Auxiliary Vector Potential

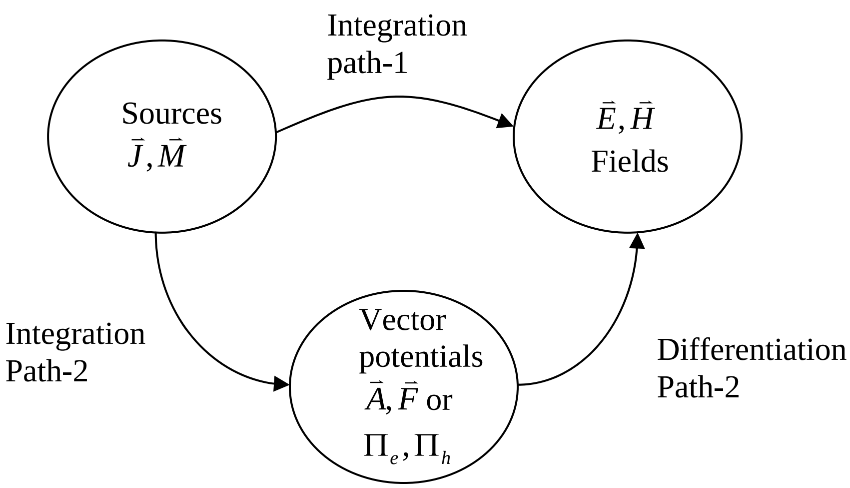
Constructing solutions using auxiliary vector potentials

• The objective of EM theory is to **find possible EM field configurations (modes) for a given boundary value problem** involving **wave propagation, radiation, scattering, or absorption.**

• This can be done **by finding the electric and magnetic fields (** $\vec{E}$ **and** $\vec{H}$ **) or equally obtaining the auxiliary vector potentials (** $\vec{A}$ **and** $\vec{F}$ **)**

• **In addition to** auxiliary vector potentials $\vec{A}$ and $\vec{F}$ **there are other possible set. For example, Hertz vector potentials (** Π_e **and** Π_h **)**. Here, we only concentrate on $\vec{A}$ and $\vec{F}$

• The **path for solving EM field** configuration is then as follows



• Depending on the problem at hand, **path-2 maybe easier than path-1**

• Traditionally $\vec{E}$ **and** $\vec{B}$ **are viewed as physical field quantities, whereas vector potential (** $\vec{A}$ **) and its scalar counter part (** ϕ_e **) are considered as mathematical constructs**. However, there are diverging views on this point!!!

• It is interesting to note that **Maxwell himself derived many of his results by using the concept of vector potential (** $\vec{A}$ **) which he called "electromagnetic momentum."** However this approach was later criticized by other practitioners such as Hertz and Heaviside.

The question of the propagation of, not merely the electric potential Ψ but the vector potential $\vec{A}$ ... when brought forward, **prove to be one of a metaphysical nature ... the electric force $\vec{E}$ and the magnetic force $\vec{H}$ ... actually represent the state of the medium everywhere.** Heaviside, *Philosophical Magazine*, 1889.

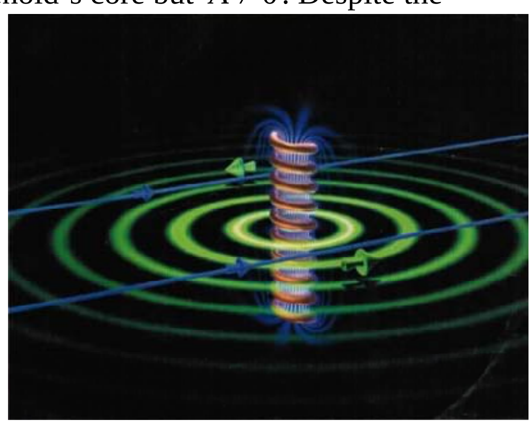
• Here is what **Hertz says about Maxwell's approach:**

I may mention the predominance of the vector potential in [Maxwell's] fundamental equations. **In the construction of new theory the potential served as a scaffolding ... it does not appear to me that any ... advantage is attained by the introduction of the vector potential in the fundamental equations**. C. **A. Mead**, *Collective electrodynamics,* 2000.

• **Here is different (more modern) point of view:**

... the **vector potentia**l which appears in quantum mechanics in an explicit form **produces a classical force which depends only on its derivatives**. In quantum mechanics what matters is the interference between nearby paths; it always turns out that the **effects depend only on how much the field $\vec{A}$ changes from point to point, and therefore only on the derivatives of $\vec{A}$ and not on the value itself.** Nevertheless, the vector potential $\vec{A}$ **(together with the scalar potential ϕ that goes with it) appears to give the most direct description of the physics.** This becomes more and more apparent the more deeply we go into the quantum theory. In the general theory of **quantum electrodynamics**, one takes the v**ector and scalar potentials as the fundamental quantities in a set of equations that replace the Maxwell equation**s: **$\vec{E}$ and $\vec{B}$ are slowly disappearing from the modern expression of physical laws, they are being replaced by $\vec{A}$ and ϕ.** **Feynman, Leighton, and Sands**, *Lectures on Physics, Vol. II*, 1984.

• **Aharonov- Bohm Effects:** What happens to an electron as it passes by an infinitely long solenoid. The $\vec{E}$ and $\vec{B}$ are zero outside the solenoid's core but $\vec{A} \neq 0$. Despite the fact that there are no EM forces outside of the solenoid, electron will experience the presence of $\vec{A}$ and its phase will be modified. For the figure shown, $\vec{A}$ will introduce a phase shift in the electrons' wave functions which can be detected by interfering the electrons.



Equations governing vector potential $\vec{A}$

• **Since** $\nabla \cdot \vec{B} = 0 \Rightarrow$ (1)

$\vec{B}_A = \nabla \times \vec{A}$ and (2)

$$\vec{H}_A = \frac{1}{\mu} \nabla \times \vec{A} \quad (3)$$

Subscript A is to remind us that $\vec{B}_A$ **and** $\vec{H}_A$ **are due to vector potential** A

• For $\vec{M} = 0$ (**no magnetic source**)

$\nabla \times \vec{E}_A = -j\omega\mu\, \vec{H}_A$ (**Faraday's Law**) (4)

• **Use (3) in (4)** $\Rightarrow$

$$\nabla \times \vec{E}_A = -j\omega\mu\left(\frac{1}{\mu}\nabla \times \vec{A}\right) \Rightarrow \nabla \times \left(E_A + j\omega\, \vec{A}\right) = 0 \quad (5)$$

• Since **curl of gradient of any scalar is zero**, i.e., $\nabla \times \left(-\nabla \phi_e\right) = 0$, then **from (5)** we have $\vec{E}_A + j\omega\vec{A} = -\nabla \phi_e \Rightarrow \vec{E}_A = -j\omega\vec{A} - \nabla \phi_e$ where (6)

$\phi_e \equiv$ **Scalar potential**

$\vec{A} \equiv$ **Vector potential**

• **Equations (6) and (3) are the expression for** $\vec{E}$ **and** $\vec{H}$ **in terms of** $\vec{A}$ **and** ϕ_e

• **We now want to find differential equations governing the behaviors of** $\vec{A}$ **and** ϕ_e

• We note that **from** $\vec{H}_A = \frac{1}{\mu}\nabla \times \vec{A}$**, for a homogeneous medium we can write**

$$\mu\nabla \times \vec{H}_A = \nabla \times \nabla \times \vec{A} \Rightarrow \qquad \mu\nabla \times \vec{H}_A = \nabla\left(\nabla \cdot \vec{A}\right) - \nabla^2 \vec{A} \quad (7)$$

• **Using Ampere's Law** $\nabla \times \vec{H}_A = \vec{J} + j\omega\varepsilon\vec{E}_A$ **in (7) we have**

$$\mu\, \vec{J} + j\omega\varepsilon\mu\, \vec{E}_A = \nabla\left(\nabla \cdot \vec{A}\right) - \nabla^2 \vec{A} \quad (8)$$

• **Previously we found the expression for** $\vec{E}_A$ **to be** $\vec{E}_A = -j\omega\vec{A} - \nabla \phi_e$. **Using this in (8) we have**

$$\mu\vec{J} + j\omega\varepsilon\mu\left[-j\omega\vec{A} - \nabla \phi_e\right] = \nabla\left(\nabla \cdot \vec{A}\right) - \nabla^2 \vec{A} \quad (9)$$

• Recall that $\omega^2 \mu\varepsilon = \beta^2$ **then (9) can be written as**

$$\nabla^2 \vec{A} + \beta^2 \vec{A} = \nabla\left(\nabla \cdot \vec{A} + j\omega\varepsilon\mu\phi_e\right) - \mu\vec{J}$$

• **We have defined the curl of** $\vec{A}$ **as** $\vec{B}_A = \nabla \times \vec{A}$, we are at **liberty to define the** $\nabla \cdot \vec{A}$.

• **In light of** $\nabla^2\vec{A}+\beta^2\vec{A}=\nabla\left(\nabla\cdot\vec{A}+j\omega\varepsilon\mu\phi_e\right)-\mu\vec{J}$ (1)

let us define the divergence of $\vec{A}$ **to be**

$$\nabla\cdot\vec{A}=-j\omega\varepsilon\mu\phi_e \tag{2}$$

• **Using (2) in (1) we have**

$$\nabla^2\vec{A}+\beta^2\vec{A}=-\mu\vec{J} \quad \text{and} \tag{3}$$

$$\phi_e=-\frac{1}{j\omega\varepsilon\mu}\nabla\cdot\vec{A} \tag{4}$$

• Finally, our **expressions for** $\vec{E}_A$ **and** $\vec{H}_A$ **in the last page [Eqs. (6) and (3)] can be written as**

$$\vec{E}_A=-\nabla\phi_e-j\omega\vec{A}=\frac{-j}{\omega\mu\varepsilon}\nabla\left(\nabla\cdot\vec{A}\right)-j\omega\vec{A} \tag{5}$$

$$\vec{H}_A=\frac{1}{\mu}\nabla\times\vec{A} \tag{6}$$

• **Now, equations (5) and (6) are expressions for** $\vec{E}_A$ **and** $\vec{H}_A$ **in terms of** $\vec{A}$ **only subject to Lorentz gauge.**

Equations governing the vector potential $\vec{F}$

• **Consider a region of space free of charges, i.e.** $q_{\text{ave}}=0$, then

$$\nabla\cdot\vec{D}=0\Rightarrow \tag{1}$$

$$\vec{D}_F=-\nabla\times\vec{F}\Rightarrow \tag{2}$$

$$\vec{E}_F=-\frac{1}{\varepsilon}\nabla\times\vec{F} \tag{3}$$

Subscript F **is to remind us** $\vec{D}_F$ **is due to vector potential** $\vec{F}$

• **Recall that Ampere's Law with** $\vec{J}=0$**, is given by**

$$\nabla\times\vec{H}_F=j\omega\,\varepsilon\,\vec{E}_F\Rightarrow\frac{1}{j\omega\varepsilon}\nabla\times\vec{H}_F=\vec{E}_F \tag{4}$$

• **Use (4) in (3) and we have**

$$\frac{1}{j\omega\varepsilon}\nabla\times\vec{H}_F=-\frac{1}{\varepsilon}\nabla\times\vec{F}\Rightarrow\nabla\times\vec{H}_F=-j\omega\nabla\times\vec{F}\Rightarrow$$

$$\nabla\times\left(\vec{H}_F+j\omega\vec{F}\right)=0 \tag{5}$$

• **Compare** $\nabla\times\left(\vec{H}_F+j\omega\vec{F}\right)=0$ **with null identity** $\nabla\times\left(-\nabla\phi_m\right)=0$, then it is clear that

$$\vec{H}_F + j\omega\vec{F} = -\nabla\phi_m \Rightarrow$$

$$\vec{H}_F = -j\omega\,\vec{F} - \nabla\phi_m \quad (1)$$

• **For homogenous media, from** $\vec{E}_F = -\frac{1}{\varepsilon}\nabla\times\vec{F}$ we have $\nabla\times\vec{E}_F = -\frac{1}{\varepsilon}\nabla\times\nabla\times\vec{F} \Rightarrow$

$$\nabla\times\vec{E}_F = -\frac{1}{\varepsilon}\nabla\left(\nabla\cdot\vec{F}\right) + \frac{1}{\varepsilon}\nabla^2\vec{F} \quad (2)$$

• **From Faraday's Law we have**

$$\nabla\times\vec{E}_F = -\vec{M} - j\omega\mu\,\vec{H}_F, \quad (3)$$

then **substitute (3) in (2)**

$$-\vec{M} - j\omega\mu\,\vec{H}_F = -\frac{1}{\varepsilon}\nabla\left(\nabla\cdot\vec{F}\right) + \frac{1}{\varepsilon}\nabla^2\vec{F} \quad (4)$$

• But we already found an expression for $\vec{H}_F$ in (1). **Use (1) in (4), and we have**

$$\beta^2\vec{F} + \nabla^2\vec{F} = -\varepsilon\vec{M} + \nabla\left(\nabla\cdot\vec{F} + j\omega\mu\varepsilon\phi_m\right) \quad (5)$$

Where again $\beta^2 = \omega^2\mu\varepsilon$

• Once again **curl of** $\vec{F}$ **is defined by** $\vec{D}_F = -\nabla\times\vec{F}$**. We are at liberty to choose the divergence of** $\vec{F}$**.** Let

$$\nabla\cdot\vec{F} = -j\omega\mu\varepsilon\phi_m \Rightarrow \quad (6)$$

$$\phi_m = \frac{-1}{j\omega\varepsilon\mu}\nabla\cdot\vec{F} \quad (7)$$

• **Using (6), (5) simplifies to**

$$\nabla^2\vec{F} + \beta^2\vec{F} = -\varepsilon\vec{M} \quad (8)$$

• **Finally, note that** $\vec{H}_F$ [Eq. (1)] **and** $\vec{E}_F$ [Eq. (3) of last page)] can be written **in terms** of $\vec{F}$ according to

$$\vec{H}_F = -j\omega\,\vec{F} - \nabla\phi_m = -j\omega\,\vec{F} - \frac{j}{\omega\varepsilon\mu}\nabla\left(\nabla\cdot\vec{F}\right)$$

$$\vec{E}_F = -\frac{1}{\varepsilon}\nabla\times\vec{F}$$

Summary

1. **Find** $\vec{A}$ **from** $\nabla^2 \vec{A} + \beta^2 \vec{A} = -\mu J$, $\beta^2 = \omega^2 \mu\varepsilon$ (1)

2. **Find** $\vec{H}_A$ **from** $\vec{H}_A = \frac{1}{\mu}\nabla \times \vec{A}$ (2)

3. **Find** $\vec{E}_A$ **from** $\vec{E}_A = -j\omega \vec{A} - j\frac{1}{\omega\mu\varepsilon}\nabla\left(\nabla \cdot \vec{A}\right)$ or $\vec{E}_A = \frac{1}{j\omega\varepsilon}\nabla \times \vec{H}_A$ (3)

4. **Find** $\vec{F}$ **from** $\nabla^2 \vec{F} + \beta^2 \vec{F} = -\varepsilon \vec{M}$ (4)

5. **Find** $\vec{E}_F$ **from** $\vec{E}_F = -\frac{1}{\varepsilon}\nabla \times \vec{F}$ (5)

6. **Find** $\vec{H}_F$ **from** $\vec{H}_F = -j\omega \vec{F} - j\frac{1}{\omega\mu\varepsilon}\nabla\left(\nabla \cdot \vec{F}\right)$ or $\vec{H}_F = \frac{-1}{j\omega\mu}\nabla \times \vec{E}_F$ (6)

7. **The total** $\vec{E}$ **is given by**

$$\vec{E} = \vec{E}_A + \vec{E}_F = -j\omega\vec{A} - j\frac{1}{\omega\mu\varepsilon}\nabla\left(\nabla \cdot \vec{A}\right) - \frac{1}{\varepsilon}\nabla \times \vec{F} \quad (7)$$

or

$$\vec{E} = \vec{E}_A + \vec{E}_F = \frac{1}{j\omega\varepsilon}\nabla \times \vec{H}_A - \frac{1}{\varepsilon}\nabla \times \vec{F} \quad (8)$$

8. **The total** $\vec{H}$ **is given by**

$$\vec{H} = \vec{H}_A + \vec{H}_F = \frac{1}{\mu}\nabla \times \vec{A} - j\omega \vec{F} - j\frac{1}{\omega\mu\varepsilon}\nabla\left(\nabla \cdot \vec{F}\right) \quad (9)$$

or

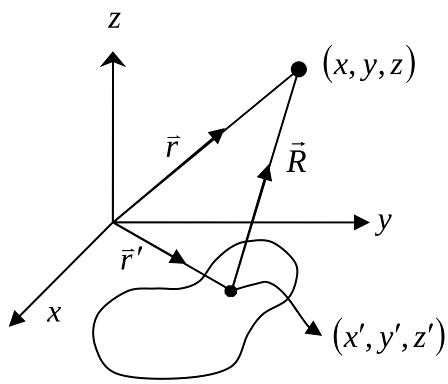
$$\vec{H} = \vec{H}_A + \vec{H}_F = \frac{1}{\mu}\nabla \times \vec{A} - \frac{1}{j\omega\mu}\nabla \times \vec{E}_F \quad (10)$$

Solutions for $\vec{A}$ and $\vec{F}$

• Recall that **governing differential equations for** $\vec{A}$ **and** $\vec{F}$ **are**

$$\nabla^2 \vec{A} + \beta^2 \vec{A} = -\mu \vec{J} \qquad (1)$$

$$\nabla^2 \vec{F} + \beta^2 \vec{F} = -\varepsilon \vec{M} \qquad (2)$$



• For **source located at** (x', y', z') and **observation** point **distance** R **from the source, the solutions** to (1) and (2) are given by

$$\vec{A}(x, y, z) = \frac{\mu}{4\pi} \iiint_{v'} \vec{J}(x', y', z') \frac{e^{-j\beta R}}{R} dv' \qquad (3)$$

$$\vec{F}(x, y, z) = \frac{\varepsilon}{4\pi} \iiint_{v'} \vec{M}(x', y', z') \frac{e^{-j\beta R}}{R} dv' \qquad (4)$$

where $\vec{J}$ and $\vec{M}$ have **dimensions proportional to 1/m^2**

• For $\vec{J}_s$ **and** $\vec{M}_s$ **dimensions proportional to 1/m** we have

$$\vec{A}(x, y, z) = \frac{\mu}{4\pi} \iint_{s'} \vec{J}_s(x', y', z') \frac{e^{-j\beta R}}{R} ds' \qquad (5)$$

$$\vec{F}(x, y, z) = \frac{\varepsilon}{4\pi} \iint_{s'} \vec{M}_s(x', y', z') \frac{e^{-j\beta R}}{R} ds' \qquad (6)$$

• For **electric and magnetic current densities** $\vec{I}_e$ **[Ampere] and** $\vec{I}_m$ **[volt]** we have

$$\vec{A}(x, y, z) = \frac{\mu}{4\pi} \int_{c} \vec{I}_e(x', y', z') \frac{e^{-j\beta R}}{R} dl' \qquad (7)$$

$$\vec{F}(x, y, z) = \frac{\varepsilon}{4\pi} \int_{c} \vec{I}_m(x', y', z') \frac{e^{-j\beta R}}{R} dl' \qquad (8)$$

TEM, TE and TM modes

• The transverse electromagnetic field configuration is a mode for **which electric and magnetic field components are transverse to a given direction**. This direction often, but not always, is the path that wave is traveling.

• **For TE mode the electric field is transverse to a given direction** and for **TM mode the magnetic field is transverse to a given direction.** Again, for TE and TM modes the **aforementioned direction is often**, but not always, **the direction of propagation**.

The conditions on auxiliary vector potentials $\vec{A}$ and $\vec{F}$ for TEM, TE and TM modes

• **Recall that $\vec{E}$ and $\vec{H}$ in terms of $\vec{A}$ and $\vec{F}$ were given by**

$$\vec{E} = \vec{E}_A + \vec{E}_F = -j\omega\vec{A} - j\frac{1}{\omega\mu\varepsilon}\nabla\left(\nabla\cdot\vec{A}\right) - \frac{1}{\varepsilon}\nabla\times\vec{F} \tag{1}$$

$$\vec{H} = \vec{H}_A + \vec{H}_F = \frac{1}{\mu}\nabla\times\vec{A} - j\omega\,\vec{F} - j\frac{1}{\omega\mu\varepsilon}\nabla\left(\nabla\cdot\vec{F}\right) \tag{2}$$

• **Let**

$$\vec{A} = A_x(x,y,z)\hat{a}_x + A_y(x,y,z)\hat{a}_y + A_z(x,y,z)\hat{a}_z \tag{3}$$

$$\vec{F} = F_x(x,y,z)\hat{a}_x + F_y(x,y,z)\hat{a}_y + F_z(x,y,z)\hat{a}_z \tag{4}$$

• **Use (3) and (4) in (1) and (2).** We get

$$\begin{aligned}\vec{E} = \hat{a}_x&\left[-j\omega A_x - j\frac{1}{\omega\mu\varepsilon}\left(\frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_y}{\partial x\partial y} + \frac{\partial^2 A_z}{\partial x\partial z}\right) - \frac{1}{\varepsilon}\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\right] + \\ \hat{a}_y&\left[-j\omega A_y - j\frac{1}{\omega\mu\varepsilon}\left(\frac{\partial^2 A_x}{\partial x\partial y} + \frac{\partial^2 A_y}{\partial y^2} + \frac{\partial^2 A_z}{\partial y\partial z}\right) - \frac{1}{\varepsilon}\left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\right] + \\ \hat{a}_z&\left[-j\omega A_z - j\frac{1}{\omega\mu\varepsilon}\left(\frac{\partial^2 A_x}{\partial x\partial z} + \frac{\partial^2 A_y}{\partial y\partial z} + \frac{\partial^2 A_z}{\partial z^2}\right) - \frac{1}{\varepsilon}\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\right]\end{aligned}$$

• **For $\vec{H}$ we have**

$$\vec{H} = \hat{a}_x\left[-j\omega F_x - j\frac{1}{\omega\mu\varepsilon}\left(\frac{\partial^2 F_x}{\partial x^2} + \frac{\partial^2 F_y}{\partial x\partial y} + \frac{\partial^2 F_z}{\partial x\partial z}\right) + \frac{1}{\mu}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\right] + \hat{a}_y\left[-j\omega F_y - j\frac{1}{\omega\mu\varepsilon}\left(\frac{\partial^2 F_x}{\partial x\partial y} + \frac{\partial^2 F_y}{\partial y^2} + \frac{\partial^2 F_z}{\partial y\partial z}\right) + \frac{1}{\mu}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\right] + \hat{a}_z\left[-j\omega F_z - j\frac{1}{\omega\mu\varepsilon}\left(\frac{\partial^2 F_x}{\partial x\partial z} + \frac{\partial^2 F_y}{\partial y\partial z} + \frac{\partial^2 F_z}{\partial z^2}\right) + \frac{1}{\mu}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\right] \quad (1)$$

• From expression for $\vec{E}$ and $\vec{H}$ in terms of $\vec{A}$ and $\vec{F}$ **we can see there are at least 3 ways for which we can obtain a TEM mode with respect to z-direction, i.e. TEM^z** (HW)

• **For example if all the condition listed below are satisfied we have a TEM^z mode**

$A_x = A_y = A_z = 0$, and $F_x = F_y = 0$, and $\frac{\partial}{\partial x} \neq 0$, and $\frac{\partial}{\partial y} \neq 0$, and

$F_z = F_z^+(x,y)e^{-j\beta z} + F_z^-(x,y)e^{+j\beta z}$,

then

$$E_z = -\overbrace{j\omega A_z}^{0} - \frac{j}{\omega\mu\varepsilon}\left(\overbrace{\frac{\partial^2 A_x}{\partial x\partial z}}^{o} + \overbrace{\frac{\partial^2 A_y}{\partial y\partial z}}^{0} + \overbrace{\frac{\partial^2 A_z}{\partial z^2}}^{0}\right) - \frac{1}{\varepsilon}\left(\overbrace{\frac{\partial F_y}{\partial x}}^{0} - \overbrace{\frac{\partial F_x}{\partial y}}^{0}\right) = 0 \quad (2)$$

$$H_z = -j\omega F_z - j\frac{1}{\omega\mu\varepsilon}\left(\overbrace{\frac{\partial^2 F_x}{\partial x\partial z}}^{0} + \overbrace{\frac{\partial^2 F_y}{\partial y\partial z}}^{0} + \frac{\partial^2 F_z}{\partial z^2}\right) + \frac{1}{\mu}\left(\overbrace{\frac{\partial A_y}{\partial x}}^{0} - \overbrace{\frac{\partial A_y}{\partial y}}^{0}\right) =$$

$$-j\omega F_z + \frac{j\beta^2}{\omega\mu\varepsilon}\left[F_z^+(x,y)e^{-j\beta z} + F_z^-(x,y)e^{j\beta z}\right] = -j\omega F_z + j\omega F_z = 0 \quad (3)$$

• **Note that from (2) and (3)** $E_z = H_z = 0$.

• **We can further calculate the** E_x**,** E_y**,** H_x**, and** H_y **to be**

$$E_x = -\overbrace{j\omega A_x}^{0} - j\frac{1}{\omega\mu\varepsilon}\left(\overbrace{\frac{\partial^2 A_x}{\partial x^2}}^{0} + \overbrace{\frac{\partial^2 A_y}{\partial x\partial y}}^{0} + \overbrace{\frac{\partial^2 A_z}{\partial x\partial z}}^{0}\right) - \frac{1}{\varepsilon}\left(\frac{\partial F_z}{\partial y} - \overbrace{\frac{\partial F_y}{\partial z}}^{0}\right) \Rightarrow$$

$$E_x = -\frac{1}{\varepsilon}\frac{\partial}{\partial y}F_z^+(x,y)e^{-j\beta z} - \frac{1}{\varepsilon}\frac{\partial}{\partial y}F_z^-(x,y)e^{j\beta z} = E_x^+ + E_x^-$$

$$E_y = \overbrace{-j\omega A_y}^{0} - j\frac{1}{\omega\mu\varepsilon}\left(\overbrace{\frac{\partial^2 A_x}{\partial x \partial y}}^{0} + \overbrace{\frac{\partial^2 A_y}{\partial y^2}}^{0} + \overbrace{\frac{\partial^2 A_z}{\partial y \partial z}}^{0}\right) - \frac{1}{\varepsilon}\left(\overbrace{\frac{\partial F_x}{\partial z}}^{0} - \frac{\partial F_z}{\partial x}\right) = \frac{1}{\varepsilon}\frac{\partial F_z}{\partial x} \Rightarrow \quad (1)$$

$$E_y = \frac{1}{\varepsilon}\frac{\partial}{\partial x}F_z^+(x,y)e^{-j\beta z} + \frac{1}{\varepsilon}\frac{\partial}{\partial x}F_z^- e^{j\beta z} = E_y^+ + E_y^- \quad (2)$$

and it can be shown

$$H_x = H_x^+ + H_x^- = -\sqrt{\frac{\varepsilon}{\mu}}E_y^+ + \sqrt{\frac{\varepsilon}{\mu}}E_y^- \text{ (HW)} \quad (3)$$

$$H_y = H_y^+ + H_y^- = \sqrt{\frac{\varepsilon}{\mu}}E_x^+ - \sqrt{\frac{\varepsilon}{\mu}}E_x^-, \text{ (HW)} \quad (4)$$

Where **expression for** $E_y^+, E_y^-, E_x^+, E_x^-$ **were given previously** (e.g.

$$E_x^+ = -\frac{1}{\varepsilon}\frac{\partial}{\partial y}F_z^+(x,y)e^{-j\beta z} \text{ and } E_x^- = -\frac{1}{\varepsilon}\frac{\partial}{\partial y}F_z^-(x,y)e^{+j\beta z}) \quad (5)$$

Transverse magnetic wave WRT *z*-direction (TM^z)

• To ensure that wave is a **transverse magnetic (TM) field WRT *z*-direction,** it is **sufficient to ensure the auxiliary vector potential** $\vec{A}$ **has only *z*-component and** $\vec{F} = 0$.

• **For TM^z**

$$\vec{A} = \hat{a}_z A_z(x,y,z) \text{ and } \vec{F} = 0 \quad (6)$$

• **The field components are then given by**

$$E_x = -j\frac{1}{\omega\mu\varepsilon}\frac{\partial^2 A_z}{\partial x \partial z} \quad (7)$$

$$E_y = -j\frac{1}{\omega\mu\varepsilon}\frac{\partial^2 A_z}{\partial y \partial z} \quad (8)$$

$$E_z = -j\frac{1}{\omega\mu\varepsilon}\left(\frac{\partial^2}{\partial z^2} + \beta^2\right)A_z \quad (9)$$

$$H_x = \frac{1}{\mu}\frac{\partial}{\partial y}A_z \quad (10)$$

$$H_y = -\frac{1}{\mu}\frac{\partial A_z}{\partial x} \quad (11)$$

$$H_z = 0 \quad (12)$$

• **All the field components of the TM^z mode can also be expressed in terms of** E_z

Transverse electric field WRT *z*-direction (TE^z)

• **To have TE^z we require $\vec{F}$ to have only z-component and** $\vec{A}=0$, i.e.,

$\vec{A}=0$ and $\vec{F}=\hat{a}_z F_z(x,y,z)$ (1)

• **The field components are given by**

$$E_x=-\frac{1}{\varepsilon}\frac{\partial F_z}{\partial y}$$

$$E_y=\frac{1}{\varepsilon}\frac{\partial F_z}{\partial x}$$

$$E_z=0$$

$$H_x=-j\frac{1}{\omega\mu\varepsilon}\frac{\partial^2 F_z}{\partial x\partial z}$$

$$H_y=-j\frac{1}{\omega\mu\varepsilon}\frac{\partial^2 F_z}{\partial y\partial z}$$

$$H_z=-j\frac{1}{\omega\mu\varepsilon}\left(\frac{\partial^2}{\partial z^2}+\beta^2\right)F_z$$

• **All the field components of the TE^z mode can also be expressed in terms of** H_z

Rectangular metallic wave guide

• Rectangular metallic waveguides are routinely **used at RF and microwave frequencie**s. Their study is not only motivated by their use as RF/microwave components, but will help us **better understand the concept of mod**e and guided wave propagation

• In studying the guided wave structures we are usually interested in parameters such as: **field configurations (modes)** that are supported by the structure, the structure **cutoff frequency, guided wavelength, wave impedance, phase constant, attenuation constant,** etc.

• For metallic rectangular waveguide, it can be shown that **although TEM field configuration is the lowest order mode, it does not satisfy the boundary conditions and as such, the waveguide does not support TEM modes**

• However, the **TE and TM modes satisfy the required boundary conditions and as such are supported by the structure**

Transverse Electric Field TE^z

• Consider the **metallic waveguide of size** $a \times b$ as shown. The **waveguide is infinite in the z-direction**

• From our previous discussion **we have seen that TE^z modes are obtained if** $\vec{A} = 0$ and $\vec{F} = \hat{a}_z F_z(x, y, z)$ which implied

$$E_x = -\frac{1}{\varepsilon}\frac{\partial F_z}{\partial y}$$

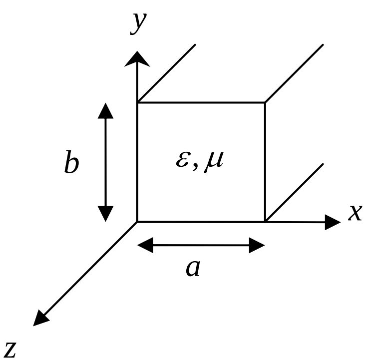
$$H_x = -j\frac{1}{\omega\mu\varepsilon}\frac{\partial^2 F_z}{\partial x \partial z}$$

$$E_y = \frac{1}{\varepsilon}\frac{\partial F_z}{\partial x}$$

$$H_y = -j\frac{1}{\omega\mu\varepsilon}\frac{\partial^2 F_z}{\partial y \partial z}$$

$$E_z = 0$$

$$H_z = -j\frac{1}{\omega\mu\varepsilon}\left(\frac{\partial^2}{\partial z^2} + \beta^2\right)F_z$$



• $\vec{F}$ **must satisfy the vector differential equation**

$$\nabla^2 \vec{F} + \beta^2 \vec{F} = 0 \Rightarrow \nabla^2 F_z(x, y, z) + \beta^2 F_z(x, y, z) = 0 \Rightarrow$$

$$\frac{\partial^2 F_z}{\partial x^2} + \frac{\partial^2 F_z}{\partial y^2} + \frac{\partial^2 F_z}{\partial z^2} + \beta^2 F_z = 0$$

• Note that $F_z(x, y, z)$ **is a scalar function that can be written a**s (using separation of variables)

$$F_z(x, y, z) = f(x)g(y)h(z)$$

• Also recall that **solutions to** $\nabla^2 F_z + \beta^2 F_z = 0$ **are either standing waves (sinusoidal)** or **traveling waves (exponential with complex argument)**

• The **particular form** (standing wave or traveling wave) **is chosen based on the boundary conditions** to be satisfied

• In the case of **our metallic waveguide solutions in** x **and** y **must be standing waves** and **solution in *z*-direction** (guide is infinite in the z-direction) **must be traveling wave**

• Hence

$$F_z(x,y,z) = f(x)g(y)h(z)$$
$$= [C_1 \cos(\beta_z x) + D_1 \sin(\beta_x x)] \cdot [C_2 \cos(\beta_y y) + D_2 \sin(\beta_y y)] \cdot [A_3 e^{-j\beta_z z} + B_3 e^{+j\beta_z z}]$$

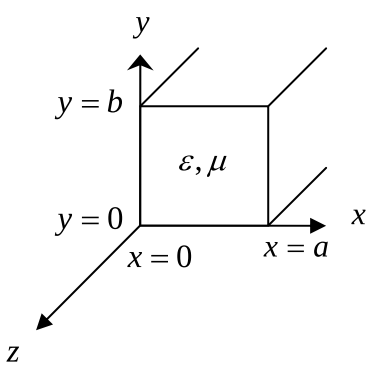
with

$$\beta_x^{\ 2} + \beta_y^{\ 2} + \beta_z^{\ 2} = \beta^2 = \omega^2 \mu\varepsilon$$

• Recall that **for** $e^{j\omega t}$ **time dependency,** $e^{-j\beta_z z}$ **is a positively traveling wave** (wave travels in positive z-direction) and $e^{+j\beta_z z}$ **is a negatively traveling wave** (wave travels in negative z-direction)

• **If source is located such that only positively traveling wave is present**, then $B_3 e^{+j\beta_z z} = 0 \Rightarrow B_3 = 0$

• **If source is located such that only negatively traveling wave is present**, then $A_3 e^{-j\beta_z z} = 0 \Rightarrow A_3 = 0$



• **If both positively and negatively traveling waves are present** (a waveguide terminated on a load that is not matched), **then both** $A_3 e^{-j\beta_z z}$ **and** $B_3 e^{+j\beta_z z}$ **must be included**

• **Here, for simplicity, we assume that only positive traveling wave exist** $\Rightarrow B_3 = 0$

•The F_z **is then given by**

$$F_z^+(x,y,z) = [C_1 \cos(\beta_x x) + D_1 \sin(\beta_x x)] \cdot [C_2 \cos(\beta_y y) + D_2 \sin(\beta_y y)] A_3 e^{-j\beta_z z} \qquad (1)$$

• We impose the **boundary conditions on the top, bottom, left and right walls of the metallic waveguide, assuming a perfect electric conductor (PEC) boundary condition, i.e.** $\vec{E}$ **and** $\vec{H}$ **tangential are zero on the walls**

• The boundary conditions are:

$E_x(0 \le x \le a, y = 0, z) = E_x(0 \le x \le a, y = b, z) = 0$, Bottom and top walls for E_x (2)

$E_z(0 \le x \le a, y = 0, z) = E_z(0 \le x \le a, y = b, z) = 0$, Bottom and top walls for E_z (3)

$E_y(x = 0, 0 \le y \le b, z) = E_y(x = a, 0 \le y \le b, z) = 0$, Left and right walls for E_y (4)

$E_z(x = 0, 0 \le y \le b, z) = E_z(x = a, 0 \le y \le b, z) = 0$, Left and right walls for E_z (5)

• Note that the **boundary conditions (3) and (5) are not independent and they represent the same boundary conditions as (2) and (4)**.

• **The necessary and sufficient conditions are to satisfy (2) OR (3) (**E_x **and** E_z **at bottom and top walls) AND (4) or (5) (**E_y **and** E_z **at the left and right walls)**

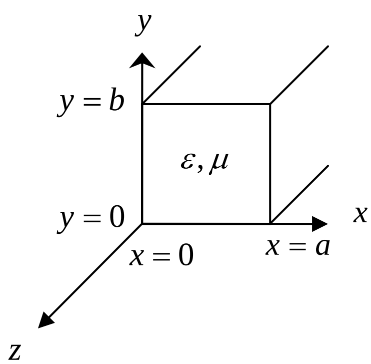
• Furthermore, note that for **TE^z modes by definition E_z is zero. This means that the necessary and sufficient B.C.'s for TE^z are (2) and (4) of the last page (** E_x **at the bottom and the top and** E_y **at the left and right walls)**

$$E_x(0 \le x \le a, y = 0, z) = E_x(0 \le x \le a, y = b, z) = 0$$
$$E_y(x = 0, 0 \le y \le b, z) = E_y(x = a, 0 \le y \le b, z) = 0$$

• Recall that the **vector potential for TE^z was given as** [Eq. (1), last page]
$F_z^+(x,y,z) = [C_1\cos(\beta_x x) + D_1\sin(\beta_x x)]\cdot[C_2\cos(\beta_y y) + D_2\sin(\beta_y y)]A_3 e^{-j\beta_z z}$. We then have

$$E_x = -\frac{1}{\varepsilon}\frac{\partial F_z^+}{\partial y} = -A_3\frac{\beta_y}{\varepsilon}[C_1\cos(\beta_x x) + D_1\sin(\beta_x x)]\cdot[-C_2\sin(\beta_y y) + D_2\cos(\beta_y y)]A_3 e^{-j\beta_z z}$$

• **From** $E_x(0 \le x \le a, y = 0, z) = 0 \Rightarrow D_2 = 0$



• **From** $E_x(0 \le x \le a, y = b, z) = 0 \Rightarrow \sin(\beta_y b) = 0 \Rightarrow$
$\beta_y b = n\pi \quad n = 0,1,2,3$ **or equally**,

$$\beta_y = \frac{n\pi}{b} \quad n = 0,1,2,3$$

• β_y is sometimes referred to as eigenvalue

• If we **use our newly found results, we have**
$$F_z^+(x,y,z) = [C_1\cos(\beta_x x) + D_1\sin(\beta_x x)]C_2\cos\left(\frac{n\pi}{b}y\right)A_3 e^{-j\beta_z z}$$

• The E_y **can be found from**

$$E_y = \frac{1}{\varepsilon}\frac{\partial F_z}{\partial x} = \frac{\beta_x}{\varepsilon}[-C_1\sin(\beta_x x) + D_1\cos(\beta_x x)]C_2\cos\left(\frac{n\pi}{b}y\right)A_3 e^{-j\beta_z x}$$

• **Boundary conditions for** E_y **at left wall** is
$E_y(x = 0, 0 \le y \le b, z) = 0 \Rightarrow D_1 = 0$

• **Boundary condition for** E_y **at right wall is**
$E_y(x = a, 0 \le y \le b, z) = 0 \Rightarrow \sin(\beta_x a) = 0 \Rightarrow$
$\beta_x a = m\pi \;\; m = 0,1,2,3$ or equally

$$\beta_x = \frac{m\pi}{a} \quad m = 0,1,2,3$$

• **Putting it all together, the vector potential** F_z^+ **is given by**

$$F_z^+(x,y,z) = C_1\cos\left(\frac{m\pi}{a}x\right)C_2\cos\left(\frac{n\pi}{b}y\right)A_3e^{-j\beta_z z} = A_{mn}\cos\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)e^{-j\beta_z z}$$

with $\begin{matrix} m = 0,1,2,3 \\ n = 0,1,2,3 \end{matrix}$ but $m = n \neq 0$

A_{mn} **is a constant =** $C_1C_2A_3$

Propagation constant (wave numbers) and wavelengths in the x, y and z direction

• From our previous discussion it is clear that **propagation constant (or wave number) along** x (β_x) **and along** y (β_y) **can be written as**

$$\beta_x = \frac{m\pi}{a} = \frac{2\pi}{\lambda_x} \Rightarrow \lambda_x = \frac{2a}{m}; m = 0, 1, 2 \qquad (1)$$

$$\beta_y = \frac{n\pi}{b} = \frac{2\pi}{\lambda_y} \Rightarrow \lambda_y = \frac{2b}{n}; n = 0, 1, 2 \text{ and with } m = n \neq 0, \qquad (2)$$

where we have also defined the **wavelength along** x **to be** λ_x **and along** y **to be** λ_y

• Recall that $\beta_x{}^2 + \beta_y{}^2 + \beta_z{}^2 = \beta^2 = \omega^2\mu\varepsilon$ or equally well:

$\frac{1}{\lambda_x{}^2} + \frac{1}{\lambda_y{}^2} + \frac{1}{\lambda_z{}^2} = \frac{1}{\lambda^2}$ where λ **is the wavelength in the medium with** ε **and** μ **(material inside the guide)**

• From (1) and (2) note that β_x **and** β_y **are discrete (one can say they are quantized), where as** β_z **is a continuous parameter.**

• Note that in principle there are **infinite numbers of possible** β_x **and** β_y (eigenvalues) hence there are **infinite number of TE^z modes that satisfy the wave equation** and the given boundary condition.

• From $\beta_x = \frac{m\pi}{a}$ and $\beta_y = \frac{n\pi}{b}$ and $\beta_x{}^2 + \beta_y{}^2 + \beta_z{}^2 = \beta^2 = \omega^2\mu\varepsilon$ we can see that

$$\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \beta_z{}^2 = \beta^2 \Rightarrow \beta^2 - \beta_z{}^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 = \beta_c{}^2,$$

where by definition: $\beta_c \equiv$ **cutoff propagation constant or cutoff wave number**

• Note that $\beta_c = \beta\big|_{\beta_z=0} = \omega\sqrt{\mu\varepsilon}\big|_{\beta_z=0} = \omega_c\sqrt{\mu\varepsilon} = 2\pi f_c\sqrt{\mu\varepsilon} = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$.

• Hence, the cutoff frequency (f_c) is given by

$$(f_c)_{mn} = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} \qquad \left.\begin{matrix} m = 0,1,2,3 \\ n = 0,1,2,3 \end{matrix}\right\} m = n \neq 0$$

• From the expression $\beta_c = \beta\big|_{\beta_z=0}$ we can see why β_c is called the cutoff propagation constant. **For this wave number, $\beta_z = 0$ and the wave no longer travels along the *z*-direction.**

• **The above can be more clearly seen from**

$$\beta_z^{\,2} = \beta^2 - \left(\beta_x^{\,2} + \beta_y^{\,2}\right) = \beta^2 - \left(\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right) = \beta^2 - \beta_c^{\,2}$$

. Clearly for

$$\beta = \beta_c \Rightarrow \beta_z = 0$$

• **The field components for TE_{mn}^{+z} are now given by**

$$E_x^{\,+} = A_{mn}\frac{\beta_y}{\varepsilon}\cos(\beta_x x)\sin(\beta_y y)e^{-j\beta_z z}$$

$$E_y^{\,+} = -A_{mn}\frac{\beta_x}{\varepsilon}\sin(\beta_x x)\cos(\beta_y y)e^{-j\beta_z z}$$

$$E_z^{\,+} = 0$$

$$H_x^{\,+} = A_{mn}\frac{\beta_x\beta_z}{\omega\mu\varepsilon}\sin(\beta_x x)\cos(\beta_y y)e^{-j\beta_z z}$$

$$H_Y^{\,+} = A_{mn}\frac{\beta_y\beta_z}{\omega\mu\varepsilon}\cos(\beta_x x)\sin(\beta_y y)e^{-j\beta_z z}$$

$$H_z^{\,+} = -jA_{mn}\frac{\beta_c^{\,2}}{\omega\mu\varepsilon}\cos(\beta_x x)\cos(\beta_y y)e^{-j\beta_z z}$$

• **To appreciate the importance of the cutoff conditions consider the following:**

$$\beta_z^{\,2} = \beta^2 - \beta_c^{\,2} = \beta^2 - \left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right] \Rightarrow$$

$$(\beta_z)_{mn} = \pm\sqrt{\beta^2 - \beta_c^{\,2}} = \pm\beta\sqrt{1 - \left(\frac{\beta_c}{\beta}\right)^2}$$

$$= \pm\beta\sqrt{1 - \left(\frac{f_c}{f}\right)^2} = \pm\beta\sqrt{1 - \left(\frac{\lambda}{\lambda_c}\right)^2}$$

for $\beta > \beta_c \Leftrightarrow f > f_c$

• $(\beta_z)_{mn} = 0$ for $\beta = \beta_c \Leftrightarrow f = f_c$

• $(\beta_z)_{mn} = \pm j\sqrt{\beta_c^{\,2} - \beta^2} = \pm j\beta\sqrt{\left(\frac{\beta_c}{\beta}\right)^2 - 1}$

$= \pm j\beta\sqrt{\left(\frac{f_c}{f}\right)^2 - 1} = \pm j\beta\sqrt{\left(\frac{\lambda}{\lambda_c}\right)^2 - 1}$ for $\beta_c > \beta \Leftrightarrow f_c > f$

• **If we only consider the positively traveling wave we must choose the sign in front of the square root appropriately, i.e.**

$$(\beta_z)_{mn} = \beta\sqrt{1 - \left(\frac{\beta_c}{\beta}\right)^2} = \beta\sqrt{1 - \left(\frac{f_c}{f}\right)^2} = \beta\sqrt{1 - \left(\frac{\lambda}{\lambda_c}\right)^2} \qquad \text{for } \beta > \beta_c \Leftrightarrow f > f_c$$

$$(\beta_z)_{mn} = 0 \qquad \text{for } \beta_c = f_c = 0$$

$$(\beta_z)_{mn} = -j\beta\sqrt{\left(\frac{\beta_c}{\beta}\right)^2 - 1} = -j\beta\sqrt{\left(\frac{f_c}{f}\right)^2 - 1} = -j\beta\sqrt{\left(\frac{\lambda}{\lambda_c}\right)^2 - 1} \qquad \text{for } \beta_c > \beta \Leftrightarrow f_c > f$$

• **Since electric and magnetic fields are proportional to** $e^{-j\beta_z z} e^{j\omega t}$ **then**

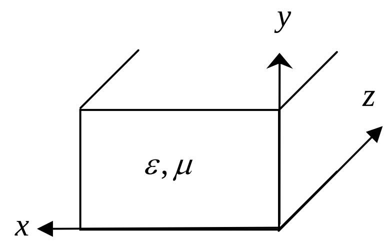
$e^{-j\beta\sqrt{1-\left(\frac{f_c}{f}\right)^2}\,z} e^{j\omega t}$ **represents a propagating wave for** $f > f_c$

$e^{j\omega t} e^{-j0z} = e^0 e^{j\omega t} = 1 e^{j\omega t}$ **represents a standing wave for** $f_c = f$

$e^{-j\left(-j\beta\sqrt{\left(\frac{f_c}{f}\right)^2 - 1}\right)z} e^{j\omega t} = e^{-\beta_z\left[\sqrt{\left(\frac{f_c}{f}\right)^2 - 1}\right]z} e^{j\omega t}$ represents an attenuated (evanescent) wave for $f_c > f$

TE^{z}_{10} Field components and Field Patterns

$E_{x(x,y,z,t)=0}$



$$E_y(x, y, z; t) = \frac{\omega\mu}{h^2}\left(\frac{\pi}{a}\right)H_0 \sin\left(\frac{\pi}{a}x\right)\sin(\omega t - \beta z),$$

$$E_z(x, y, z; t) = 0,$$

$$H_x(x, y, z; t) = -\frac{\beta}{h^2}\left(\frac{\pi}{a}\right)H_0 \sin\left(\frac{\pi}{a}x\right)\sin(\omega t - \beta z),$$

$$H_y(x, y, z; t) = 0,$$

$$H_z(x, y, z; t) = H_0 \cos\left(\frac{\pi}{a}x\right)\cos(\omega t - \beta z),$$