad (9)$$

i.e from (9) and (8): $\cancel{\sqrt{Z_0}}\,a_1(1+S_{11}) = \cancel{\sqrt{Z_1}}\,(e^{j\theta}+\Gamma_L e^{-j\theta})\,\dfrac{\cancel{\sqrt{Z_0}}\,b_2}{\cancel{\sqrt{Z_1}}\,(1+\Gamma_L)}$

Therefore, $$a_1 (1+S_{11})(1+\Gamma_L) = b_2\left(e^{j\theta} + \Gamma_L e^{-j\theta}\right)$$

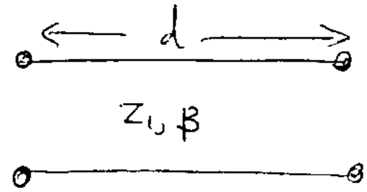
$$\Rightarrow \quad S_{21} = \frac{b_2}{a_1} = \frac{(1+S_{11})(1+\Gamma_L)}{\left(e^{j\theta} + \Gamma_L e^{-j\theta}\right)}$$

But
$$1+S_{11} = \frac{2Z_0Z_1\cos\theta + 2jZ_1^2\sin\theta}{2Z_0Z_1\cos\theta + j(Z_0^2+Z_1^2)\sin\theta}$$

from which
$$\boxed{S_{21} = \frac{2Z_0Z_1}{2Z_0Z_1\cos\theta + j(Z_0^2+Z_1^2)\sin\theta}}$$

Second-solution:

ABCD for a line:



$$\begin{bmatrix} \cos\beta d & jZ_1 \sin\beta d \\ j\frac{\sin\beta d}{Z_1} & \cos\beta d \end{bmatrix} \qquad \theta = \beta d$$

Convert to [S]:

$$S_{22} = S_{11} = \frac{A + B/Z_0 - CZ_0 - D}{A + B/Z_0 + CZ_0 + D} = \frac{\cancel{\cos\beta d} + j\frac{Z_1}{Z_0}\sin\beta d - j\frac{Z_0}{Z_1}\sin\beta d - \cancel{\cos\beta d}}{\cos\beta d + j\frac{Z_1}{Z_0}\sin\beta d + j\frac{Z_0}{Z_1}\sin\beta d + \cos\beta d}$$

$$= \frac{+j\left(\frac{Z_1}{Z_0} - \frac{Z_0}{Z_1}\right)\sin\beta d}{2\cos\beta d + j\left(\frac{Z_1}{Z_0} + \frac{Z_0}{Z_1}\right)\sin\beta d} = \frac{j\left(Z_1^2 - Z_0^2\right)\sin\theta}{2Z_0Z_1\cos\theta + j\left(Z_0^2 + Z_1^2\right)\sin\theta}$$

$$S_{12} = S_{21} = \frac{2}{A + B/Z_0 + CZ_0 + D} = \frac{2}{2\cos\beta d + j\left(\frac{Z_1}{Z_0} + \frac{Z_0}{Z_1}\right)\sin\beta d} = \frac{2Z_0Z_1}{2Z_0Z_1\cos\theta + j\left(Z_0^2 + Z_1^2\right)\sin\theta}$$

Problem 2 :

Design a reciprocal, lossless Two-port with scattering matrix:

$$S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \quad \text{where} \quad S_{11} = S_{22} = 0.2\angle 108^\circ \text{ and } S_{12} = S_{21} \text{ at } f = 1\text{GHz}$$

(1)

Since the two-port is lossless $\tilde{S}\tilde{S}^{*T} = \tilde{I}$ and once S_{11} is defined so is S_{12}. In our case

$$\tilde{S}S^{*T} = \begin{bmatrix} S_{11} & S_{12} \\ S_{12} & S_{11} \end{bmatrix}\begin{bmatrix} S_{11}^* & S_{12}^* \\ S_{12}^* & S_{11}^* \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow$$

$$|S_{11}|^2 + |S_{12}|^2 = 1 \text{ and} \quad \ldots (1)$$

$$S_{11}S_{12}^* + S_{12}S_{11}^* = 0 \quad \ldots (2)$$

From (1) $S_{12} = \sqrt{1-|S_{11}|^2} = \sqrt{1-(0.2)^2} = 0.98$

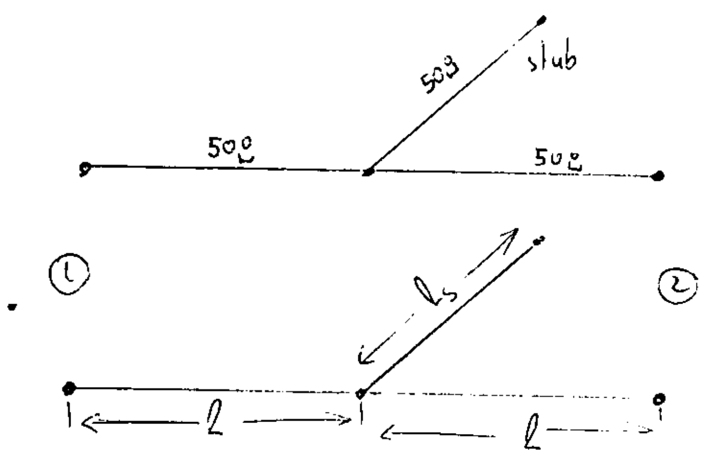
From (2) $|S_{11}||S_{12}|e^{j(\phi_{11}-\phi_{12})} = -|S_{11}||S_{12}|e^{j(\phi_{12}-\phi_{11})} \Rightarrow$

$$\phi_{11}-\phi_{12} = \phi_{12}-\phi_{11}-\pi \Rightarrow \boxed{\phi_{12} = \frac{2\phi_{11}+\pi}{2}} \quad \ldots (3)$$

i.e. $\phi_{12} = \dfrac{2(108)+180}{2} = 198^\circ \; (-162^\circ)$

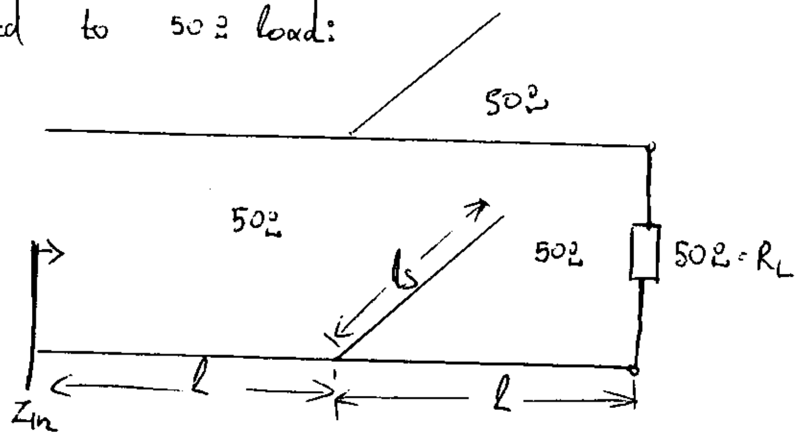
Therefore $\boxed{S_{12} = 0.98\angle -162^\circ}$ $\ldots (4)$

(a) The desired [S] is symmetrical (i.e $S_{11}=S_{22}$), reciprocal and lossless. A possible configuration would be:



The stub provides the magnitude of S_{11} and the line its phase.

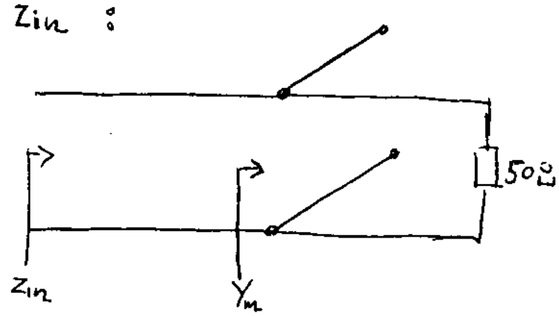
(b) From the definition of the scattering matrix we get $S_{11} \triangleq \frac{b_1}{a_1}\Big|_{a_2=0}$. If we are to connect the two-port to 50Ω lines then $a_2=0$ when the output port is terminated to 50Ω load:



Then $$S_{11} = \Gamma_{in} = \frac{Z_{in}-50}{Z_{in}+50} \quad \ldots\ldots \quad (5)$$

i.e S_{11} is the input reflection coefficient of the above terminated two-port if considered as a one port.

To find Z_{in} :



$$0.2e^{+j\angle 108^\circ} = \frac{Z_{in}-50}{Z_{in}+50} \Rightarrow Z_{in} = 50\frac{S_{11}+1}{1-S_{11}} = \frac{50\left(1+0.2e^{+j\angle 108^\circ}\right)}{\left(1-0.2e^{j\angle 108^\circ}\right)} = 41.25 + 16.35j$$

Now we know that $\bar{Y}_{in} = \bar{Y}_{stub} + 1 = j\bar{B}+1$. Therefore $Re[\bar{Y}_{in}] = 1$.
Therefore, starting from $\bar{Z}_{in}$ on the Smith Chart and moving towards the load on the constant SWR circle we should reach the point $\bar{Y}_{in}$ with $Re[\bar{Y}_{in}]=1$. The constant SWR circle intersects the $Re[Y]=1$ circle at two points. One of them is very close to $\bar{Y}_{in}$ leading to a very short l to be practically designed.

Ie the sensitivity of [S] with small percentile variations of the length l will be very large. (For example : $\frac{|\Delta S_{11}|/|S_{11}|}{|\Delta l|/l}$, Large)

The second point of intersection is at $\bar{Y}_m = 1 + 0.41j$ which corresponds to a length $l = (0.36 - 0.153)\lambda = 0.207\lambda$

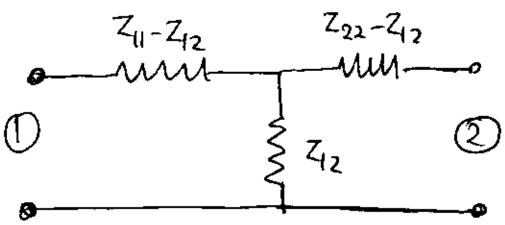
Now $\bar{Y}_m = 1 + j\bar{B}$ therefore $\bar{B} = 0.41$ Ie we need a capacitive (open-circuited) stub with $\bar{B} = 0.41$. Starting from the point $\bar{Y} = 0$ on the outermost circle of the smith chart and moving towards the generator untill $\bar{Y} = 0.41j$ we find that $l_s = (0.061 - 0.000)\lambda = 0.061\lambda$

Therefore the designed characteristics of our Two-port are the following:

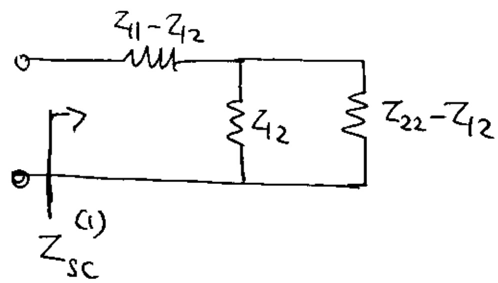
1. All lines with $Z_0 = 50\Omega$
2. $l = 0.207\lambda$
3. $l_s = 0.061\lambda$

Problem 4.8 :

Use the equivalent model for a 2-port :



Then shorting port 2 :

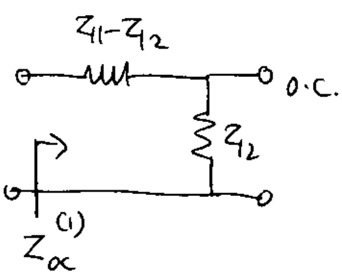


$$Z_{sc}^{(1)} = Z_{11}-Z_{12} + \frac{Z_{12}(Z_{22}-Z_{12})}{Z_{22}} = Z_{11} - Z_{12}^{2}/Z_{22} \quad \cdots\cdots (1)$$

Similarly

$$Z_{sc}^{(2)} = Z_{22} - Z_{12}^{2}/Z_{11}$$

For an open circuited port 2:



$Z_{oc}^{(1)} = Z_{11}$, similarly $Z_{oc}^{(2)} = Z_{22}$

Hence $Z_{12}^{2} = -\left(Z_{sc}^{(1)} - Z_{11}\right) Z_{22} = \left(Z_{oc}^{(1)} - Z_{sc}^{(1)}\right) Z_{oc}^{(2)}$ (from (1)).

I.e: By measuring $Z_{oc}^{(1)}, Z_{oc}^{(2)}$ and $Z_{sc}^{(1)}$ we can identify the equivalent 2-port circuit and corresponding impedance-matrix.

(4.16)

a) To be lossless, [S] must be unitary:

$|S_{11}|^2 + |S_{12}|^2 + |S_{13}|^2 + |S_{14}|^2 = (.1)^2 + (.6)^2 + (.6)^2 = 0.73 \neq 1$

Thus the network is <u>not lossless</u>

b) To be reciprocal, [S] must be symmetric:
Since $S_{13} \neq S_{31}$, the network is <u>not reciprocal</u>.

c) When all other ports are matched, $\Gamma = S_{11}$
Thus, $RL = -20 \log |\Gamma| = -20 \log(.1) = \underline{20\,dB}$ at port 1. ✓

d) When all other ports are matched, the insertion loss from port 2 to port 4 is,

$IL = -20 \log |S_{42}| = -20 \log(.6) = \underline{4.4\,dB}$. Phase delay $= \underline{45°}$ ✓

e) For a short circuit at port 3, and matched loads at other ports, we have,

$$V_2^+ = V_4^+ = 0\,, \quad V_3^+ = -V_3^-$$
$$V_1^- = S_{11}V_1^+ + S_{13}V_3^+ = S_{11}V_1^+ - S_{13}V_3^-$$
$$V_3^- = S_{31}V_1^+ + S_{33}V_3^+ = S_{31}V_1^+ - S_{33}V_3^-$$

Solving the second equation for V_3^-:

$$V_3^- = \frac{S_{31}V_1^+}{1+S_{33}}$$

Then,

$$\Gamma^{(1)} = \frac{V_1^-}{V_1^+} = S_{11} - \frac{S_{13}S_{31}}{1+S_{33}} \checkmark$$

$$= .1j - \frac{(.6\angle 45°)(.6\angle -45°)}{1+0} = .1j - .36 = 0.374\underline{\angle 164°} \checkmark$$

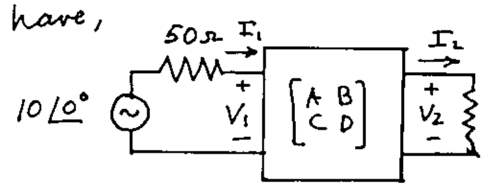
(verified with supercompact)

(4.24) Using Table 4.1, the ABCD matrix of the cascade of four components is,

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 40+j30 \\ 0 & 1 \end{bmatrix}}_{R+jX} \underbrace{\begin{bmatrix} 3 & 0 \\ 0 & 1/3 \end{bmatrix}}_{\text{TRANSF.}} \underbrace{\begin{bmatrix} 0 & j75 \\ j/75 & 0 \end{bmatrix}}_{\text{T-LINE}} = \begin{bmatrix} (-.1333+j.1778) & j225 \\ j/225 & 0 \end{bmatrix}$$

CHECK: $AD - BC = 1$ ✓

Then we have,



$I_2 = V_2/60.$

$$V_1 = AV_2 + BI_2 = (A + B/60)\, V_2$$
$$I_1 = CV_2 + DI_2 = (C + D/60)\, V_2$$

$$Z_{in} = \frac{V_1}{I_1} = \frac{(A+B/60)}{(C+D/60)} = \frac{(-.1333+j.1778+j225/60)}{j/225} = 883.8 + j30.0\ \Omega \checkmark$$

$$V_L = V_2 = \frac{I_1}{(C+D/60)} = \frac{V_g}{(C+D/60)(Z_{in}+50)} = \frac{10(-j225)}{933.8+j30} = 2.41\angle{-92°}$$