Maxwell's Equations in Differential Form

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} - \vec{M}_i = -\vec{M}_d - \vec{M}_i$$

$$\nabla \times \vec{H} = \vec{J}_i + \vec{J}_c + \frac{\partial \vec{D}}{\partial t} = \vec{J}_i + \vec{J}_c + \vec{J}_d$$

$$\nabla \cdot \vec{D} = \rho_{ev}$$

$$\nabla \cdot \vec{B} = \rho_{mv}$$

$$\vec{M}_d = \frac{\partial \vec{B}}{\partial t}, \qquad \vec{J}_d = \frac{\partial \vec{D}}{\partial t}$$

$\vec{E} \equiv$ Electric field intensity [V/m]

$\vec{B} \equiv$ Magnetic flux density [Weber/m^2 = V s/m^2 = Tesla]

$\vec{M}_i \equiv$ Impressed (source) magnetic current density [V/m^2]

$\vec{M}_d \equiv$ Magnetic displacement current density [V/m^2]

$\vec{H} \equiv$ Magnetic field intensity [A/m]

$\vec{J}_i \equiv$ Impressed (source) electric current density [A/m^2]

$\vec{D} \equiv$ Electric flux density or electric displacement [C/m^2]

$\vec{J}_c \equiv$ Electric conduction current density [A/m^2]

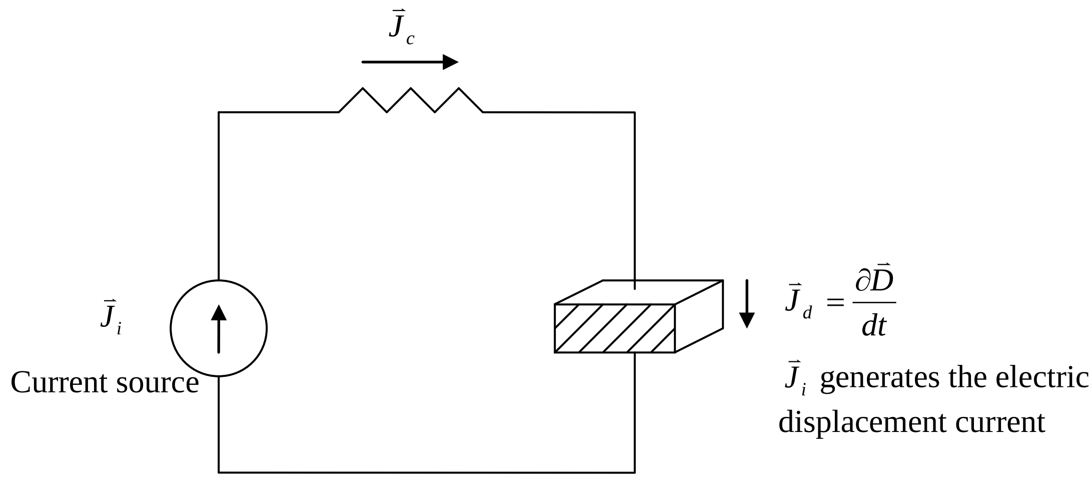
$\vec{J}_d \equiv$ Electric displacement current density [A/m^2]

$\rho_{ev} \equiv$ Electric charge volume density [C/m^3]

$\rho_{mv} \equiv$ Magnetic charge volume density [Weber/m^3]

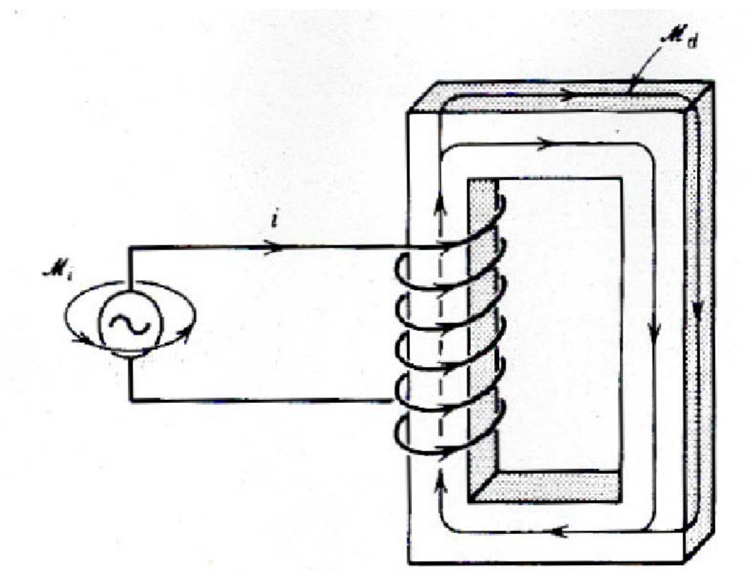
Remarks:

1. Impressed magnetic current density ($\vec{M}_i$) and magnetic charge density (ρ_{mv}) are unphysical quantities introduced through "generalized" current to balance Maxwell's equations.
2. Although unphysical, $\vec{M}_i$, ρ_{mv} similar to $\vec{J}_i$ and ρ_{ev} can be considered as energy sources that generate the fields.



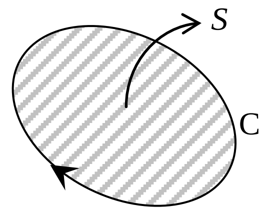
3. Through "equivalent principle" $\vec{M}_i$ and ρ_{mv} can be used to simplify the solutions to some boundary value problems.
4. $\vec{M}_d = \frac{\partial \vec{B}}{\partial t}$ (magnetic displacement current density [V/m^2]) is introduced

   analogous to $\vec{J}_d = \frac{\partial \vec{D}}{\partial t}$ (electric displacement current density [A/m^2])



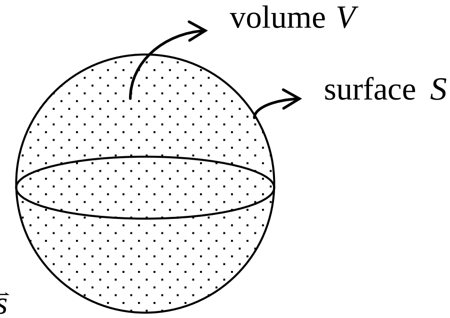
Integral form of Maxwell's Equations

Elementary vector calculus:



Stokes' Theorem: $\iint_S \left(\nabla \times \vec{A}\right) \cdot d\vec{s} = \oint_C \vec{A} \cdot d\vec{l}$

• It says that if you want to know what is happening in the interior of a surface bounded by a curve just go around the curve and add up the field contributions.



Divergence Theorem: $\iiint_V \left(\nabla \cdot \vec{A}\right) dV = \oiint_S \vec{A} \cdot d\vec{s}$

• In simple words, divergence theorem states that if you want to know what is happening within a volume of V just go around the surface S (bounding volume V) and add up the field contributions.

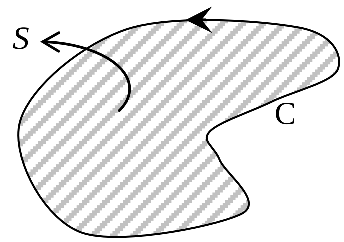
Null Identities:

$$\nabla \cdot \left(\nabla \times \vec{A}\right) = 0 \Leftrightarrow \nabla \cdot \vec{B} = 0 \Leftrightarrow \vec{B} = \nabla \times \vec{A}$$

$$\nabla \times \left(\nabla \phi\right) = 0 \Leftrightarrow \underset{\text{(electrostatic)}}{\nabla \times \vec{E} = 0} \Leftrightarrow \vec{E} = -\nabla \phi$$

• The Divergence and Stokes' theorems can be used to obtain the integral forms of the Maxwell's Equations from their differential form.

• $\nabla \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t} - \vec{M}_i \Rightarrow \iint_S \nabla \times \vec{E} \cdot d\vec{s} = -\dfrac{\partial}{\partial t} \iint_S \vec{B} \cdot d\vec{s} - \iint_S \vec{M}_i \cdot d\vec{s}$



$$\Rightarrow \oint_C \vec{E}\cdot d\vec{l} = -\frac{\partial}{\partial t}\iint_S \vec{B}\cdot d\vec{s} - \iint_S \vec{M}_i \cdot d\vec{s}$$

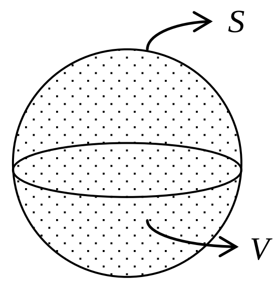
- $\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + \sigma\vec{E} + \vec{J}_i \Leftrightarrow \iint_S \nabla \times \vec{H}\cdot d\vec{s} = \iint_S \frac{\partial \vec{D}}{\partial t}\cdot d\vec{s} + \iint_S \sigma\vec{E}\cdot d\vec{s} + \iint_S \vec{J}_i \cdot d\vec{s}$

$$\Rightarrow \oint_C \vec{H}\cdot d\vec{l} = \frac{\partial}{\partial t}\iint_S \vec{D}\cdot d\vec{s} + \iint_S \vec{J}_c \cdot d\vec{s} + \iint_S \vec{J}_i \cdot d\vec{s}$$

Where $\vec{J}_c = \sigma \vec{E}$

- $\nabla\cdot\vec{D} = \rho_{ev} \Leftrightarrow \iiint_V \nabla\cdot\vec{D}\ dv = \iiint_V \rho_{ev}\ dv \Rightarrow \oiint_S \vec{D}\cdot d\vec{s} = \iiint_V \rho_{ev}\ dv = Q_e$

- $\nabla\cdot\vec{B} = \rho_{mv} \Leftrightarrow \iiint_V \nabla\cdot\vec{B}dv = \iiint_V \rho_{mv}dv \Rightarrow \oiint_S B\cdot d\vec{s} = Q_m = \iiint_V \rho_{mv}dv$



Since $Q_m = 0 \Rightarrow \oiint_S \vec{B}\cdot d\vec{s} = 0$

Helmholtz Theorem

• Traditionally, Newtonian mechanic is formulated in terms of force ($\vec{F}$) and torque ($\vec{\tau}$),

$\vec{F} = \dfrac{d\vec{P}}{dt}$, $\quad \vec{\tau} = \dfrac{d\vec{L}}{dt}$ where $\vec{L}$ is the angular momentum.

However such an approach to classical electromagnetism will be unnecessarily cumbersome. Instead, the description of electromagnetics starts with Maxwell's equations which are written in terms of curls and divergences. The question is then whether or not such a description (in terms of curls and divergences) is sufficient and unique? The answer to this question is provided by Helmholtz Theorem

• A vector field is determined to within an additive constant if both its divergence and its curl are specified everywhere.

• Equivalent statement: A vector field is uniquely specified by giving its divergence and its curl within a region and its normal component over the boundary, that is if:
S and $\vec{C}$ are known and given by
$\nabla \cdot \vec{M} = S$,
$\nabla \times \vec{M} = \vec{C}$
and $\vec{M}_n$ (the normal component of $\vec{M}$ on the boundary) is also known; then $\vec{M}$ is uniquely defined.

Remark: Helmholtz's theorem allows us to appreciate the importance of the Maxwell's equations in which $\vec{E}$ and $\vec{H}$ are defined by their divergence and curl.
Ex.: $\nabla \times \vec{E} = -\dfrac{\partial}{\partial t}\vec{B}$ and $\nabla \cdot \vec{E} = \dfrac{\rho_{ev}}{\varepsilon}$

Irrotational & Solenoidal Fields (Use of Helmholtz Theorem)

Definition:
• A field is irrotational if its curl is zero
$\nabla \times \vec{F}_i = 0 \equiv F_i$ is irrotational
• A field is solenoidal (divergenceless) if its divergence is zero
$\nabla \cdot \vec{F}_s = 0 \equiv \vec{F}_s$ is solenoidal

Theorem:
• A vector field which its divergence and curl vanishes at infinity can be written as the sum of an irrotational & a solenoidal fields.

• According to the theorem stated above, the vector field $\vec{M}$ can be written as

(1) $\vec{M} = \vec{F}_i + \vec{F}_s$

• Since $\vec{F}_i$ is irrotational then $\nabla \times \vec{F}_i = 0 \Rightarrow \vec{F}_i = -\nabla V$ where V is a scalar function.

• Since $\vec{F}_s$ is solenoidal then $\nabla \cdot \vec{F}_s = 0 \Rightarrow \vec{F}_s = \nabla \times \vec{A}$ then (1) $\Rightarrow \vec{M} = -\nabla V + \nabla \times \vec{A}$

Constitutive Relations

$\vec{D} = \varepsilon \, \vec{E}$

$\vec{B} = \mu \, \vec{H}$

$\varepsilon = \varepsilon_r \, \varepsilon_0$

$\mu = \mu_r \, \mu_0$

$\varepsilon \equiv$ permittivity [F/m]

$\varepsilon_o \equiv$ vacuum permittivity = 8.85×10^{-12} [F/m]

$\varepsilon_r \equiv$ Relative permittivity or dielectric constant [#]

$\mu \equiv$ permeability [H/m]

$\mu_0 \equiv$ free space permeability = $4\pi \times 10^{-7}$ [H/m]

$\mu_r \equiv$ relative permeability [#]

• We also write

(1) $\mu_r = 1 + \chi_m$

(2) $\varepsilon_r = 1 + \chi_e$

Where χ_m and χ_e are the magnetic and electric susceptibility, respectively. χ_m, χ_e are dimensionless.

• Index of refraction is defined as

(3) $n = \sqrt{\varepsilon_r \, \mu_r}$

$n \equiv$ index of refraction or phase index [#]

• If we are mostly concerned with non-magnetic materials then

$\mu_r \approx 1 \Rightarrow \mu = \mu_0 \Rightarrow n = \sqrt{\varepsilon_r}$

Polarization and Magnetization

• Polarization vector $\vec{P}$ and magnetization vector $\vec{M}$ are related to $\vec{D}$ and $\vec{E}$ and $\vec{B}$ and $\vec{H}$ according to:

(4) $\vec{D} = \vec{P} + \varepsilon_0 \vec{E}$

(5) $\vec{B} = \mu_0 \vec{H} + \mu_0 \vec{M} = \mu_0 \left(\vec{H} + \vec{M} \right)$

• Assuming $\vec{P} = \varepsilon_0 \chi_e \vec{E}$, then:

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P} = \varepsilon_0 \vec{E} + \varepsilon_0 \chi_e \vec{E} = \varepsilon_0 (1 + \chi_e) \vec{E} = \varepsilon_0 \varepsilon_r \vec{E} \Rightarrow$$
$$\vec{D} = \varepsilon \vec{E}$$

• Assuming $\vec{M} = \chi_m \vec{H}$ then

$$\vec{B} = \mu_0 \vec{H} + \mu_0 \vec{M} = \mu_0 (1 + \chi_m) \vec{H} = \mu_o \mu_r \vec{H} \Rightarrow$$
$$\vec{B} = \mu \vec{H}$$

• ε and μ describe the macroscopic response of the media. ε characterizes the electric response while μ describes the magnetic response. In the following we assume our medium is nonmagnetic.

Homogeneous vs. Inhomogeneous, Isotropic vs. Anisotropic, Linear vs. Non-Linear

• If ε depends on position, i.e. $\varepsilon(r)$, media is non-homogeneous.

• If ε depends on the direction of the applied field, i.e., $\vec{D}$ and $\vec{E}$ are not co-linear, then the medium is said to be anisotropic. Examples of anisotropic materials are Calcite (uniaxial) or topaz (biaxial).
• In the case of anisotropic medium ε is a tensor (for our purposes a matrix of rank 2). We then write:

(1) $D = \overset{=}{\varepsilon} \vec{E}$ where

(2) $$\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \varepsilon_0 \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}$$

• If ε depends on the magnitude of the applied field, i.e. $\varepsilon\left(\left|\vec{E}\right|\right)$, we say medium is nonlinear. Note that in this case even though permittivity is a function of the filed strength, it can still be a scalar function.

• An example of non-linear medium is when

(3) $$\frac{\varepsilon}{\varepsilon_0} = \left[1 + \frac{1}{b^2} \left(c^2 B^2 - E^2 \right) \right]^{-1/2},$$

Where b is the maximum field strength.

• Interesting thing about (3) is the fact that it describes the response of the vacuum, (proposed by Born & Infeld) in order to address the problem of vacuum infinite self-energy.

Infinite Self Energy

• A charge particle can be thought as the localization of the charge density. As a charge distribution localizes to a point charge, its electromagnetic energy grows more and more and becomes unbounded[1]. To avoid this infinite self-energy we can think that some saturation of field strength takes place, i.e., field strength has an upper bound. This classical non-linear effect is given by $\frac{\varepsilon}{\varepsilon_0}=\left[1+\frac{1}{b^2}\left(c^2B^2-E^2\right)\right]^{1/2}$

• However, there are few problems with Born & Infeld classical non-linear vacuum response. (1) The theory suffers from arbitrariness in the manner in which the nonlinearities occur. (2) There are problems with transitions to the quantum domain. (3) So far, there has been no experimental evidence of the existence of this kind of classical nonlinearities.

• As to the last point, we may note that in the orbits of electrons in atoms, field strengths of 10^{11}-10^{17} V/m are present. For heavier atoms, these fields can be even as large as 10^{21} V/m at the edge of the nucleus; yet ordinary quantum theory with linear superposition is sufficient to describe the observed phenomena with a high degree of accuracy.

HW: Consider a hydrogen atom unexcited and in thermal equilibrium. Calculate the magnitude of the electric field due to its nucleus at the site of its electron.

Temporal dispersion

• If ε depends on frequency, i.e. $\varepsilon(\omega)$, we say the medium is dispersive (frequency dispersion)

$$(1)\quad \varepsilon_r=\frac{\varepsilon}{\varepsilon_0}=1+\frac{\omega_p{}^2}{\omega_0{}^2+j\,\gamma\,\omega-\omega^2}$$

• Note that from (1) we can write

$$(2)\quad \varepsilon=\varepsilon'-j\varepsilon''$$

• Remarks: Temporal dispersion means that the parameters describing the medium response (e.g. ε and μ) are functions of time derivatives. Spatial dispersion means that the parameters describing the medium response (e.g. ε and μ) are functions of space derivatives.

• If a medium is linear, homogeneous, and isotropic, we say the medium is simple.

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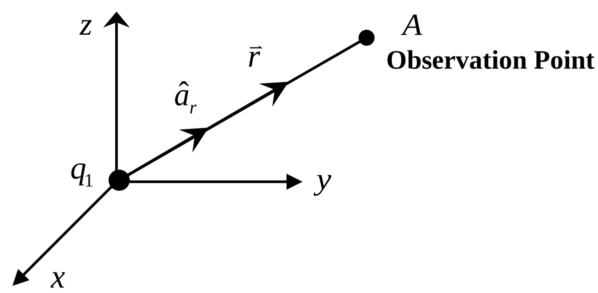
[1] Recall that the potential energy (U) corresponding to two charges q_1 and q_2 separated by a distance r is given by: $U = q_1\, q_2 / 4\, \pi\, \varepsilon_0\, r$

Electric Field

• Electric field due to a point charge in origin

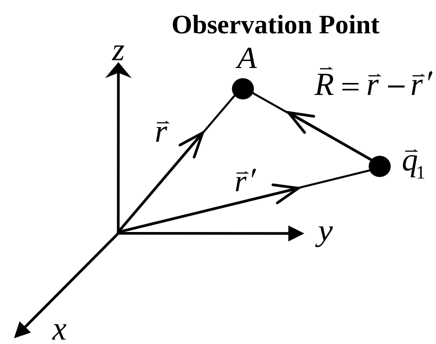
$$\boxed{\vec{E} = \frac{q_1}{4\pi\varepsilon_0}\frac{\hat{a}_r}{|\vec{r}|^2} = \frac{q_1}{4\pi\varepsilon_0}\frac{\vec{r}}{|\vec{r}|^3} = \frac{q_1}{4\pi\varepsilon_0}\frac{\vec{r}}{r^3}}$$ where

$\hat{a}_r = \frac{\vec{r}}{|\vec{r}|} = \frac{\vec{r}}{r}$ and we use the shorthand notation $|\vec{r}| = r$.



• Electric filed due to a point charge not at the origin

$$\boxed{\vec{E} = \frac{q_1}{4\pi\varepsilon_0}\frac{\hat{a}_R}{|\vec{R}|^2} = \frac{q_1}{4\pi\varepsilon_0}\frac{\vec{R}}{|\vec{R}|^3} = \frac{q_1}{4\pi\varepsilon_0}\frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3}}$$



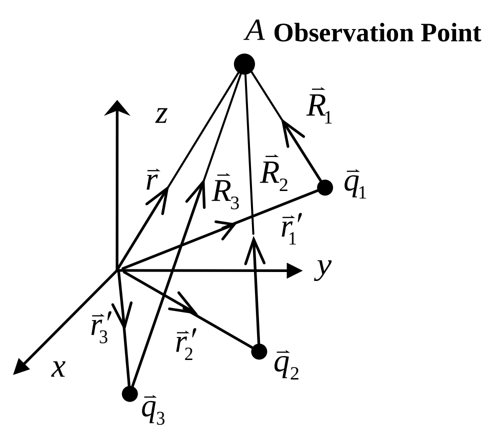
• Superposition principle

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\left[q_1\frac{\hat{a}_{R1}}{|\vec{R}_1|^2} + q_2\frac{\hat{a}_{R2}}{|\vec{R}_2|^2} + q_3\frac{\hat{a}_{R3}}{|\vec{R}_3|^2} + \cdots\right]$$

$$= \frac{1}{4\pi\varepsilon_0}\left[q_1\frac{\vec{R}_1}{|\vec{R}_1|^3} + q_2\frac{\vec{R}_2}{|\vec{R}_2|^3} + q_3\frac{\vec{R}_3}{|\vec{R}_3|^3} + \cdots\right]$$

$$= \frac{1}{4\pi\varepsilon_0}\left[q_1\frac{(\vec{r}-\vec{r}_1')}{|\vec{r}-\vec{r}_1'|^3} + q_2\frac{(\vec{r}-\vec{r}_2')}{|\vec{r}-\vec{r}_2'|^3} + q_3\frac{(\vec{r}-\vec{r}_3')}{|\vec{r}-\vec{r}_3'|^3} + \cdots\right]$$

$$\boxed{E = \frac{1}{4\pi\varepsilon_0}\sum_{k=1}^{N} q_k\frac{\vec{r}-\vec{r}_k'}{|\vec{r}-\vec{r}_k'|^3}}$$



Electric Field & Potential due to Continuous Charge Distribution

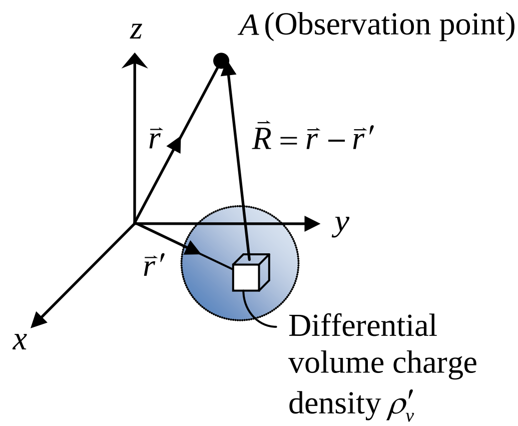
• Volume charge density, $\rho_v'(\vec{r}\,') = \rho_v'(x', y', z')$

$$d\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{\vec{R}}{|\vec{R}|^3}\rho_v'\,dv' = \frac{1}{4\pi\varepsilon_0}\frac{\hat{a}_R}{|\vec{R}|^2}\rho_v'\,dv'$$

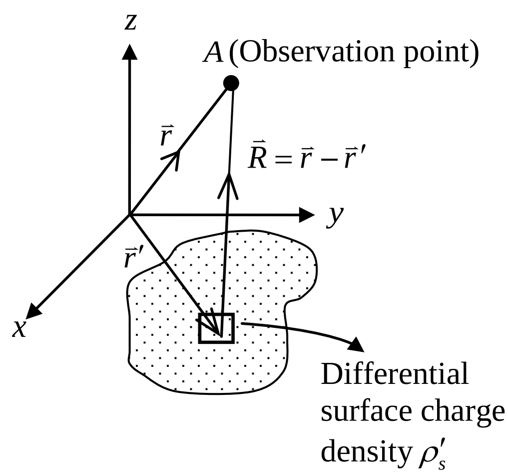
$$= \frac{1}{4\pi\varepsilon_0}\frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3}\rho_v'\,dv'$$

$$\boxed{\vec{E} = \iiint_{v'}\frac{1}{4\pi\varepsilon_0}\frac{\hat{a}_R}{|\vec{R}|^2}\rho_v'\,dv' = \iiint_{v'}\frac{1}{4\pi\varepsilon_0}\frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3}\rho_v'\,dv'}$$

$$\boxed{V = \iiint_{v'}\frac{1}{4\pi\varepsilon_0}\frac{\rho_v'\,dv'}{|\vec{R}|} = \iiint_{v'}\frac{1}{4\pi\varepsilon_0}\frac{\rho_v'\,dv'}{R} = \iiint_{v'}\frac{1}{4\pi\varepsilon_0}\frac{\rho_v'\,dv'}{|\vec{r}-\vec{r}\,'|}}$$



• Surface charge density, $\rho_s'(\vec{r}\,') = \rho_s'(x', y', z')$



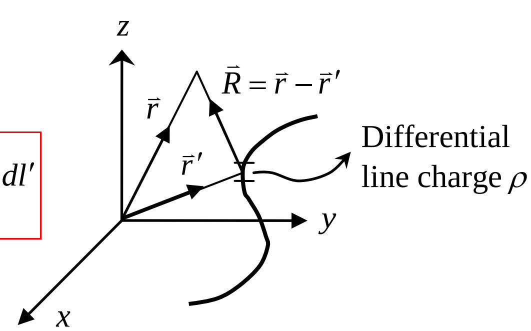
$$\boxed{\vec{E} = \iint_{s'}\frac{1}{4\pi\varepsilon_0}\frac{\hat{a}_R}{|\vec{R}|^2}\rho_s'\,ds' = \iint_{s'}\frac{1}{4\pi\varepsilon_0}\frac{\vec{R}}{|\vec{R}|^3}\rho_s'\,ds' = \frac{1}{4\pi\varepsilon_0}\iint_{s'}\frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3}\rho_s'\,ds'}$$

$$\boxed{V = \frac{1}{4\pi\varepsilon_0}\iint_{s'}\frac{\rho_s'\,ds'}{|\vec{R}|} = \frac{1}{4\pi\varepsilon_0}\iint_{s'}\frac{\rho_s'\,ds'}{R} = \frac{1}{4\pi\varepsilon_0}\iint_{s'}\frac{\rho_s'\,ds'}{|\vec{r}-\vec{r}\,'|}}$$

• $\vec{E}$ and V due to a line charge, $\rho_l'(\vec{r}\,') = \rho_l'(x', y', z')$

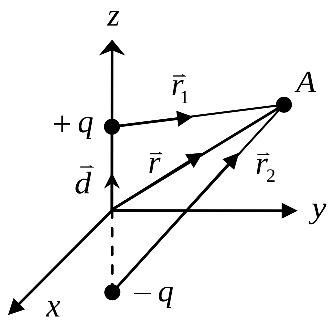
$$\boxed{\vec{E} = \frac{1}{4\pi\varepsilon_0}\int_{l'}\frac{\hat{a}_R}{|\vec{R}|^2}\rho_l'\,dl' = \frac{1}{4\pi\varepsilon_0}\int_{l'}\frac{\vec{R}}{|\vec{R}|^3}\rho_l'\,dl' = \frac{1}{4\pi\varepsilon_0}\int_{l'}\frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3}\rho_l'\,dl'}$$

$$\boxed{V = \frac{1}{4\pi\varepsilon_0}\int_{l'}\frac{\rho_l'\,dl'}{|\vec{R}|} = \frac{1}{4\pi\varepsilon_0}\int_{l'}\frac{\rho_l'\,dl'}{R} = \frac{1}{4\pi\varepsilon_0}\int_{l'}\frac{\rho_l'\,dl'}{|\vec{r}-\vec{r}\,'|}}$$



Remark: If you have forgotten the differential length, surface, and volume elements for rectangular, cylindrical, or spherical, you may want to revisit these. See also the end of this note set.

Electric Field of a Dipole



$$\vec{E} = \frac{q}{4\pi\varepsilon_0}\left[\frac{\vec{r}_1}{|\vec{r}_1|^3} - \frac{\vec{r}_2}{|\vec{r}_2|^3}\right]$$

$$\vec{r}_1 = \vec{r} - \frac{\vec{d}}{2}$$

$$\vec{r}_2 = \vec{r} + \frac{\vec{d}}{2} \quad \& \quad |\vec{r}| >> |\vec{d}|$$

Then

$$\vec{E} = \frac{q}{4\pi\varepsilon_0}\left[\frac{\vec{r} - \vec{d}/2}{|\vec{r} - \vec{d}/2|^3} - \frac{\vec{r} + \vec{d}/2}{|\vec{r} + \vec{d}/2|^3}\right]$$

$$|\vec{r} - \vec{d}/2|^{-3} \approx |\vec{r}|^{-3}\left[1 + \frac{3}{2}\frac{\vec{r}\cdot\vec{d}}{|\vec{r}|^2}\right]$$

$$|\vec{r} + \vec{d}/2|^{-3} \approx |\vec{r}|^{-3}\left[1 - \frac{3}{2}\frac{\vec{r}\cdot\vec{d}}{|\vec{r}|^2}\right]$$

$$\vec{E} \approx \frac{q}{4\pi\varepsilon_0}|\vec{r}|^{-3}\left[\left(\vec{r} - \vec{d}/2\right)\left(1 + \frac{3}{2}\frac{\vec{r}\cdot\vec{d}}{|\vec{r}|^2}\right) - \left(\vec{r} + \vec{d}/2\right)\left(1 - \frac{3}{2}\frac{\vec{r}\cdot\vec{d}}{|\vec{r}|^2}\right)\right]$$

$$\vec{E} = \frac{q|\vec{r}|^{-3}}{4\pi\varepsilon_0}\left[3\frac{\vec{r}\cdot\vec{d}}{|\vec{r}|^2}\vec{r} - \vec{d}\right]$$

Recall $q\vec{d} = \vec{p}$ is the **dipole moment,** then

$$\vec{E} = \frac{1}{4\pi\varepsilon_0|\vec{r}|^3}\left[\frac{3(\vec{r}\cdot\vec{p})}{|\vec{r}|^2}\vec{r} - \vec{p}\right]$$

• For our coordinate system $\vec{p} = p\,\hat{a}_z$

• $\vec{r}$ is the position vector in spherical coordinate, then let us express $\vec{E}$ in the spherical coordinate

$$\vec{A} = A_r(r,\theta,\phi)\hat{a}_r + A_\theta(r,\theta,\phi)\hat{a}_\theta + A_\phi(r,\theta,\phi)\hat{a}_\phi$$

$$\vec{A} = A_x(x,y,z)\hat{a}_x + A_y(x,y,z)\hat{a}_y + A_z(x,y,z)\hat{a}_z$$

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} \text{ with}$$

$$x = r\sin\theta\cos\phi$$

$$y = r\sin\theta\sin\phi$$

$$z = r\cos\theta$$

$$\begin{bmatrix} p_r \\ p_\theta \\ p_\phi \end{bmatrix} = \begin{bmatrix} \cos\theta\ p_z \\ -\sin\theta\ p_z \\ 0 \end{bmatrix} \Rightarrow \vec{p} = p\ \hat{a}_z = (\cos\theta\ \hat{a}_r - \sin\theta\ \hat{a}_\theta)\ p$$

• Or finally from

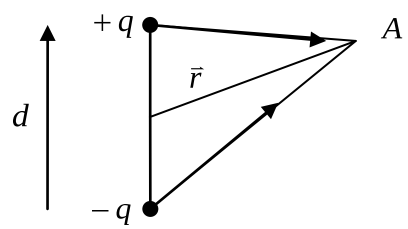
$$\vec{E} = \frac{1}{4\pi\varepsilon_0|\vec{r}|^3}\left[\frac{3(\vec{r}\cdot\vec{p})}{|\vec{r}|^2}\vec{r} - \vec{p}\right]$$

we get

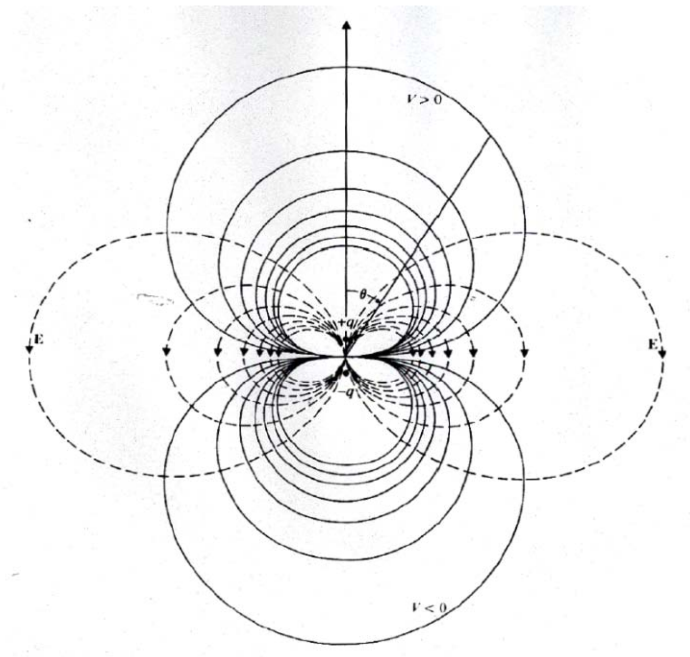
$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{1}{|\vec{r}|^3}\left[\frac{3r\ \hat{a}_r\cdot(\cos\theta\ \hat{a}_r - \sin\theta\ \hat{a}_\theta)\ p}{|\vec{r}|^2} r\ \hat{a}_r - (\cos\theta\ \hat{a}_r - \sin\theta\ \hat{a}_\theta)\ p\right] \Rightarrow$$

$$\boxed{\vec{E} = \frac{p}{4\pi\varepsilon_0|\vec{r}|^3}[2\cos\theta\ \hat{a}_r + \sin\theta\ \hat{a}_\theta]}, \text{ where } |\vec{r}| = r$$

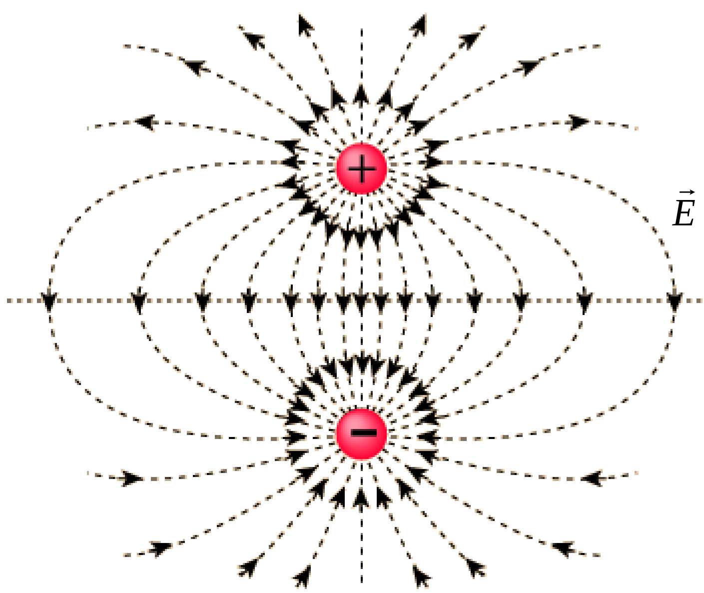
HW: Show that potential at point A for an electric dipole is given by



$$\boxed{V = \frac{\vec{p}\cdot\hat{a}_r}{4\pi\varepsilon_0|\vec{r}|^2} = \frac{\vec{p}\cdot\hat{a}_r}{4\pi\varepsilon_0 r^2}}$$



V > 0
θ
+q
E
E
−q
V < 0



+
−
$\vec{E}$

Electric Polarization $\vec{P}$

$$\boxed{\vec{P} = \lim_{\Delta v' \to 0} \frac{\sum_{k=1}^{N\Delta v'} \vec{p}_k}{\Delta v'}}$$

$\vec{p}$ [C·m] Electric dipole moment

$\vec{P}$ [C/m^2] Electric polarization vector

N [#/m^3] is the number of dipoles per unit volume

• $\vec{P}$ [C/m^2] is the volume density of electric dipole moment $\vec{p}$ [C·m]

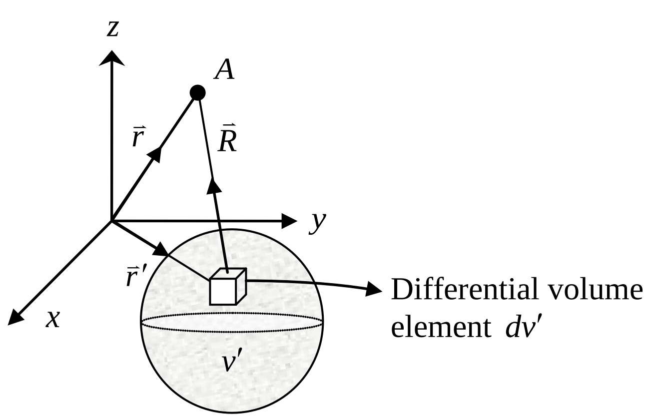
Note $\vec{P}$ and $\vec{D}$ have the same units [C/m^2]: $\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$

• Polarization vector $\vec{P}$ may come to exist due to (a) induced dipole moment, (b) alignment of the permanent dipole moments, or (c) migration of ionic charges.

• In differential form: $\vec{P} = \dfrac{d\vec{p}}{dv'}$

Potential due to Bound (Polarized) Surface & Volume Charge Densities

A dielectric of volume v' is polarized. We want to calculate the potential V [Volt] set up by this polarized dielectric.



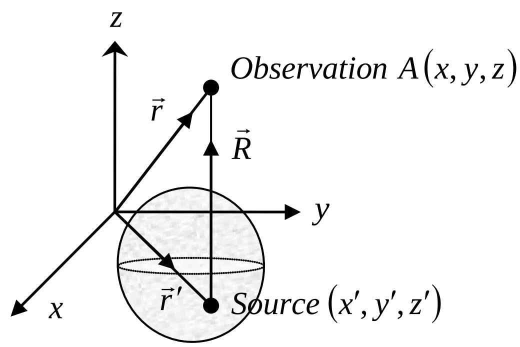
• Potential due to a single dipole

$$\boxed{V = \frac{\vec{p} \cdot \hat{a}_R}{4\pi\varepsilon_0 R^2}}$$

• An elemental electric dipole, having a differential electric dipole moment of $d\vec{p}$ [C·m], will set up a differential potential $dV = \dfrac{d\vec{p}\cdot\hat{a}_R}{4\pi\varepsilon_0 R^2}$

But from our definition of polarization we $d\vec{p} = \vec{P}\,dv' \Rightarrow$

$$dV = \frac{d\vec{p}\cdot\hat{a}_R}{4\pi\varepsilon_0 R^2} = \frac{\vec{P}\cdot\hat{a}_R}{4\pi\varepsilon_0 R^2}dv'$$



Total potential V is found by integrating the above:

$$V = \frac{1}{4\pi\varepsilon_0}\iiint_{v'}\frac{\vec{P}\cdot\hat{a}_R}{R^2}dv'$$

Where $R^2 = \left|\vec{R}\right|^2 = (x-x')^2 + (y-y')^2 + (z-z')^2$

Note that $\nabla'\left(\dfrac{1}{R}\right) = \dfrac{\hat{a}_R}{R^2}$ (see Remarks below) then

$$\boxed{V = \frac{1}{4\pi\varepsilon_0}\iiint_{v'}\vec{P}\cdot\nabla'\left(\frac{1}{R}\right)dv'}$$

and furthermore

$$\nabla'\cdot\left(f\ \vec{A}\right) = f\ \nabla'\cdot\vec{A} + \vec{A}\cdot\nabla' f \Rightarrow \vec{A}\cdot\nabla' f = \nabla'\cdot\left(f\ \vec{A}\right) - f\ \nabla'\cdot\vec{A}$$

* Let $\vec{A} = \vec{P}$ and $f = \dfrac{1}{R}$ then

$$\iiint_{v'}\vec{P}\cdot\nabla'\left(\frac{1}{R}\right)dv' = \iiint_{v'}\nabla'\cdot\left(\frac{\vec{P}}{R}\right)dv' - \iiint_{v'}\left(\frac{1}{R}\nabla'\cdot\vec{P}\right)dv'$$

Use divergence theorem $\Rightarrow$

$$\iiint_{v'}\vec{P}\cdot\nabla'\left(\frac{1}{R}\right)dv' = \oiint_{s'}\frac{\vec{P}}{R}\cdot d\vec{s}' - \iiint_{v'}\left(\frac{1}{R}\nabla'\cdot\vec{P}\right)dv'$$

The potential then can be written as

$$\boxed{V = \frac{1}{4\pi\varepsilon_0}\left[\oiint_{S'}\frac{\vec{P}\cdot\hat{a}'_n}{R}ds' - \iiint_{v'}\frac{\nabla'\cdot\vec{P}}{R}dv'\right]},$$

Where $\hat{a}'_n$ is perpendicular to surface S' bounding volume v'.

Compare above to the previously obtained expressions for V due to surface and volume charge densities, i.e.:

$$V = \iint_{s'}\frac{1}{4\pi\varepsilon_0}\frac{\rho'_s\,ds'}{R}\text{ and } V = \iiint_{v'}\frac{1}{4\pi\varepsilon_0}\frac{\rho'_v\,dv'}{R} \Rightarrow$$

$$\vec{P}\cdot\hat{a}'_n = \rho'_s$$

$$-\nabla'\cdot\vec{P} = \rho'_v$$

Or in general, dropping the prim notation since we know that integration is carried with respect to the prim coordinate, we define

• Bound or polarized surface charge density: $\boxed{\rho_{sP} = \vec{P} \cdot \hat{a}_n}$ [C/m^2]

• Bound or polarized volume charge $\boxed{\rho_{vP} = -\nabla \cdot \vec{P}}$ [C/m^3]

• A polarized dielectric can be replaced by a bound (polarized) surface and volume charge densities (ρ_{sP} & ρ_{vP}). The potential setup by these bound charges then can be calculated.

Remarks: Few useful identities

$$\nabla'\left(\frac{1}{R}\right) = \frac{\hat{a}_R}{R^2}$$

$$\nabla\left(\frac{1}{R}\right) = \frac{-\hat{a}_R}{R^2}$$

$$\nabla R = \hat{a}_R = \frac{\vec{R}}{|\vec{R}|} = \frac{\vec{R}}{R}$$

$$\nabla f(R) = \hat{a}_R \frac{\partial f(R)}{\partial R}$$

$$-\nabla^2 \frac{1}{|\vec{R}|} = 4\pi\delta^3(\vec{R}), \text{ or } -\nabla^2 \frac{1}{|\vec{r}-\vec{r}\,'|} = 4\pi\delta^3(\vec{r}-\vec{r}\,')$$

Generalized Gauss' Law & Constitutive Relation $\vec{D} = \varepsilon\,\vec{E}$

• In free space $\nabla \cdot \vec{E} = \dfrac{\rho_v}{\varepsilon_0}$.

• When a medium is polarized we must take into account the effects of the bound charges, hence

$$\nabla \cdot \vec{E} = \frac{\rho_v}{\varepsilon_0} + \frac{\rho_{vp}}{\varepsilon_0} = \frac{\rho_v}{\varepsilon_0} - \frac{\nabla \cdot \vec{P}}{\varepsilon_0} \Rightarrow$$

$$\nabla \cdot \varepsilon_0 \vec{E} + \nabla \cdot \vec{P} = \rho_v \Rightarrow \nabla \cdot \left(\varepsilon_0 \vec{E} + \vec{P}\right) = \rho_v$$

Let's define $\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$ then

$\boxed{\nabla \cdot \vec{D} = \rho_v}$ $\leftarrow$ Generalized Gauss' Law

• Also note that for $\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$ if $\vec{P} = \varepsilon_0 \chi_e \vec{E}$ then $\vec{D} = \varepsilon_0 (1+\chi_e)\vec{E} = \varepsilon_0 \varepsilon_r \vec{E}$

Where $\varepsilon_r = 1 + \chi_e$ then

$\boxed{D = \varepsilon_0 \varepsilon_r \vec{E} = \varepsilon\,\vec{E}}$ where $\varepsilon = \varepsilon_0 \varepsilon_r$

Magnetization & Permeability

• Magnetic materials exhibit magnetic polarization ($\vec{M}$, magnetization) when subjected to an applied magnetic field

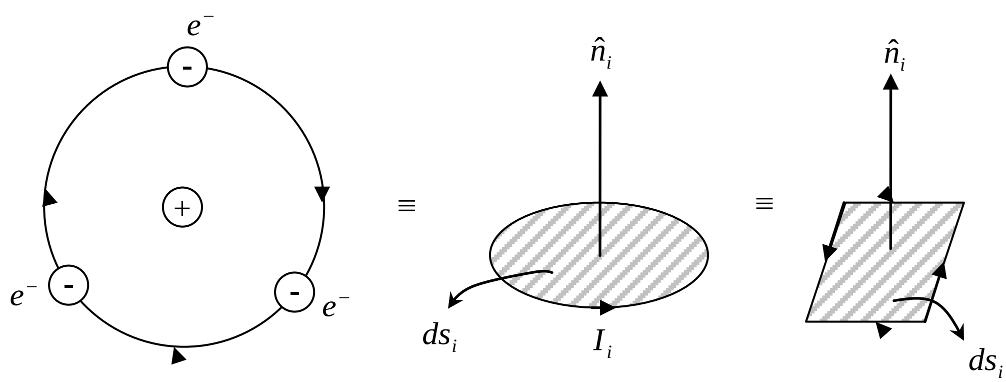
• This magnetization is the result of alignment of the magnetic dipoles of material with the applied magnetic field. This is similar to electric polarization which is the result of alignment of electric dipoles of the material with the applied electric field.

Magnetic Dipole & Magnetic Dipole Moment

• To accurately describe magnetic behavior of materials *quantum theory of matter is needed*. However, accurate *qualitative and quantitative description can be found using simple atomic model* (*semi-classical*)

• The *electron orbiting* the nuclei can be thought of as a small current loop of area ds_i with current I_i

• As long as loop is small, its shape c*an be circular, square, or any other closed curve*



• The *magnetic dipole moment* is given by

$\boxed{d\vec{m}_i = \hat{n}_i I_i ds_i}$ [A·m^2],

where $\hat{n}_i$ is perpendicular to the loop surface.

• The magnetic field of the current carrying loop at large distance is similar to the field of a *linear bar magnet, i.e., a magnetic dipole*.

• For a material of volume Δv which contains N_m magnetic dipoles (orbiting electrons) per unit volume, the *total magnetic moment* is given by

$$\vec{m}_t = \sum_{i=1}^{N_m \Delta v} d\vec{m}_i = \sum_{i=1}^{N_m \Delta v} \hat{n}_i I_i ds_i$$

• *The magnetic polarization, i.e. magnetization* ($\vec{M}$) is given by

$$\boxed{\vec{M} = \underset{\Delta v \to 0}{\text{limt}} \left[\frac{1}{\Delta v} \vec{m}_t \right] = \underset{\Delta v \to 0}{\text{limt}} \left[\frac{1}{\Delta v} \sum_{i=1}^{N_m \Delta v} d\vec{m}_i \right] = \underset{\Delta v \to 0}{\text{limt}} \left[\frac{1}{\Delta v} \sum_{i=1}^{N_m \Delta v} \hat{n}_i I_i ds_i \right]} \text{ [A/m]}$$

• Note that magnetization ($\vec{M}$) is the volume density of the total magnetic dipole moment ($\vec{m}_t$), and also the fact that magnetization ($\vec{M}$) has the *same units as the magnetic field intensity*, $\vec{H}$ [A/m].

• In absence of an applied field ($\vec{B}_a = 0$) the magnetic dipoles point in random directions. However, when $\vec{B}_a \neq 0$, the *dipoles will experience a torque* given by

$$\Delta \vec{\tau} = d\vec{m}_i \times \vec{B}_a = |d\vec{m}_i||\vec{B}_a| \sin(\angle d\vec{m}_i, \vec{B}_a) = |I_i ds_i||\vec{B}_a| \sin(\angle \hat{n}_i, \vec{B}_a) = |I_i ds_i B_a| \sin \Psi_i$$

• Subjected to the above torque, the magnetic dipoles *realign themselves such that their moment ($d\vec{m}_i$) is collinear with $\vec{B}_a$* (see figure in the next page)

$$\Psi_i \to 0 \Rightarrow \Delta \vec{\tau} \to 0$$

Remark: Comparing the similarities between the torque & potential energy for electric & magnetic dipoles

$$\boxed{\begin{aligned} \Delta \vec{\tau}_B &= d\vec{m} \times \vec{B}_a \\ \Delta \vec{\tau}_E &= d\vec{p} \times \vec{E}_a \end{aligned}} \qquad \boxed{\begin{aligned} \Delta U_B &= -\, d\vec{m} \cdot \vec{B}_a \\ \Delta U_E &= -\, d\vec{p} \cdot \vec{E}_a \end{aligned}}$$

• From next page figure we see that in *absence of an applied magnetic field*, we can write

(1) $\vec{B} = \mu_0 \vec{H}_a$.

But, when a magnetic material is present, a magnetic polarization ($\vec{M}$) is also present and an additional term must be added to (1). In order to take into account the influence of the material, we write

(2) $B = \mu_0 \vec{H}_a + \mu_0 \vec{M} = \mu_0 \left(\vec{H}_a + \vec{M} \right)$

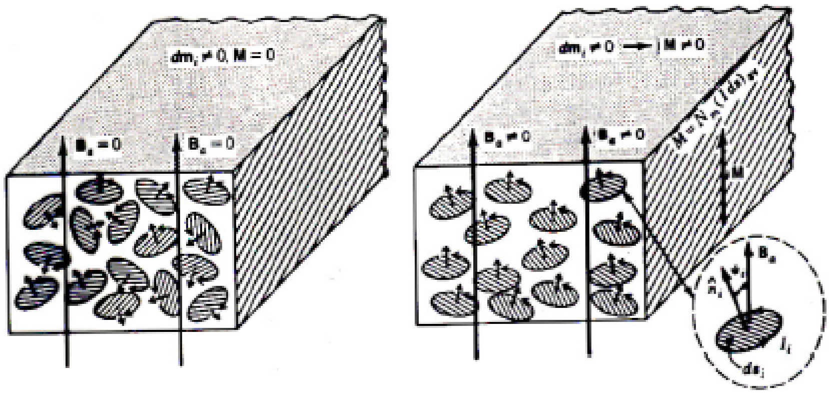
• However, $\vec{M}$ *is ultimately related to the applied field* $\vec{H}_a$. If we assume

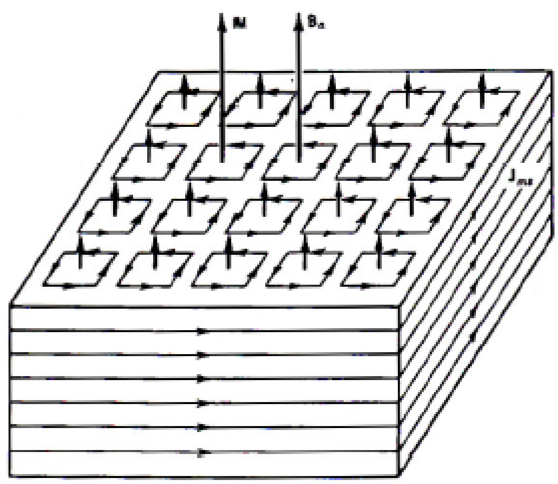
(3) $\vec{M} = \chi_m \vec{H}_a$,

Where χ_m is a scalar (or tensor) function then we have

(4) $B = \mu_0 [1 + \chi_m] \vec{H}_a = \mu_0 \, \mu_r \, \vec{H}_a = \mu \, \vec{H}_a$,

Where $\mu_r = 1 + \chi_m$ is the *relative permeability and* μ *is the permeability*.


dm_i ≠ 0, M = 0
B_a = 0
B_a = 0
dm_i ≠ 0 → M ≠ 0
B_a ≠ 0
B_a ≠ 0
M = N_m (I ds)_av
M
B_a
I_i
ds_i


M
B_a
J_ms

Bound Magnetization Current Density

• Recall that for an electric field applied to a medium we had

$\rho_{sP} = \vec{P} \cdot \hat{a}_n$

$\rho_{vP} = -\nabla \cdot \vec{P}$, where $\vec{P}$ is the electric polarization, ρ_{vP} and ρ_{sP} are the volume and surface bound charges, and $\hat{a}_n$ is the normal to the surface.

• Similarly, for magnetic field applied to a medium we have

$\vec{J}_{sm} = \vec{M} \times \hat{a}_n$

$\vec{J}_{vm} = \nabla \times \vec{M}$

• Here, $\vec{J}_{sm}$ is the *bound magnetization surface current density* [A/m], $\vec{J}_{vm}$ is the *bound magnetization volume current density* [A/m^2], and $\hat{a}_n$ is the normal to the surface.

Remark:

The origin of magnetization ($\vec{M}$) can also be visualized by the following:

• When $\vec{B}_a \neq 0$, the magnetic moments line up with $\vec{B}_a$ to *minimize the potential energy* as shown in the figure.

• Since the number of dipoles is very large and therefore they are *closely packed*, in the limit, the currents of the loops within the *interior part of the medium will cancel* each other and only a surface current ($\vec{J}_{sm}$) on the exterior of the slab remains.

• This *bound magnetization surface current density* ($\vec{J}_{sm}$) is responsible for producing the magnetization ($\vec{M}$).

• So far we have only considered the magnetic moment of the orbiting electron; however, a *magnetic moment can also be assigned to the spin* of electron.

• Only electrons in the atomic *shell that are not completely filled* will contribute to the spin magnetic moment.

• In general the magnitude of the spin magnetic moment is $\approx \pm 9 \times 10^{-24}$ [A·m^2].

• There is also a *magnetic moment associated with the nucleus* .

DC Conductivity

• Consider a small cylinder containing N electrons per unit volume, where electrons are moving with velocity $\vec{v}$.

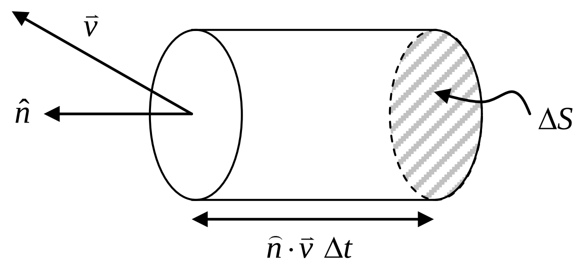
$N \equiv$ Number of electrons per unit volume [1/m^3]

$\vec{v} \equiv$ Velocity vector of electrons

$e \equiv$ Electron charge
$\hat{n} \equiv$ Normal to the surface
$\Delta V \equiv$ Volume of the cylinder

• The total chare (ΔQ) contained within the volume (ΔV) is given by
$\Delta Q = N\ e\ \Delta V$, where $\Delta V = \Delta S\ \hat{n} \cdot \vec{v}\ \Delta t$
hence
$\Delta Q = N\ e\ \Delta S\ \hat{n} \cdot \vec{v}\ \Delta t$. This implies

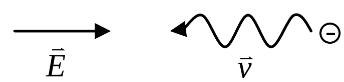
$$\frac{\Delta Q}{\Delta t} = \Delta I = N\ e\ \Delta S\ \hat{n} \cdot \vec{v}$$



• We define $N\ e\ \vec{v} = \vec{J}$, where $\vec{J}$ *is the current density vector* [A/m^2] and $\Delta Q / \Delta t = \Delta I$; then we have $\Delta I = \vec{J} \cdot \hat{n}\ \Delta S$

Remark:

• $\Delta I = \vec{J} \cdot \hat{n}\ \Delta S$ can be written as $dI = \vec{J} \cdot \hat{n}\ ds$ in *differential form*, which implies



$\Rightarrow I = \iint \vec{J} \cdot \hat{n}\ ds \leftarrow$ This is our *standard equation for calculating current from current density*.

• Let us assume *a linear relationship between velocity* ($\vec{v}$) *and electric filed* ($\vec{E}$), i.e., $\vec{v} = -\mu\ \vec{E}$, where μ is called mobility [m^2/V·s] (note $\vec{E}$ and $\vec{v}$ are anti-parallel)

• Then $\vec{J} = N\ e\ \vec{v} = -N\ e\ \mu\ \vec{E}$, for electron $e = -q = -1.602 \times 10^{-19}$ [C]
$\Rightarrow \vec{J} = q\ N\ \mu\ \vec{E}$

• Compare the above to $\vec{J} = \sigma_s \vec{E} \Rightarrow \sigma_s = q\ N\ \mu$. This says that static conductivity is the product of electron charge, electron density, and electron mobility.

• In our analysis so far we have only considered the electrons, however when *positive charges (ions of holes)* are present we must consider the contributions of both carriers to the conductivity. The static conductivity is then modified according to:

$$\sigma_s = q\ N_e\ \mu_e + q\ N_h\ \mu_h$$

$\mu_e \equiv$ Electron mobility
$\mu_h \equiv$ Hole mobility
N_e and N_h are electron and holes densities [1/m^3]

Time Harmonic or Sinusoidal Steady State Electromagnetic Fields

• Assuming time harmonic fields, the **instantaneous field** $\vec{E}(x,y,z,t)$ and the **complex spatial field** $\vec{E}(x,y,z)$ are related by

$$\vec{E}(x,y,z,t) = \mathrm{Re}\left[\vec{E}(x,y,z)\,e^{j\omega t}\right]$$
$$\vec{H}(x,y,z,t) = \mathrm{Re}\left[\vec{H}(x,y,z)\,e^{j\omega t}\right]$$
$$\vdots$$

Conductivity (DC & AC)

• The Amper's law given by $\nabla \times \vec{H}(x,y,z,t) = \vec{J}_i + \sigma_s\ \vec{E} + \dfrac{\partial \vec{D}}{\partial t} = \vec{J}_i + \vec{J}_c + \vec{J}_d$ can be written as

$$\nabla \times \vec{H}(x,y,z) = \vec{J}_i + \vec{J}_C + \vec{J}_D = \vec{J}_i + \sigma_s \vec{E}(x,y,z) + j\omega\varepsilon\ \vec{E}(x,y,z)$$

• Where ε in *general is complex:* $\varepsilon = \varepsilon' - j\varepsilon''$ and σ_s *is the static* (*DC*) *conductivity* (this is due to free carriers; i.e., electrons at $\omega = 0$).

• The Ampere law then can be written as:

$$\nabla \times \vec{H} = \vec{J}_i + \sigma_s \vec{E} + j\omega\left(\varepsilon' - j\varepsilon''\right)\vec{E} = \vec{J}_i + \left(\sigma_s + \omega\,\varepsilon''\right)\vec{E} + j\omega\varepsilon'\vec{E} = \vec{J}_i + \sigma_e \vec{E} + j\omega\varepsilon'\vec{E}$$

where we have defined the followings:

$\sigma_e = \sigma_s + \omega\,\varepsilon'' = \sigma_s + \sigma_a$; where σ_e is the *equivalent* (*effective*) *conductivit*y [1/Ω·m]

$\sigma_a = \omega\varepsilon'' \equiv$ *Alternating* (*AC*) *conductivity* [1/Ω·m]

$$\sigma_s = \begin{cases} \mu_e N_e q \text{ (for conductors)} \\ \mu_e N_e q + \mu_h N_h q \text{ (for semiconductors)} \end{cases}$$ = *static* (*DC*) *conductivity* [1/Ω·m]

• Note σ_s is due to *free charges at* $\omega = 0$ (a signature of true conductors).

• σ_a is due to *"resistance" of the dipoles* as they attempt to align (rotate) themselves with the applied field.

• The phenomenon of dipole rotation, which contributes to σ_a is sometimes called dielectric hysteresis.

• For good dielectrics such as glass or plastic $\sigma_s \approx 0$, but these materials when exposed to alternating fields ($\sigma_a \neq 0$) can dissipate large amount of energy. Example of large σ_a and its application are:

- microwave cooking
- selective heating of human tissue
- removing sulfur from mineral coal to produce clean coal (selective heating)

• Note $\nabla \times \vec{H} = \vec{J}_i + \sigma_e \vec{E} + j\omega\varepsilon' \vec{E} = \vec{J}_i + \vec{J}_{ce} + \vec{J}_{de}$

$\vec{J}_i \equiv$ Impressed current density

$\vec{J}_{ce} \equiv \sigma_e \vec{E} = (\sigma_s + \sigma_a)\vec{E} = (\sigma_s + \omega\varepsilon'')\vec{E}$: Effective conduction current density

$\vec{J}_{de} \equiv j\omega\varepsilon' \vec{E}$: Effective displacement current density

Loss Tangent

• Note that Amper's law given by $\nabla \times \vec{H} = \vec{J}_i + \sigma_e \vec{E} + j\omega\varepsilon' \vec{E}$ can be rewritten as:

$\nabla \times \vec{H} = \vec{J}_i + j\omega\varepsilon' \left(1 - j\frac{\sigma_e}{\omega\varepsilon'}\right)\vec{E} = \vec{J}_i + j\omega\varepsilon'(1 - j\tan\delta_e)\,\vec{E}$, where

$\tan\delta_e \equiv$ *Effective electric loss tangent*

$$\tan\delta_e = \frac{\sigma_e}{\omega\varepsilon'} = \frac{\sigma_s}{\omega\varepsilon'} + \frac{\sigma_a}{\omega\varepsilon'} = \frac{\sigma_s}{\omega\varepsilon'} + \frac{\omega\varepsilon''}{\omega\varepsilon'} = \frac{\sigma_s}{\omega\varepsilon'} + \frac{\varepsilon''}{\varepsilon'}$$

$= \tan\delta_s + \tan\delta_a$

with

$\tan\delta_s = \dfrac{\sigma_s}{\omega\varepsilon'}$: *Static* (*DC*) *loss tangent*

$\tan\delta_a = \dfrac{\varepsilon''}{\varepsilon'}$: *Alternating* (*AC*) *loss tangent*

• Manufacturer usually provides loss tangent or the conductivity.

• Note that in the above discussion we have expressed the *conduction* (*DC*) *and dielectric losses* (*AC*) in terms of effective conductivity (σ_e) or effective loss tangent ($\tan\delta_e$). We could have also formulated the problem in terms of *complex permittivity*.

• To see this we write: $\nabla \times \vec{H} = \vec{J}_i + j\omega\varepsilon' \left(1 - j\frac{\sigma_e}{\omega\varepsilon'}\right)\vec{E} = \vec{J}_i + j\omega\varepsilon_c \vec{E}$, where

$$\varepsilon_c = \varepsilon' \left(1 - j\frac{\sigma_e}{\omega\varepsilon'}\right) = \varepsilon' - j(\frac{\sigma_s + \sigma_a}{\omega}) = \varepsilon' - j(\frac{\sigma_s}{\omega} + \varepsilon'')$$

• In the expression for ε_c the free carrier losses and dielectric losses are clearly evident.

• Remark: The presence of static conductivity as a separate mechanism of loss in addition to the dielectric loss (ε'') can also be observed in the Kramers-Kronig relations which connects the real and imaginary parts of the dielectric constant. When a medium has static conductivity σ_s then Kramers-Kronig relations are given by

$$\varepsilon''(\omega) = \mathrm{Im}[\varepsilon(\omega)] = \frac{\sigma_s}{\omega} - \frac{2\omega}{\pi} P \int_0^{+\infty} \frac{\{\mathrm{Re}[\varepsilon(\omega')]/\varepsilon_0\} - 1}{\omega'^2 - \omega^2}\, d\omega'$$

,

$$\varepsilon'(\omega) = \mathrm{Re}[\varepsilon(\omega)] = 1 + \frac{2}{\pi} P \int_0^{+\infty} \frac{\omega' \{\mathrm{Im}[\varepsilon(\omega')]/\varepsilon_0\}}{\omega'^2 - \omega^2}\, d\omega'$$

where P stands for the principle value integral.

Boundary Conditions

• Maxwell's equations in differential forms are *point equations*; i.e. they are valid when fields are: *single valued, bounded, continuous, and have continuous derivatives*.

• When boundaries are present, fields are discontinuous; hence to find the fields *we must rely on their integral form*.

• **Boundary conditions for tangential $\vec{H}$:**

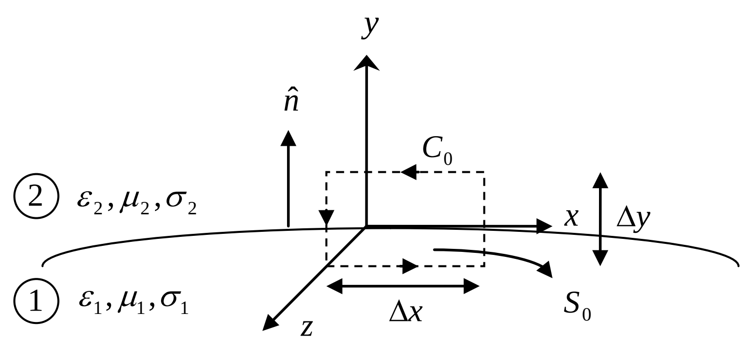
Assume finite conductivity $(\sigma_1, \sigma_2 \neq \infty)$ and no sources on boundary $(\vec{M}_i = 0, \vec{J}_i = 0)$

$$\oint_{C_0} \vec{H} \cdot d\vec{l} = \iint_{S_0} \sigma\, \vec{E} \cdot d\vec{s} + \frac{\partial}{\partial t} \iint_{S_0} \vec{D} \cdot d\vec{s} \quad (1)$$

• Taking the limit of the both sides of Eq. (1), the *Left hand side* (*LHS*) can be written as:

$$\lim_{\Delta y \to 0} \oint_{C_0} \vec{H} \cdot d\vec{l} = \lim_{\Delta y \to 0} \left[\int \vec{H}_1 \cdot d\vec{l}_1 + \int \vec{H}_2 \cdot d\vec{l}_2 \right]$$

$$= \vec{H}_1 \cdot \Delta x\, \hat{a}_x - \vec{H}_2 \cdot \Delta x\, \hat{a}_x = \left(\vec{H}_1 - \vec{H}_2\right) \cdot \Delta x\, \hat{a}_x$$



• *The first term on the right hand side* (RHS) of Eq. (1) can be written as:

$$\lim_{\Delta y \to 0} \iint_{S_0} \sigma \vec{E} \cdot d\vec{s} = \lim_{\Delta y \to 0} \iint_{S_0} \sigma \vec{E} \cdot dxdy\, \hat{a}_z$$

$$= \lim_{\Delta y \to 0} \left[\sigma \vec{E} \Delta x \Delta y \cdot \hat{a}_z \right] = 0$$

• *The second term on the RHS* of Eq. (1) can be written as:

$$\lim_{\Delta y \to 0} \frac{\partial}{\partial t} \iint_{S_0} \vec{D} \cdot d\vec{s} = \lim_{\Delta y \to 0} \frac{\partial}{\partial t} \iint_{S_0} \vec{D} \cdot dxdy\, \hat{a}_z = \lim_{\Delta y \to 0} \frac{\partial}{\partial t} (\vec{D} \cdot \Delta x \Delta y\, \hat{a}_z) = 0$$

• *Putting it all together*:

$$\hat{a}_x \cdot \left(\vec{H}_1 - \vec{H}_2\right) \Delta x = 0 \Rightarrow \hat{a}_x \cdot \left(\vec{H}_2 - \vec{H}_1\right) = 0\,.$$

• Note that:

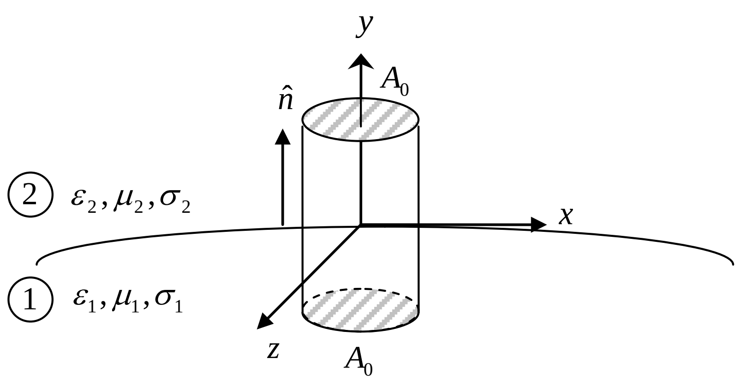
$\hat{a}_x \cdot H_2 \equiv$ *Tangential component of* $\vec{H}_2$ WRT the interface,

$\hat{a}_x \cdot \vec{H}_1 \equiv$ *Tangential component of* $\vec{H}_1$ WRT the interface.

• Also the fact that we can carry the *same analysis in the y-z plane* which results in $\hat{a}_z \cdot \left(\vec{H}_2 - \vec{H}_1\right) = 0$, with $\hat{a}_z \cdot H_2$ and $\hat{a}_z \cdot H_1$ designating the tangential components of the $\vec{H}$ fields. The conclusion is then the following: *tangential components of* $\vec{H}$ *are continuous across the boundary between two dielectrics. This all can be summarized as* $\hat{n} \times \left(\vec{H}_2 - \vec{H}_1\right) = 0$

Boundary Condition on Normal Components (Not corrected)

Medium (1) and (2) are non conductors (dielectrics) ($\sigma_1, \sigma_2 \neq \infty$) and there are no sources at the boundary $\rho_{es} = \rho_{ms} = 0$



$$\oiint \vec{D} \cdot d\vec{s} = \iiint_v \rho_v dv$$

LHS:

$$\lim_{\Delta y \to 0} \oiint \vec{D} \cdot d\vec{s} = \lim_{\Delta y \to 0} \left[\iint \vec{D}_2 \cdot d\vec{s} + \iint \vec{D}_1 \cdot d\vec{s} \right] = \lim_{\Delta y \to 0} \left[\iint \vec{D}_2 \cdot dxdz\hat{a}_y - \iint \vec{D}_1 \cdot dxdz\hat{a}_y \right]$$

$$= \lim_{\Delta y \to 0} \left(\vec{D}_2 A_0 \hat{a}_y - \vec{D}_1 A_0 \hat{a}_y \right)$$

RHS:

$$\lim_{\Delta y \to 0} \iiint_v \rho_v dv = \lim_{\Delta y \to 0} \left[\rho_v \Delta y A_0 \right] = A_0 \lim_{\Delta y \to 0} \rho_v \Delta y = A_0 \rho_s = 0$$

Then

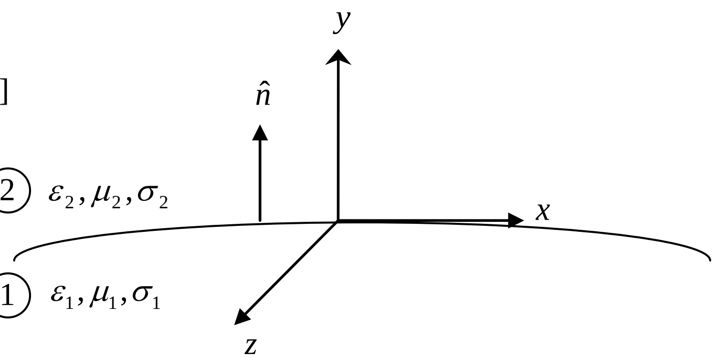
$$\left(\vec{D}_2 - \vec{D}_1 \right) \cdot \hat{a}_y = 0 \Leftrightarrow \hat{n} \cdot \left(\vec{D}_2 - \vec{D}_1 \right) = 0 \Leftrightarrow \hat{n} \cdot \left(\varepsilon_2 \vec{E}_2 - \varepsilon_1 \vec{E}_1 \right) = 0$$

Summary of boundary conditions

General Case:

- $\hat{n}\times\left(\vec{E}_2-\vec{E}_1\right)=-\vec{M}_s$

M_s: Fictitious magnetic current density [V/m]



- $\hat{n}\times\left(\vec{H}_2-\vec{H}_1\right)=\vec{J}_s$

$\vec{J}_s$: Electric surface current density [A/m]

- $\hat{n}\cdot\left(\vec{D}_2-\vec{D}_1\right)=\rho_{es}$

ρ_{es}: Electric surface charge density [C/m^2]

- $\hat{n}\cdot\left(\vec{B}_2-\vec{B}_1\right)=\rho_{ms}$

ρ_{ms}: Fictitious magnetic surface charge density [Weber/m^2]

Boundary Conditions Between to Perfect Dielectrics:

$\hat{n}\times\left(\vec{E}_2-\vec{E}_1\right)=0$,
$\hat{n}\times\left(\vec{H}_2-\vec{H}_1\right)=0$,
$\hat{n}\cdot\left(\vec{D}_2-\vec{D}_1\right)=0$,
$\hat{n}\cdot\left(\vec{B}_2-\vec{B}_1\right)=0$

Boundary Conditions for Two Media in which One Medium Is a Perfect Conductor ($\sigma_1=\infty$), With no Sources Present ($\vec{M}_s=0$, $\rho_{ms}=0$):

- In medium-1, *since perfect conductor* $\Rightarrow \vec{E}_1=\vec{D}_1=0$ then $\nabla\times\vec{E}_1=-\frac{\partial}{\partial t}\vec{B}_1 \Rightarrow 0=\frac{\partial}{\partial t}\vec{B}_1$

. But this means that $\vec{B}_1$ must be a constant function of time which *contradicts the assumption of time varying electric and magnetic fields*; i.e. the electrodynamics assumption. Therefore, $\vec{B}_1=\vec{H}_1=0$

- $\hat{n}\times\left(\vec{E}_2-\vec{E}_1\right)=-\vec{M}_s \Rightarrow \hat{n}\times\vec{E}_2=0$

Electric filed has *no tangential component on the boundary between perfect conductor and dielectric.*

- $\hat{n}\times\left(\vec{H}_2-\vec{H}_1\right)=\vec{J}_s \Rightarrow \hat{n}\times\vec{H}_2=\vec{J}_s$

Tangential component of $\vec{H}$ *is discontinuous by amount of surface current* $\vec{J}_s$ at the boundary between perfect conductor and dielectric.

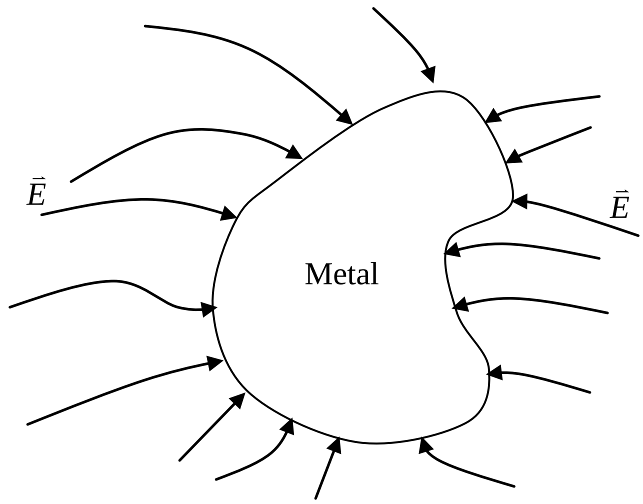
- J_s is the *surface current due to the free charges* on the metal (not the bound charges)

• $\hat{n}\cdot\left(\vec{D}_2-\vec{D}_1\right)=\rho_{es} \Rightarrow \hat{n}\cdot\vec{D}_2=\rho_{es}$

Electric field has *only normal component* on the boundary between perfect conductor and dielectric.

• $\hat{n}\cdot\left(\vec{B}_2-\vec{B}_1\right)=\rho_{ms} \Rightarrow \hat{n}\cdot\vec{B}_2=0$

Magnetic field has *no normal component* on the boundary between perfect conductor and dielectric.



Boundary Conditions Between Two Medium one of which Is a Perfect Magnetic Material (the medium has infinite magnetic conductivity, i.e. $\vec{H}_{1t}=0$) and no sources are present ($\rho_{es}=0$, $\vec{J}_s=0$)

• Here $\vec{H}_1=0 \Rightarrow \vec{B}_1=0$, $\vec{E}_1=\vec{D}_1=0$

• $\hat{n}\times\left(\vec{E}_2-\vec{E}_1\right)=-\vec{M}_s \Rightarrow \hat{n}\times\vec{E}_2=-\vec{M}_s$

Electric filed is tangential to the boundary

• $\hat{n}\times\left(\vec{H}_2-\vec{H}_1\right)=\vec{J}_s \Rightarrow \hat{n}\times\vec{H}_2=0$

Magnetic filed has no tangential component on the boundary

• $\hat{n}\cdot\left(\vec{D}_2-\vec{D}_1\right)=\rho_{es} \Rightarrow \hat{n}\cdot\vec{D}_2=0$

Electric filed has no normal component at the boundary

• $\hat{n}\cdot\left(\vec{B}_2-\vec{B}_1\right)=\rho_{ms} \Rightarrow \hat{n}\cdot\vec{B}_2=\rho_{ms}$

Magnetic field is normal to the boundary

Differential length elements

Rectangular Coordinate System:

Cylindrical Coordinate System:

Spherical Coordinate System: