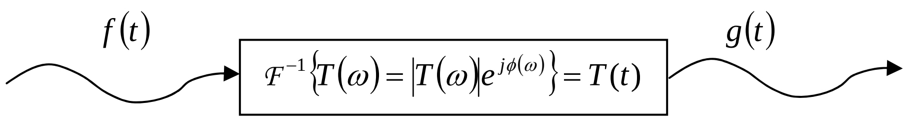
The propagation of electromagnetic wave packets (pulses): Group Delay

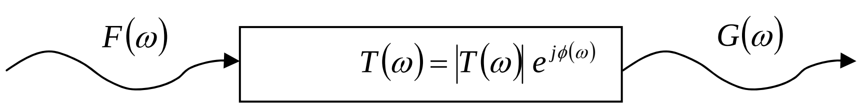
• In discussing wave propagation in a medium we **may describe the phenomenon in terms of velocity or equally well in terms of the time required for the propagation.** For example, we will see that **the peak of a well-behaved EM wave packet (pulse) propagating in a medium of index** $n(\omega)$**, moves with group velocity** $\vec{v}_g$. We can also formulate this problem in terms of the **time it takes for the packet to travel a given distance through the medium. This time is referred to as the group delay.**

• Let us consider the following. **A medium characterized by its transmission function** $T(\omega)$ is excited by an **incident pulse** $f(t) = f_e(t)\cos(\omega_0 t)$, where the **output of the system is** $g(t)$**.**

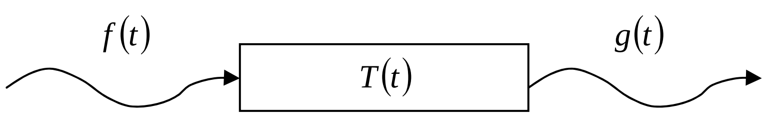


• $f_e(t)$ **is the envelope of the modulated incident wave packet** $f(t)$**, and** ω_0 **is the frequency of modulation.**

• In the language of the **linear system theory,** $T(\omega)$ **is the network function** or the system response.

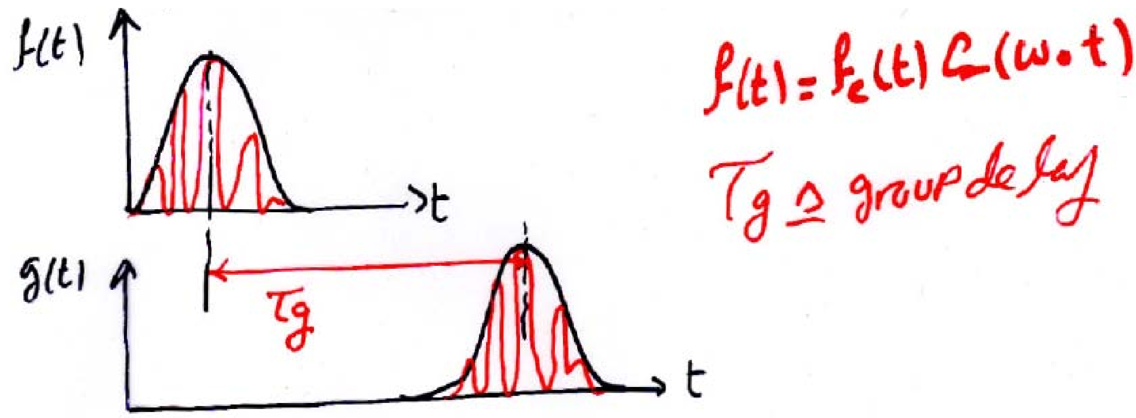


$$G(\omega) = F(\omega)\,T(\omega)$$



$$g(t) = \underbrace{\int_{-\infty}^{+\infty} f(\tau)\,T(t-\tau)\,d\tau}_{\text{convolution integral}}$$

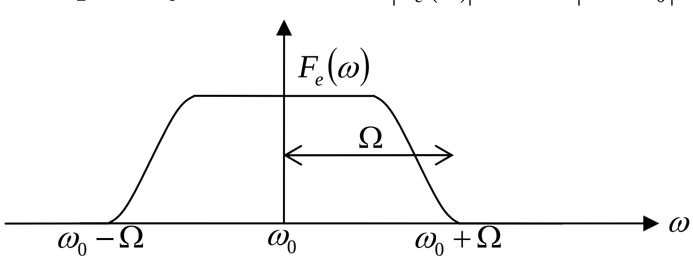
• For a **well-behaved wave packet** and a medium that is **not too dispersive (or equally for narrow bandwidth wave packet)** the relation between $f(t)$ and $g(t)$ is shown below



• In the following we shall see that the **output [$g(t)$] is indeed a delayed version of the input [$f_e(t)$]** by an **amount given by the group delay (** τ_g **)**, and we will obtain an expression for the group delay.

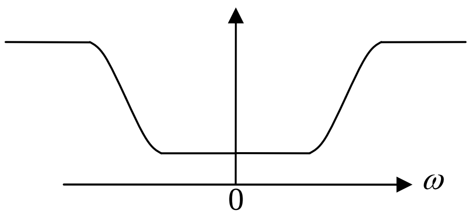
• For the input $f(t) = f_e(t)\cos(\omega_0 t)$ we assume f_e **(the envelope) is narrowband**, i.e. $|F_e(\omega)| = 0$ **for** $|\omega - \omega_0| > \Omega$ **and** $\omega_0 > \Omega$

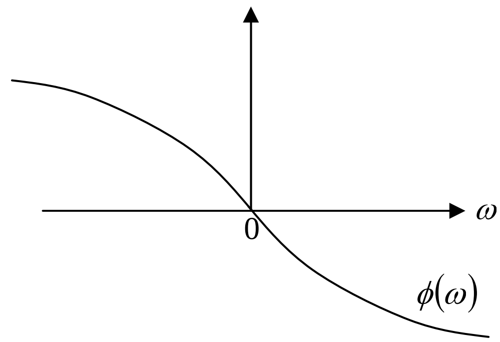
• **Graphically, the condition** $|F_e(\omega)| = 0$ **for** $|\omega - \omega_0| > \Omega$ **is shown here**



• We also assume $T(\omega)=|T(\omega)|\,e^{j\phi(\omega)}$ **has the following properties:**

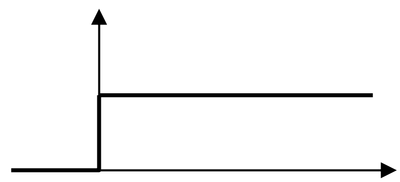
$$\begin{cases} |T(\omega)| \text{ is even about origion} \\ \phi(\omega) \text{ is odd about origion} \end{cases}$$





• For symmetric systems such as above, **the network function** $T(\omega)$ **can be expressed as the sum of two terms:**
$T_1(\omega)$ **for positive frequencies**
$T_2(\omega)$ **for negative frequencies**

• $T(\omega)=T_1(\omega)+T_2(\omega)$
$T_1(\omega)=T(\omega)U(\omega)$
$T_2(\omega)=T(\omega)U(-\omega)$

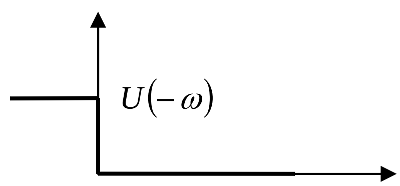


• U **is the step function (Heaviside function)**

$$U(\omega)=\begin{cases} 1 & \text{for } \omega>0 \\ 0 & \text{for } \omega<0 \end{cases}$$

and

$$U(-\omega)=\begin{cases} 1 & \text{for } \omega<0 \\ 0 & \text{for } \omega>0 \end{cases}$$



• **Since inverse Fourier transform of** $T(\omega)$**, i.e.,** $\mathcal{F}^{-1}[T(\omega)]$**, (the impulse response) must be real**, **then we can show** $T(-\omega)=T^*(\omega)$ **which can be used to show** $T_1(-\omega)=T_2^*(\omega)$ **(HW)**

• But what is $T_1(\omega)$? **let us now find an expression for :** $T_1(\omega)=A(\omega)\,e^{j\phi(\omega)}$

• **At the vicinity of** ω_0**,** we approximate $A(\omega)$ by $A(\omega)\approx A_0$ **and** $\phi(\omega)$ **by:**

$$\phi(\omega)=\phi(\omega_0)+\left.\frac{d\phi}{d\omega}\right|_{\omega_0}(\omega-\omega_0)+\cdots$$
$$=-\left\{-\frac{\omega_0\phi(\omega_0)}{\omega_0}-\left.\frac{d\phi}{d\omega}\right|_{\omega_0}(\omega-\omega_0)-\cdots\right\}$$

• Let us define

$\tau_p \equiv -\dfrac{\phi(\omega_0)}{\omega_0}$ **as the phase delay and**

$\tau_g \equiv -\left.\dfrac{d\phi}{d\omega}\right|_{\omega_0}$ **as the group delay,**

then,

$$\phi(\omega)=-\left\{-\frac{\omega_0\phi(\omega_0)}{\omega_0}-\left.\frac{d\phi}{d\omega}\right|_{\omega_0}(\omega-\omega_0)-\cdots\right\}=-\{\omega_0\tau_p+(\omega-\omega_0)\tau_g+\cdots\}$$

• **Hence to second order approximation**

$$T_1(\omega)=A(\omega)e^{j\phi(\omega)}\approx A_0e^{-j\left[\omega_0\tau_p+(\omega-\omega_0)\tau_g\right]}=A_0e^{-j\omega_0\tau_p}e^{-j(\omega-\omega_0)\tau_g}$$

• **Knowing** $T_1(\omega)$ **from above,** $T_2(\omega)$ **can be evaluated from** $T_1(-\omega)=T_2^{\,*}(\omega)$ **to be (HW)**

$$T_2(\omega)\approx A_0e^{j\omega_0\tau_p}e^{-j(\omega+\omega_0)\tau_g}$$

• Now we are ready to **calculate the output from linear system theory**

$$\xrightarrow{F(\omega)}\boxed{T(\omega)}\xrightarrow{G(\omega)}$$

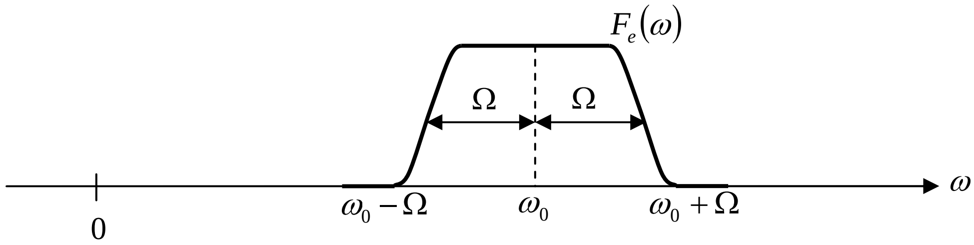
$G(\omega)=F(\omega)T(\omega)\Rightarrow G(\omega)=\mathcal{F}\left[f_e(t)\cos(\omega_0t)\right]\left[T_1(\omega)+T_2(\omega)\right]$ **where** $\mathcal{F}$ **means the Fourier transform**

• **From Fourier transform we know** $\mathcal{F}\left[f_e(t)\cos(\omega_0t)\right]=\dfrac{F_e(\omega+\omega_0)}{2}+\dfrac{F_e(\omega-\omega_0)}{2}$, where $F_e(\omega)=\mathcal{F}\left[f_e(t)\right]$

• **Then** $G(\omega)$ **can be written as**

$$G(\omega)=\left[\frac{F_e(\omega+\omega_0)}{2}+\frac{F_e(\omega-\omega_0)}{2}\right]\left[T_1(\omega)+T_2(\omega)\right]$$
$$=\frac{F_e(\omega+\omega_0)T_1(\omega)}{2}+\frac{F_e(\omega+\omega_0)T_2(\omega)}{2}+\frac{F_e(\omega-\omega_0)T_1(\omega)}{2}+\frac{F_e(\omega-\omega_0)T_2(\omega)}{2}$$

• **Note that from our condition on** $F_e(\omega)$**, i.e.** $|F_e(\omega)| = 0$ **for** $|\omega - \omega_0| > \Omega$ **with** $\omega_0 > \Omega$**, we can conclude that** $F_e(\omega + \omega_0) = 0$**, hence** $F_e(\omega + \omega_0)\, T_1(\omega) = 0$.



• **Similar reasoning implies that** $\dfrac{F_e(\omega - \omega_0) T_2(\omega)}{2} = 0$

• **Then**

$$g(\omega) \approx \frac{F_e(\omega + \omega_0)\, T_2(\omega)}{2} + \frac{F_e(\omega - \omega_0)\, T_1(\omega)}{2}$$

• **In the above we substitute the expressions for** $T_1(\omega)$ **and** $T_2(\omega)$ **from the last page, and we have**

$$G(\omega) = \frac{F_e(\omega + \omega_0)}{2} A_0 e^{j\omega_0 \tau_p} e^{-j(\omega+\omega_0)\tau_g} + \frac{F_e(\omega - \omega_0)}{2} A_0 e^{-j\omega_0 \tau_p} e^{-j(\omega-\omega_0)\tau_g}$$

• **With some careful thinking and a table of Fourier transforms we can see that**

$$g(t) = \mathcal{F}^{-1}[G(\omega)] = f_e(t - \tau_g)\cos[\omega_0 (t - \tau_p)]$$

• τ_g **is the time by which the envelope of the wave packet is delayed (or advanced) and is a measure of the group velocity.**

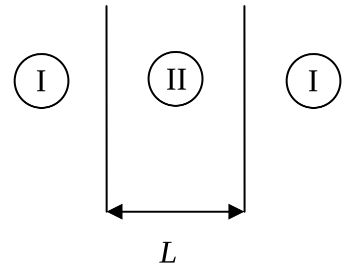
• τ_p **is the time by which the phase of elementary excitations (harmonics) are delayed (or advanced) and is a measure of the phase velocity.**

• **This means that the wave packet moves with group velocity** $v_g = \dfrac{L}{\tau_g}$**, where** L **is the physical thickness of the medium characterized by** $T(\omega)$

• **This means that the elementary excitations (harmonics) moves with phase velocity** $v_p = \dfrac{L}{\tau_p}$**, where** L **is the physical thickness of the medium characterized by** $T(\omega)$

Group velocity and group delay in the matched medium case

• Recall that **transmission coefficient for a slab was given by**



$T^{\mathrm{TE}} = \dfrac{t_{12} t_{21} e^{+j\phi}}{1-(r_{21})^2 e^{2j\phi}}$, where $\phi = -k_{2z} L = \dfrac{\omega}{c} n_2 \cos\theta_2 \; L$ with

$T^{\mathrm{TE}} \equiv$ **Field transmission coefficient, and**

$$r_{21} = \frac{\mu_1 k_{2z} - \mu_2 k_{1z}}{\mu_1 k_{2z} + \mu_2 k_{1z}} = -r_{12}$$

$$t_{12} = \frac{2\mu_2 k_{1z}}{\mu_2 k_{1z} + \mu_1 k_{2z}}$$

$$t_{21} = \frac{2\mu_1 k_{2z}}{\mu_1 k_{2z} + \mu_2 k_{1z}}$$

• Under the **matched matching:** $r_{21} = r_{12} = 0 \Rightarrow \mu_1 k_{2z} = \mu_2 k_{1z} \Rightarrow t_{12} = t_{21} = 1$
then we have $T^{\mathrm{TE}}(\omega) = e^{j\phi}$. **Therefore, transmission only introduces a phase term**

• let us **calculate the group delay**

$$\tau_g = -\frac{\partial\phi}{\partial\omega} = -\frac{\partial}{\partial\omega}\left[-k_{2z} L\right] = L\frac{\partial k_{2z}}{\partial\omega} = \frac{L}{V_g} \Rightarrow V_g = \frac{\partial\omega}{\partial k_{2z}} = \frac{L}{\tau_g}.$$

This is sometimes called **generalized group velocity**

• **Note that in general** $T(\omega) = \dfrac{t_{12} t_{21} e^{j\phi}}{1-(r_{21})^2 e^{2j\phi}} = |T(\omega)| e^{j\phi'}$**. Since** $t_{12}, t_{21}, r_{21}, \phi$ **are all complex then**

$$v_g = \frac{L}{\tau_g} = \frac{L}{-\dfrac{\partial}{\partial\omega}\left[\text{Trans. phase}\right]} = \frac{L}{-\dfrac{\partial}{\partial\omega}\left[\phi'\right]}.$$

Above expression also takes the interface effects (mismatch effects) into account.

Free Particles (free electrons and photons)

• In the following we will derive some of the **general properties of the waves, or more specifically wave packets (pulses or wave functions), which are applicable to both electromagnetic and also electronic waves. We begin our study be considering free particles (free electrons or photons).**

Electron Case:

• Quantum mechanics Postulates that the **state of a particl**e (all you need to know about a particle) **at time** t **and position** $\vec{r}$ **is determined by the wave function** $\Psi(\vec{r},t)$, where $\Psi(\vec{r},t)$ is **called the probability amplitude**. In other words, **the probability of finding the particle, at time** t**, in a volume element** $dr^3 = dxdydz$ **situated at the point** $\vec{r}$ **is given by**

$$d\rho(\vec{r},t) = C\left|\Psi(\vec{r},t)\right|^2 dr^3$$

• Quantum mechanics postulates that the **time evolution of the wave function** $\Psi(\vec{r},t)$ **is governed by Schrödinger equation**

$$i\hbar\frac{\partial}{\partial t}\Psi(\vec{r},t) = \frac{-\hbar^2}{2m}\nabla^2\Psi(\vec{r},t) + V(r)\Psi(\vec{r},t)$$

where $V(r)$ **is the time independent potential.**

• **In Schrödinger equation let** $\Psi(\vec{r},t) = \phi(\vec{r})\chi(t)$ **then**

$$i\hbar\phi(\vec{r})\frac{d\chi}{dt} = \frac{-\hbar^2}{2m}\chi(t)\nabla^2\phi(\vec{r}) + V(\vec{r})\phi(\vec{r})\chi(t) \Rightarrow \qquad (1)$$

$$\underbrace{\frac{i\hbar}{\chi(t)}\frac{d\chi(t)}{dt}}_{\substack{\text{only a function of}\\ \text{time=constant}=\hbar\omega}} = \underbrace{-\frac{\hbar^2}{2m}\frac{1}{\phi(\vec{r})}\nabla^2\phi(\vec{r}) + V(\vec{r})}_{\text{only a function of position=constant}=\hbar\omega} \Rightarrow$$

$$\frac{i\hbar}{\chi(t)}\frac{d\chi(t)}{dt} = \hbar\omega \Rightarrow \frac{d\chi(t)}{dt} = -i\omega\chi(t) \Rightarrow$$

$$\chi(t) = Ae^{-i\omega t} \Rightarrow \Psi(\vec{r},t) = A\phi(\vec{r})\,e^{-i\omega t} \equiv \underbrace{\phi(\vec{r})}_{\substack{\text{we have absorbed } A\\ \text{in } \phi(\vec{r})}}e^{-i\omega t}$$

• Then **RHS of (1) can be written a**s

$$\frac{-\hbar^2}{2m}\nabla^2\phi(\vec{r}) + V(\vec{r})\phi(\vec{r}) = \hbar\omega\phi(\vec{r}) = E\phi(\vec{r})$$

where $E = \hbar\omega$ **is the energy. This is called Schrödinger equation in the stationary form**.

• $\Psi(\vec{r},t) = \phi(\vec{r})\,e^{-i\omega t}$ **is called the stationary solution of the Schrödinger equation**

• **In stationary state form, the probability density** $\left|\Psi(\vec{r},t)\right|^2 = \left|\phi(\vec{r})\right|^2$ **is time independent**

• **A stationary state is a state with well defined energy** $E = \hbar\omega$

• In classical mechanics if **potential is time independent, then total energy is a constant of the motion**. **In quantum mechanics, this means that a well determined energy state exists.**

• Now, let us **focus on the stationary form of the Schrödinger equation.**

$-\frac{\hbar^2}{2m}\nabla^2\phi(\vec{r}) + V(\vec{r})\phi(\vec{r}) = E\,\phi(\vec{r})$. Suppose the **solution to this be written as**

$\phi(\vec{r}) = Ae^{i\vec{k}_e\cdot\vec{r}}$ **, then it can be shown that**

$-\frac{\hbar^2}{2m}\left(-k_e^2\right) + V = E \Rightarrow \frac{\hbar^2 k_e^2}{2m} = E - V$ or

$$k_e^2 = \frac{2m}{\hbar^2}(E - V)$$

• **If over a region of space** $V = 0$ **or constant, i.e.** $F = -\nabla V = 0$ **(no force)**, then we say the **particle is free and we have**

$E = \frac{\hbar^2 k_e^2}{2m}$ (free electron dispersion relation)

• **Using Broglie relation** $P = \hbar k$, where now k must be replaced by k_e, we can write

$E = \frac{\hbar^2 k_e^2}{2m} = \frac{P^2}{2m}$ **which is our classical result for free electron**

• **Also using Einstein relation** $E = \hbar\omega$, **for free electron we can write** ω **in terms of** k_e **according to**

$$\omega = \frac{\hbar\, k_e^2}{2m}$$

• Finally, note that **for the stationary state** $\Psi(\vec{r},t) = A\,e^{i\vec{k}\cdot\vec{r}}e^{-i\omega t}$, **the probability density is given by** $\left|\Psi(\vec{r},t)\right|^2 = A^2$ **which is a constant**

• **Hence for the plane wave** $\Psi(\vec{r},t) = A\,e^{i\vec{k}\cdot\vec{r}}e^{-i\omega t}$ **the probability of finding the electron at any point in space is the same, however this cannot represent a physical situation.**

Photon Case:

• **The dynamical wave equation governing the behavior of electromagnetic waves (photons) in a simple medium in absence of charges and conduction current is the Helmholtz's wave equation given by**

$$\nabla^2 \vec{\mathcal{E}}(\vec{r},t) - \mu\varepsilon \frac{\partial^2}{\partial t^2} \vec{\mathcal{E}}(\vec{r},t) = 0,$$

where $\vec{\mathcal{E}}(\vec{r},t)$ **is the electric filed.**

• In comparing the wave equations for electrons and photons note **the order of the time derivatives and the vector nature of the electric field.**

• The **stationary form of the Helmholtz's wave equation** is given by
$\nabla^2 \vec{E}(\vec{r}) + \Omega^2 \mu\varepsilon\, \vec{E}(\vec{r}) = 0$, where Ω **is the optical frequency**

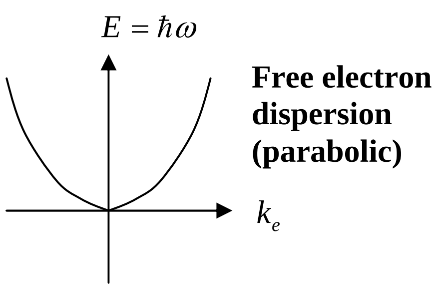
• Suppose the **solution to this be written as** $\vec{E} = \vec{E}_0\, e^{i\vec{k}_p \cdot \vec{r}}$ **, then it can be shown that**

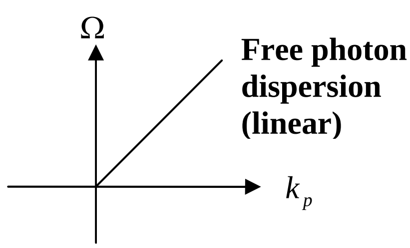
$$-k_p^2 + \Omega^2 \mu\varepsilon = 0 \Rightarrow k_p = \Omega \sqrt{\mu\varepsilon} = \frac{\Omega}{c} n(\Omega)$$

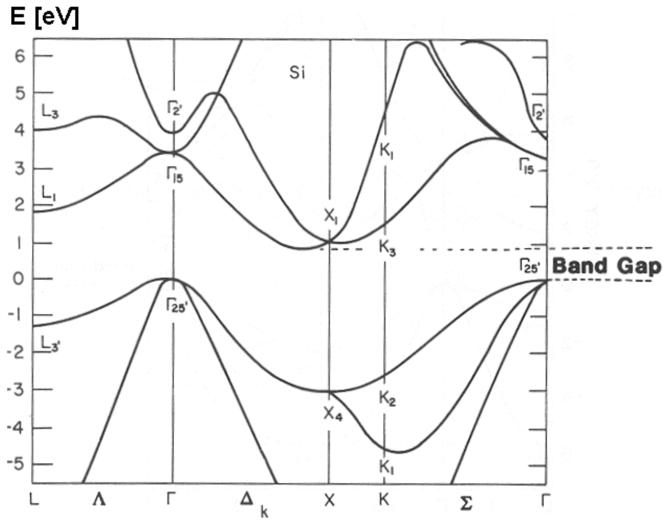
• **The dispersion relation for free photons is then given by**

$k_p = \frac{\Omega}{c} n(\Omega)$ or equally $\Omega = \frac{k_p\, c}{n(k_p)}$

•



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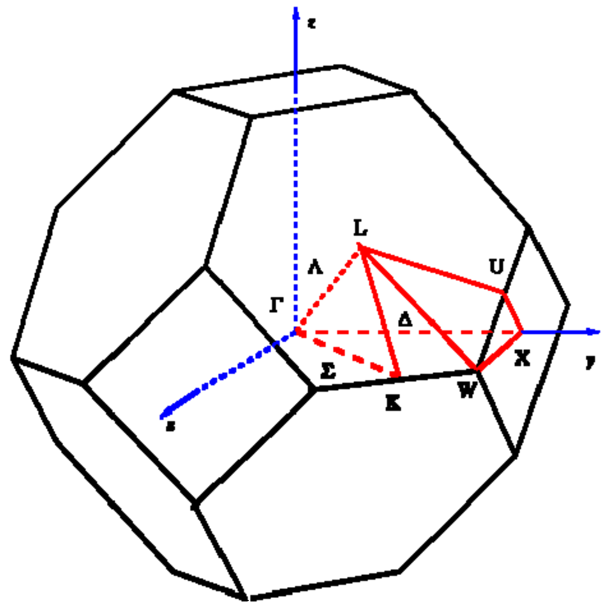
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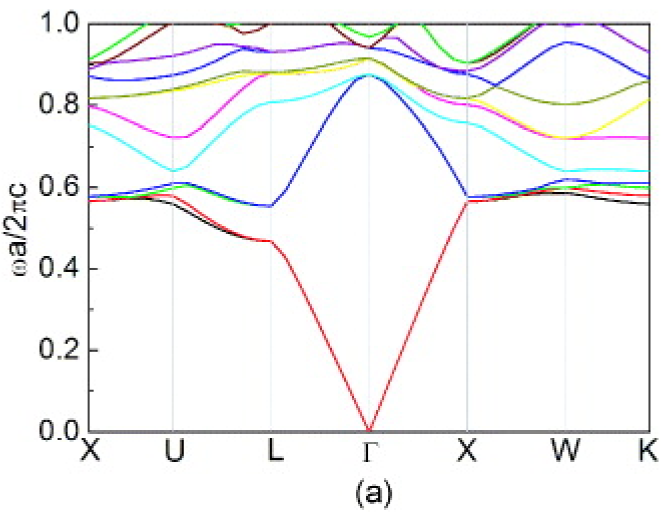
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Principle of Superposition

• The discussion to follow is applicable to **both electrons and photons, i.e. the Schrödinger or Helmholtz's wave equations in the stationary state.** We take $\psi(\vec{r},t)$ to be **either the electronic wave function or any components of the electric or magnetic fields**. We also represent the **frequencies (electron or EM waves) with** ω, and k **is the corresponding wave propagation for electrons (**k_e**) or photons (**k_p**).** Clearly, the **main difference between the two cases is the parabolic or linear relations for** $\omega(k)$.

• If the plane wave $\psi(\vec{r},t)=Ae^{i\vec{k}\cdot\vec{r}-i\omega t}$ **is a solution of the wave equations** then all plane waves **satisfying the corresponding** $k-\omega$ relations are also **possible solutions** and all their **linear combination is also a solution** $\Rightarrow$ general solution is then given by

$$\psi(\vec{r},t)=\frac{1}{(2\pi)^{3/2}}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} g(\vec{k})\, e^{i\vec{k}\cdot\vec{r}-i\omega t}dk_x dk_y dk_z$$

• In the above $g(k)$ **is in general complex**. It is also **smooth enough to allow differentiation inside the integral**

• It can be shown that **any square integrable solution of the wave equations** can also **be written** as $\iiint_{-\infty}^{+\infty} g(k)e^{i\vec{k}\cdot\vec{r}-i\omega t}d^3k$, where $d^3k\equiv dk_x dk_y dk_z$

• $\psi(\vec{r},t)=\frac{1}{(2\pi)^{3/2}}\iiint_{-\infty}^{+\infty} g(k)e^{i\vec{k}\cdot\vec{r}}e^{-i\omega t}d^3k$ which is a **superposition of many plane waves**, is called a **wave packet (in EM we often use the term pulse)**.

• Now, consider the case of **one-dimensional propagation**. Suppose that our wave packet is **propagating parallel to x-axis**, then

$$\psi(x,t)=\frac{1}{\sqrt{2\pi}}\int g(k)e^{ikx-i\omega t}dk$$

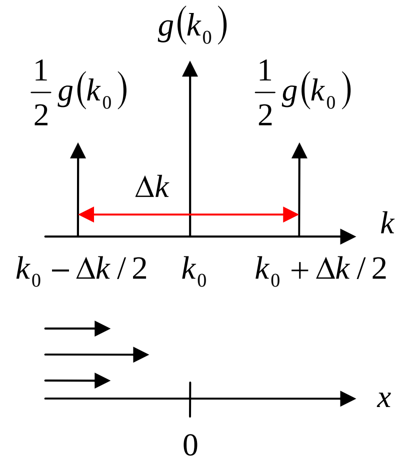
• At a **given time** (say $t=0$), $\psi(x,0)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} g(k)e^{ikx}dk$. This can be used to **show**

$$g(k)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\psi(x,0)e^{-ikx}dx$$

• $g(k)$ is sometimes **called the spectrum of the wave packet** or the pulse

• From above note that $\psi(x,0)=\mathcal{F}[g(k)]$ and $g(k)=\mathcal{F}^{-1}[\psi(x,0)]$

Spatial form of a wave packet (pulse)



• To understand the idea of **superposition** and how a **wave packet comes about** as the result of superposition, consider the following **one dimensional case**. Let $\psi(x,0)$ be **composed only of three plane waves** with $k = k_0$, $k = k_0 + \Delta k/2$ and $k = k_0 - \Delta k/2$ and **all** $\vec{k}$ **vectors pointing along x-axis**

• Then

$$\psi(x,0) = \left[\frac{1}{2}g(k_0)e^{i\left(k_0 - \frac{\Delta k}{2}\right)x} + \frac{1}{2}g(k_0)e^{i\left(k_0 + \frac{\Delta k}{2}\right)x} + g(k_0)e^{ik_0 x}\right]\frac{1}{\sqrt{2\pi}}$$

• Since the **spectrum is discrete the integration has been replaced by a summation**,

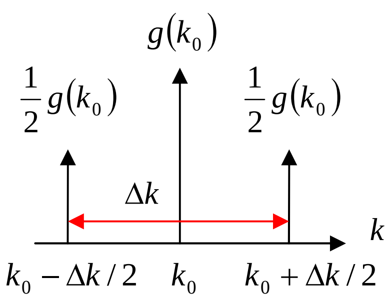
i.e. $\psi(x,0) = \frac{1}{\sqrt{2\pi}}\int g(k)\,e^{ikx}dk \Rightarrow \psi(x,0) = \frac{1}{\sqrt{2\pi}}\sum_{l=1}^{N} g(k_l)\,e^{ik_l x}$

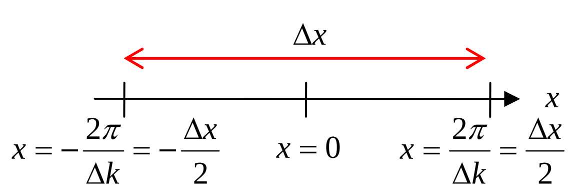
• The $\psi(x,0)$ above can be **further simplified** to

$$\psi(x,0) = \frac{g(k_0)}{\sqrt{2\pi}}\left[1 + \frac{e^{i\frac{\Delta k}{2}x}}{2} + \frac{e^{-i\frac{\Delta k}{2}x}}{2}\right]e^{ik_0 x} \Rightarrow$$

$$\psi(x,0) = \frac{g(k_0)}{\sqrt{2\pi}}e^{ik_0 x}\left[1 + \cos\left(\frac{\Delta k\ x}{2}\right)\right]$$

• We want to **find** x **for which** $\psi(x,0) = 0 \Rightarrow \frac{\Delta kx}{2} = \pi$ or $\frac{\Delta kx}{2} = -\pi \Rightarrow x = 2\pi/\Delta k$ or $x = -2\pi/\Delta k$

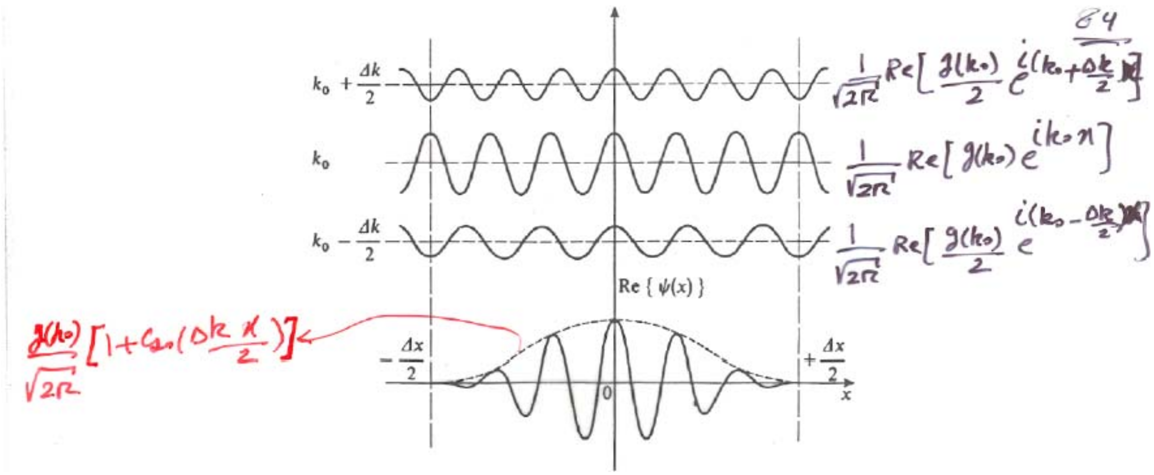




• Note that $\frac{\Delta k\ x}{2} = \pi$ **can also be written as** $\frac{\Delta k}{2}\frac{\Delta x}{2} = \pi \Rightarrow \Delta k\ \Delta x = 4\pi$

• This means that **for this example** as $\Delta k \downarrow$, $\Delta x \uparrow$ or as $\Delta k \uparrow$, $\Delta x \downarrow$, since **the product must be conserved**

• $\psi(x,0)$ given above **is a periodic function** and has **series of maximum and minimum**. This is because **we only considered superposition of 3 plane waves. As the number of plane waves is increased, there will be only one maximum point.**



$$\Psi(x,0) = \frac{g(k_0)}{\sqrt{2\pi}} e^{ik_0 x}\left[1+\cos\left(\frac{\Delta k\ x}{2}\right)\right]$$

• We observe that the **waves constructively interfere at** $x = 0$ and **we have a peak there.** The position at which **maximum of the wave packet occurs is** $x_M(t=0) = x = 0$

• **Waves destructively interfere at** $x = \frac{\Delta x}{2}$ **and we have a minimum**

• We say the **plane waves are in phase at** $x = 0$ **and out of phase at** $x = \frac{\Delta x}{2}$

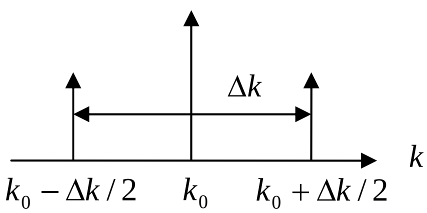
• For **electrons** $|\psi(x,0)| = \frac{g(k_0)}{\sqrt{2\pi}}\left[1+\cos\left(\frac{\Delta k\ x}{2}\right)\right]$ gives the **probability of finding the particle at some interval.**

Position of the centre (maximum) point of a wave packet

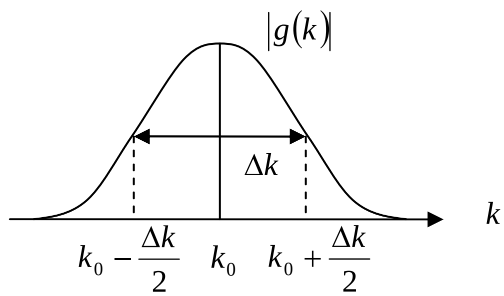
• In the **previous** example we **consider a discrete spectrum** $\left(k_0, k_0+\frac{\Delta k}{2}, k_0-\frac{\Delta k}{2}\right)$ and observed that **a maximum for** $\psi(x,0)$ **occurred when plane waves interfere constructively**. Now, let us return to the **more general case** of

$\psi(x,0)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} g(k)e^{ikx}dk$ **with** (1)

$g(k)=|g(k)|e^{i\alpha(k)}$ (2)



Discrete



Continuous

• **Assume** $\alpha(k)$ **varies smoothly between** $\left[k_0-\frac{\Delta k}{2}, k_0+\frac{\Delta k}{2}\right]$ then

$$\alpha(k)=\alpha(k_0)+(k-k_0)\left.\frac{d\alpha}{dk}\right|_{k=k_0}+\cdots$$

• Let $x_0=-\left.\frac{d\alpha}{dk}\right|_{k=k_0}$ hence $\alpha(k)=\alpha(k_0)-(k-k_0)\,x_0$ **substitute this and (2) in (1). We have**

$$\psi(x,0)=\frac{1}{\sqrt{2\pi}}\int |g(k_0)|e^{i[\alpha(k_0)-(k-k_0)x_0]}\,e^{ikx}dk$$

• **We will add and subtract** $ik_0\,x$ **to the phase** so

$$\psi(x,0)=\frac{1}{\sqrt{2\pi}}\int |g(k)|e^{i[\alpha(k_0)-kx_0+k_0x_0+kx+k_0x-k_0x]}dk$$ which then can be written as

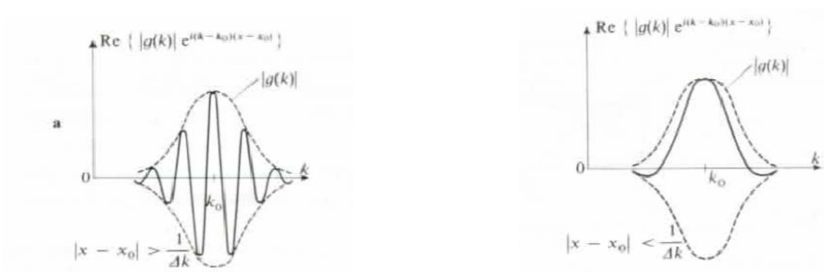
$$\psi(x,0)=\frac{e^{i[\alpha(k_0)+k_0x]}}{\sqrt{2\pi}}\int |g(k)|e^{i[x_0(k_0-k)-x(k_0-k)]}dk$$ or

$$\psi(x,0)=\frac{e^{i[\alpha(k_0)+k_0x]}}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}|g(k)|e^{i[(k-k_0)(x-x_0)]}dk, \qquad (1)$$

where we recall $x_0=-\left.\frac{d\alpha}{dk}\right|_{k=k_0}$ and $g(k)=|g(k)|e^{i\alpha(k)}$

• If $|x-x_0|>>\frac{1}{|k-k_0|}==\frac{1}{\Delta k}$ then $e^{i(k-k_0)(x-x_0)}$ **oscillates very rapidly such that the integration in (1) produces small number** $\Rightarrow \psi(x,0)$ for this case **is small**

• If $|x-x_0|<<\frac{1}{|k-k_0|}=\frac{1}{\Delta k}$ then $|g(k)|e^{i(k-k_0)(x-x_0)}$ **oscillates slowly and the result of integration is not negligible** $\Rightarrow \psi(x,0)$ **is measurable.**



• Above discussion can be summarized as the following: **when** x **is far away from** $x_0=-\left.\frac{d\alpha}{dk}\right|_{k=k_0}$ **the integrand** $\left(e^{i(k-k_0)(x-x_0)}\right)$ **oscillates rapidly** (plane waves **interfere destructively**) and $\psi(x,0)$ **is practically zero**. **When** $x\approx x_0=-\left.\frac{d\alpha}{dk}\right|_{k=k_0}$ **the integrand varies slowly** (plane waves **interfere constructively**) and $\psi(x,0)$ **is large**. This means that the **maximum point of the wave packet (centre of the wave packet) occurs at position** x_M **given by**:

$$x_M(t=0)=x_0=-\frac{d\alpha}{dk}$$

• For $\psi(x,0)\approx\frac{e^{i(\alpha(k_0)+k_0x)}}{\sqrt{2\pi}}\int|g(k)|e^{i(k-k_0)(x-x_0)}dk$ when x **deviates from** x_0, $|\psi(x,0)|$ decreases. **This decrease can be appreciable when**
$(k-k_0)(x-x_0)\geq 1\Rightarrow \Delta k\ \Delta x\geq 1$.
This inequality should remind us of the **Heisenberg uncertainty**.

Stationary Phase Condition

• The **position of the maximum** point can also be **obtained by applying the stationary phase condition** to

$$\psi(x,0)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} g(k)e^{ikx}dk=\int |g(k)|e^{i(kx+\alpha(k))}dk$$

• Over the interval $\left[k_0-\frac{\Delta k}{2},k_0+\frac{\Delta k}{2}\right]$, $\psi(x,0)$ **is large when the phase of the integrand varies slowly with respect to** k (evaluated at centre wave number k_0), i.e.

$$\frac{d}{dk}\left(kx+\alpha(k)\right)\Big|_{k=k_0}=0\Rightarrow x+\frac{d\alpha(k)}{dk}\Big|_{k=k_0}=0\Rightarrow$$ **(Stationary Phase Condition, SPC)**

$$x_M(t=0)=-\frac{d\alpha(k)}{dk}\Big|_{k=k_0}=x_0.$$

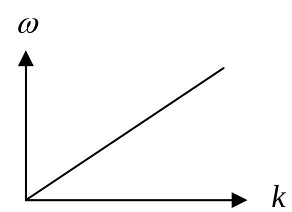
x_M **is the position at which the wave packet has a maximum**

Time Evolution of free wave packet

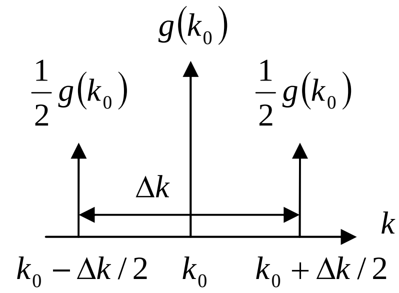
• In our previous discussions **we have shown that a plane wave** $e^{i(kx-\omega t)}$ **moves with phase velocity** $v_p=\frac{\omega}{k}$, obtained from the condition $kx-\omega t=$constant, i.e.

$$\Rightarrow k\ x-\omega\ t=\text{constant}\Rightarrow\frac{dx}{dt}=\frac{\omega}{k}=v_p$$

• If ω **is a linear function of** k (or the other way around), **as shown in the figure**, the **wave packet which is a linear superposition of many plane waves** will also **propagate with the velocity** v_p **without distortion or broadening**. Hence the **wave packet maximum (** x_M **) also travels with velocity** v_p.



• If the **relation between** ω **and** k **deviates from what is shown in the figure**, then different plane waves propagate at **different phase velocities**. This means that the **wave packet** (and hence the position of its maximum) **does not propagate at** $\frac{\omega_0}{k_0}$ **(the phase velocity of the main component** or average phase velocity) **but at some other velocity, we will call group velocity**.



• In order to **see the time evolution of the wave packet** let us start with **our previous example of three plane waves** propagating along the x-axis with k_0, $k_0 + \frac{\Delta k}{2}$ and $k_0 - \frac{\Delta k}{2}$

• In the **previous example we studied** ψ **at a particular time** $t = 0$, i.e. $\psi(x, t = 0)$, **here we study** $\psi(x,t)$

$$\psi(x,t) = \frac{g(k_0)}{\sqrt{2\pi}} e^{ik_0 x - i\omega_0 t} + \frac{g(k_0)}{2\sqrt{2\pi}} e^{i\left(k_0 + \frac{\Delta k}{2}\right)x} e^{-i\left(\omega_0 + \frac{\Delta\omega}{2}\right)t} + \frac{g(k_0)}{\sqrt{2\pi}} e^{i\left(k_0 - \frac{\Delta k}{2}\right)x} e^{-i\left(\omega_0 - \frac{\Delta\omega}{2}\right)t}$$

$$= \frac{g(k_0)}{\sqrt{2\pi}} e^{i(k_0 x - \omega_0 t)} \left[1 + \frac{1}{2} e^{i\left(\frac{\Delta k}{2}x - \frac{\Delta\omega}{2}t\right)} + e^{-i\left(\frac{\Delta k}{2}x - \frac{\Delta\omega}{2}t\right)}\right] \Rightarrow$$

$$\psi(x,t) = \frac{g(k_0)}{\sqrt{2\pi}} e^{i(k_0 x - \omega_0 t)} \left[1 + \cos\left(\frac{\Delta k}{2}x - \frac{\Delta\omega}{2}t\right)\right]$$

• $\psi(x,t)$ **maximum occur at** $\frac{\Delta k}{2}x - \frac{\Delta\omega}{2}t = 0 \Rightarrow x_M(t) = \frac{\Delta\omega}{\Delta k}t$

• Note that the **position of maximum is not simply** $\frac{\omega_0}{k_0}t$ **but** $\frac{\Delta\omega}{\Delta k}t$.

• Let us look at the **location for the maximum of** $\psi(x,t)$ from the point of view of **constructive and destructive interferences** among the plane waves.

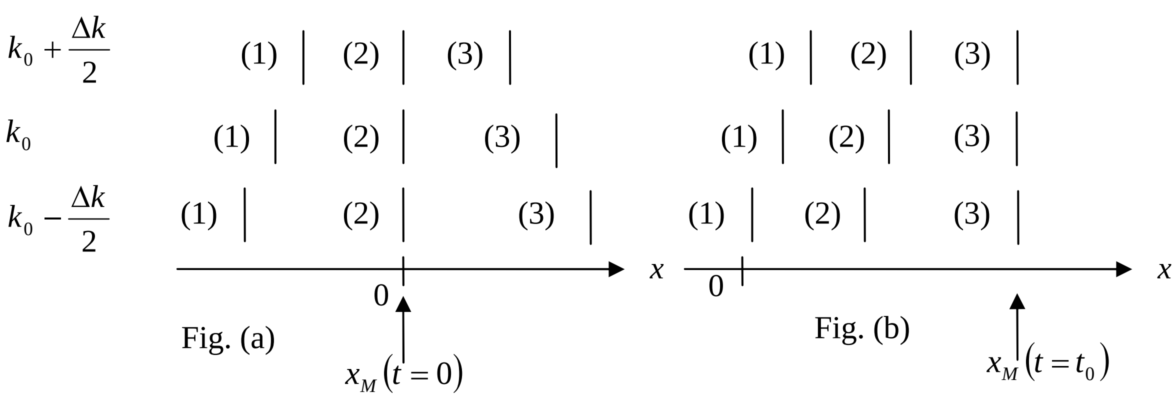


Fig. (a) Fig. (b)

• Figure shows the **position of the maximums for the three-plane-waves at** $t = 0$ **(Fig. a) and at** $t = t_0$ **(Fig. b)**

• **At time** $t = 0$, the **plane waves maximums** (designated by the digit 2) **are all at position** $x = 0$. In other words, for $t = 0$ **waves interference constructively at** $x = 0$ and we have **the wave packet maximum also at** $x = 0$.

• **At time** $t=t_0$, wave $k_0+\frac{\Delta k}{2}$ has caught up with wave k_0 which in turn has caught up with wave $k_0-\frac{\Delta k}{2}$. **The plane waves maximums (designated by the digit 3) now all line up at** $x=x_M(t_0)=\frac{\Delta\omega}{\Delta k}t_0$, which is clearly **not the same as** $\frac{\omega_0}{k_0}t_0$. Recall that $x_M(t_0)$ **designates the location of the wave packet maximum at time** t_0.

Stationary phase condition

• In our **previous discussion we showed that for** $\psi(x,0)$ given by $\psi(x,0)=\int g(k)e^{ikx}dk=\int|g(k)|e^{i(kx+\alpha(k))}dk$ where $g(k)=|g(k)|e^{i\alpha(k)}$ the **position of maximum at** $t=0$ **was given by** $x_M(t=0)=-\left.\frac{d\alpha}{dk}\right|_{k=k_0}=x_0$

• The **position of the maximum for** $\psi(x,t)=\frac{1}{\sqrt{2\pi}}\int g(k)e^{ikx-i\omega t}dk$ **at any time can be obtained with similar reasoning if we note that** $\alpha(k)\to\alpha(k)-\omega(k)t$**, then**

$$x_M(t)=-\frac{d}{dk}[\alpha(k)-\omega(k)t]_{k=k_0}=\left.\frac{d\omega(k)}{dk}\right|_{k=k_0}t-\left.\frac{d\alpha(k)}{dk}\right|_{k=k_0}\Rightarrow$$

$x_M(t)=\left.\frac{d\omega(k)}{dk}\right|_{k_0}t+x_0$. Compare[1] this with $x=v\,t+x_1\Rightarrow$

We can define our new velocity (the group velocity) as $\left.\frac{d\omega(k)}{dk}\right|_{k=k_0}=v_g$

Summary of Velocities for free electrons and photons

• **For free electrons** $\omega=\frac{\hbar k^2}{2m}\Rightarrow v_p=\frac{\omega}{k}=\frac{\hbar k}{2m}\equiv\frac{P}{2m}$

• **For free photons** $\omega=\frac{ck}{n(k)}\Rightarrow v_p=\frac{\omega}{k}=\frac{c}{n(k)}$

• **For free electrons** $v_g=\frac{d\omega}{dk}=\frac{d}{dk}\left(\frac{\hbar k^2}{2m}\right)=\frac{\hbar k}{m}=2v_p\equiv\frac{P}{m}$

---

[1] Or note that $dx_M(t)/dt=d\omega(k)/dk=v_g$

• **For free photons** $v_g = \frac{d\omega}{dk} = c\left[\frac{n(k) - \frac{dn(k)}{dk}k}{n^2(k)}\right]$ or $v_g = \frac{c}{n(\omega) + \omega\frac{dn(\omega)}{d\omega}} = \frac{c}{n_g(\omega)}$

General case of 3D wave packet

• In the beginning of our discussion, considering the case of free particles, we stated the principle of spectrum decomposition for propagation in three dimensions as

$$\psi(\vec{r},t) = \frac{1}{(2\pi)^{3/2}}\iiint g(k)e^{i\vec{k}\cdot\vec{r} - i\omega(\vec{k})t}d^3k\ , \qquad (1)$$

• However, (1) may not always hold for non-free particles. For example, for an arbitrary potential [$V(\vec{r}) \neq$ constant $\Rightarrow \vec{F} \neq 0$] (1) does not necessarily hold, and in addition the potential (or index) may also depend on time.

• Eq. (1) then needs to be modified according to

$$\Psi(\vec{r},t) = \frac{1}{(2\pi)^{2/3}}\iiint g(\vec{k},t)\,e^{i\vec{k}\cdot\vec{r}}d^3k$$

• The arbitrary time dependent potential is now introduced via $g(\vec{k},t)$

• Also a priori, there is no reason to believe $g(\vec{k},t)$ can be expressed in terms of products, i.e. $g(\vec{k},t) \neq g_1(k_x,t)\,g_2(k_y,t)\,g_3(k_z,t)$

• We make the following hypothesis: $\left|g(\vec{k},t)\right|$ at a given time t, has a pronounced peak at $\vec{k} = \vec{k}_0$ and is negligible when the tip of $\vec{k}$ leaves the domain D_k centered at $\vec{k}_0$

• We write $g(\vec{k},t) = \left|g(\vec{k},t)\right|e^{i\alpha(\vec{k},t)}$, then

$$\psi(\vec{r},t) = \frac{1}{(2\pi)^{3/2}}\iiint g(\vec{k},t)e^{i\vec{k}\cdot\vec{r}}d^3k = \frac{1}{(2\pi)^{3/2}}\iiint\left|g(\vec{k},t)\right|e^{i\xi(\vec{k},r,t)}d^3k\ ,$$

where $\xi(\vec{k},\vec{r},t) = \alpha(\vec{k},t) + \vec{k}\cdot\vec{r} = \alpha(\vec{k},t) + k_x x + k_y y + k_z z$ and $\xi(\vec{k},\vec{r},t)$ is the phase of the wave

• Note that $\psi(\vec{r},t) = \frac{1}{(2\pi)^{3/2}}\iiint\left|g(\vec{k},t)\right|\,e^{i\xi(\vec{k},\vec{r},t)}d^3k$ has a maximum when all the waves for which $\vec{k} \in D_k$ are in phase $\equiv$ when $\xi(\vec{k},\vec{r},t)$ varies little in the domain D_k

• We now examined the variation of $\xi(\vec{k},\vec{r},t)$ about $\vec{k}_0$. We define the difference vector $\delta\vec{k}$ as $\delta\vec{k} = \vec{k} - \vec{k}_0$ where

$$\delta\vec{k} = \vec{k} - \vec{k}_0 = (k_x - k_{0x})\hat{a}_x + (k_y - k_{0y})\hat{a}_y + (k_z - k_{0z})\hat{a}_z$$
$$= \delta k_x \hat{a}_x + \delta k_y \hat{a}_y + \delta k_z \hat{a}_z$$

• Then the variation of the phase [$\xi(\vec{k},\vec{r},t)$], to the first order is given by

$$\delta\xi(\vec{k},\vec{r},t) = \delta k_x \left[\frac{\partial}{\partial k_x}\xi(\vec{k},\vec{r},t)\right]_{k=k_0} + \delta k_y \left[\frac{\partial}{\partial k_y}\xi(\vec{k},\vec{r},t)\right]_{k=k_0} + \delta k_z \left[\frac{\partial}{\partial k_z}\xi(\vec{k},\vec{r},t)\right]_{k=k_0} \quad (1)$$

• (1) can be written as

$$\delta\xi(\vec{k},\vec{r},t) = \delta\vec{k}\cdot\left[\nabla_k \xi(\vec{k},\vec{r},t)\right]_{k=k_0} = \delta\vec{k}\cdot\nabla_k\left[\alpha(\vec{k},t) + \vec{k}\cdot\vec{r}\right]_{k=k_0} = \delta\vec{k}\cdot\left[\vec{r} + \nabla_k\alpha(\vec{k},t)\right]_{k=k_0} \quad (2)$$

where we have used $\xi(\vec{k},\vec{r},t) = \alpha(\vec{k},t) + \vec{k}\cdot\vec{r}$

• To minimize $\delta\xi$, we set the bracket equal to zero

$$\vec{r} + \nabla_k\alpha(\vec{k},t)\Big|_{k=k_0} = 0 \Rightarrow \vec{r} = \vec{r}_M(t) = -\nabla_k\alpha(\vec{k},t)\Big|_{k=k_0}$$

$\vec{r}_M(t)$ is the position of the wave packet maximum

• Let us define $\delta\vec{r} = \vec{r} - \vec{r}_M$ in the D_r domain

$\delta\vec{r} \equiv$ Variation of position vector $(\vec{r})$ in the domain D_r, from the position of the wave packet maximum ($\vec{r}_M$)

$\delta\vec{k} \equiv$ Variation of $\vec{k}$ vector in the D_k domain from $\vec{k}_0$, where $\vec{k}_0$ is the value for which $\left|g(\vec{k},t)\right|$ is maximum

• Then (2) can be written as $\delta\xi(\vec{k},\vec{r},t) = \delta\vec{k}\cdot\left[\vec{r} + \nabla_k\alpha(\vec{k},t)\right]_{k_0} = \delta\vec{k}\cdot\left[\vec{r} - \vec{r}_M(t)\right] = \delta\vec{k}\cdot\delta\vec{r}$

Remark: When variation of the phase $\delta\xi(\vec{k},\vec{r},t)$ is ≥ 1, the wave function $\left|\psi(\vec{r},t)\right|$ is much smaller than $\max\left[\left|\psi(\vec{r},t)\right|\right]$, but

$$\delta\xi(\vec{k},\vec{r},t) \geq 1 \equiv \delta\vec{k}\cdot\delta\vec{r} \geq 1 \Leftrightarrow \begin{cases} \Delta k_x \cdot \Delta x \geq 1 \\ \Delta k_y \cdot \Delta y \geq 1 \\ \Delta k_z \cdot \Delta z \geq 1 \end{cases}$$

•From the expression for the position of the maximum we had:

$$\vec{r}_M(t) = -\nabla_k\alpha(\vec{k},t)\Big|_{k=k_0}.$$

The velocity of the maximum point is given by the time derivative of $\vec{r}_M(t)$. This is the velocity by which the wave packet (a grouping of the plane waves) propagates. This velocity if called the group velocity

$$\vec{V}_g = \frac{d}{dt}\vec{r}_M(t) = -\frac{d}{dt}\left[\nabla_k \alpha(\vec{k},t)\right]_{\vec{k}=\vec{k}_0}$$

• In the case of a free wave packet that necessary does not satisfy $g(k) = g(k_1)g(k_2)g(k_3)$ we can write:
$\alpha(\vec{k},t) = \alpha(\vec{k},0) - \omega(\vec{k})t$, such that

$$\psi(\vec{r},t) = \frac{1}{(2\pi)^{3/2}}\iiint g(\vec{k},t)e^{i\vec{k}\cdot\vec{r}}d^3k = \frac{1}{(2\pi)^{3/2}}\iiint \left|g(\vec{k},t)\right| e^{i\alpha(\vec{k},t)}e^{i\vec{k}\cdot\vec{r}}d^3k$$

$$= \frac{1}{(2\pi)^{3/2}}\iiint \left|g(\vec{k},t)\right| e^{i\alpha(k)}e^{i\left[\vec{k}\cdot\vec{r}-\omega(\vec{k})t\right]}d^3k$$

• Back to the expression for $\vec{V}_g = \frac{d}{dt}\vec{r}_M(t) = -\frac{d}{dt}\left[\nabla_k \alpha(\vec{k},t)\right]_{\vec{k}=\vec{k}_0}$, with
$\alpha(\vec{k},t) = \alpha(\vec{k},0) - \omega(\vec{k})t$, we now have

$$\vec{V}_g = -\frac{d}{dt}\left[\nabla_k \alpha(\vec{k},t)\right]_{\vec{k}=\vec{k}_0} = -\frac{d}{dt}\left\{\nabla_k\left[\alpha(\vec{k}) - \omega(\vec{k})t\right]\right\}_{\vec{k}=\vec{k}_0} = -\frac{d}{dt}\left[\nabla_k \alpha(k)\right]_{\vec{k}=\vec{k}_0} + \frac{d}{dt}\left[\nabla_k \omega(\vec{k})t\right]_{\vec{k}=\vec{k}_0}$$

$$= 0 + \left[\nabla_k \omega(\vec{k})t\right]_{\vec{k}=\vec{k}_0} \Rightarrow \vec{V}_g = \nabla_k \omega(\vec{k})\Big|_{\vec{k}=\vec{k}_0}$$

• If the dispersion relation $\omega(\vec{k})$ is known, the group velocity can be obtained from
$\vec{V}_g = \nabla_k \omega(\vec{k})\Big|_{\vec{k}=\vec{k}_0}$

• Surfaces $\omega(\vec{k})\Big|_{\vec{k}=\vec{k}_0} = \omega(k_x,k_y,k_z)\Big|_{k_x=k_{x0},\,k_y=k_{y0},\,k_z=k_{z0}}$ are the surfaces of equi-frequency

• In one-dimension for which $k_x = k_y = 0$, and $k_z \equiv k$ the group velocity is given by
$\vec{V}_g = \nabla_k \omega(\vec{k}) = \frac{\partial\omega}{\partial k_z}(k_z)\hat{a}_z = \frac{d\omega}{dk}\hat{a}_z$, the well known result in 1D

• For electrons $\omega(\vec{k}) = \frac{\hbar k^2}{2m} = \frac{\hbar}{2m}\left[k_x^{\,2} + k_y^{\,2} + k_z^{\,2}\right]$

$$\vec{V}_g = \nabla_k \omega(\vec{k})\Big|_{k_0} = \frac{\hbar}{m}\left[k_x\hat{a}_x + k_y\hat{a}_y + k_z\hat{a}_z\right]\Big|_{\vec{k}=\vec{k}_0} = \frac{\hbar\vec{k}}{m}\Big|_{\vec{k}=\vec{k}_0} = \frac{\hbar\vec{k}_0}{m}$$

• For photons $\omega(\vec{k}) = \frac{ck}{n(\vec{k})}$

$$\vec{V}_g = \nabla_k \omega(\vec{k})\Big|_{\vec{k}_0} = \nabla_k\left[\frac{ck}{n(k)}\right]_{\vec{k}=\vec{k}_0} = c\left[\frac{n(\vec{k})\nabla_k k - k\nabla_k n(\vec{k})}{n^2(\vec{k})}\right]_{\vec{k}=\vec{k}_0} = c\left[\frac{\hat{k}\,n(\vec{k}) - k\nabla_k n(\vec{k})}{n^2(\vec{k})}\right]_{\vec{k}=\vec{k}_0}$$

• For dispersion-less medium $n(\vec{k}) = \text{constant} \Rightarrow$

$$\nabla_k n(\vec{k}) = 0 \Rightarrow \vec{V}_g = \frac{c}{n(\vec{k})}\hat{k} = \vec{V}_p \Rightarrow \vec{V}_g \parallel \vec{V}_p \parallel \vec{k}$$

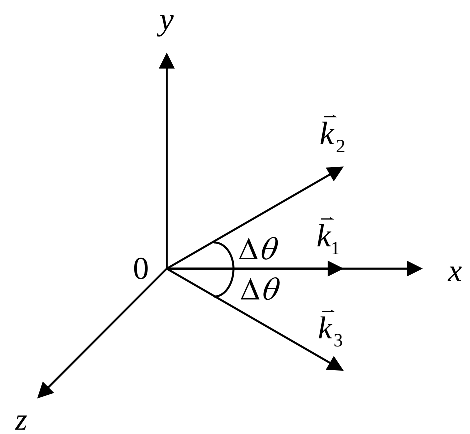
• Note when dispersion is present, $\nabla_k n(\vec{k}) \neq 0$, then $\vec{V}_g$ is not necessarily parallel to $\hat{k}$

• In the above we developed the general concept of wave propagation. As an example, let us see what happens to waves propagating in two-dimensions.

Two dimensional wave propagation

• We assume our wave packet is composed of 3 plane waves, with propagation vectors $\vec{k}_1, \vec{k}_2,$ and $\vec{k}_3$.

• We also assume that all the three plane waves are in the xy plane and $|\vec{k}_1| = |\vec{k}_2| = |\vec{k}_3| = k$ (see figure).



• From the figure it is easy to see that

$$\vec{k}_1 = k_{1x}\hat{a}_x$$

$$\vec{k}_2 = k_{2x}\hat{a}_x + k_{2y}\hat{a}_y$$

$$\vec{k}_3 = k_{3x}\hat{a}_x + k_{3y}\hat{a}_y$$

• For the three plane waves that are nearly co-linear, i.e. $\Delta\theta$ is small, we have

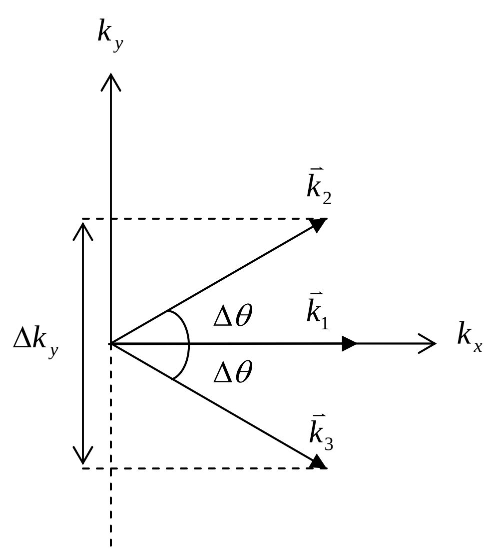
$$k_{1x} = |\vec{k}_1| = k$$

$$k_{2x} = |\vec{k}_2|\cos\Delta\theta \approx k_2 = k$$

$$k_{2y} = |\vec{k}_2|\sin\Delta\theta \approx |\vec{k}_2|\Delta\theta = k\Delta\theta$$

$$k_{3x} = |\vec{k}_3|\cos\Delta\theta \approx |\vec{k}_3| = k$$

$$k_{3y} = -|\vec{k}_3|\sin\Delta\theta \approx -|\vec{k}_3|\Delta\theta = -k\Delta\theta$$



• Then

$$\vec{k}_1 = k\hat{a}_x$$

$$\vec{k}_2 = k\hat{a}_x + k\Delta\theta\,\hat{a}_y$$

$$\vec{k}_3 = k\hat{a}_x - k\Delta\theta\,\hat{a}_y$$

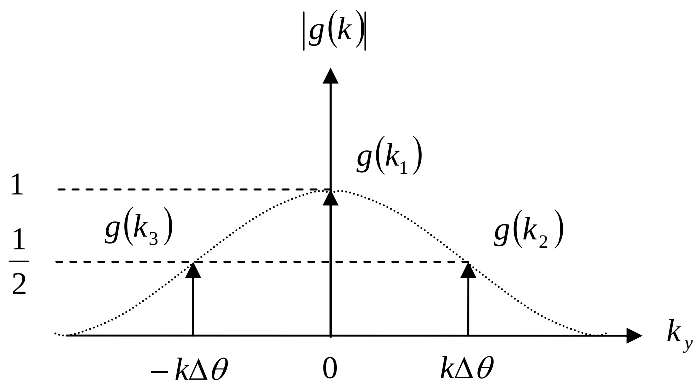
• Note that $\vec{k}_1, \vec{k}_2$, and $\vec{k}_3$ are $\perp$ to the z -axis and from the figure $\Delta k_y = 2k_{2y} = 2k\ \Delta\theta$

• Recall that from the principle of spectrum decomposition for continuous and discrete components we had

$\psi(\vec{r}, t=0) = \iiint g(\vec{k}) e^{i\,\vec{k}\cdot\vec{r}} d^3k$ for a continuous distribution of k

$\psi(\vec{r}, t=0) = \sum_{l=1} g(k_l) e^{i\,\vec{k}_l\cdot\vec{r}}$ for a discreet distribution of k

• Here $|g(k)| = |g(k_y)|$ and we suppose (similar to our previous consideration) that it has a shape as shown in the figure



$g(k_2) = \frac{1}{2} g(k_1)$

$g(k_3) = \frac{1}{2} g(k_1)$

• The wave packet is then given by

$$\psi(x, y, t=0) = \sum_{l=1}^{3} g(k_l) e^{i\vec{k}_l\cdot\vec{r}} = g(\vec{k}_1) e^{i\vec{k}_1\cdot\vec{r}} + g(\vec{k}_2) e^{i\vec{k}_2\cdot\vec{r}} + g(k_3) e^{i\vec{k}_3\cdot\vec{r}} \Rightarrow$$

$$\psi(x, y, t=0) = g(\vec{k}_1)\left[e^{i\vec{k}_1\cdot\vec{r}} + \frac{1}{2} e^{i\vec{k}_2\cdot\vec{r}} + \frac{1}{2} e^{i\vec{k}_3\cdot\vec{r}}\right].$$

Using $\vec{k}_1 = k\hat{a}_x$, $\vec{k}_2 = k\hat{a}_x + k\Delta\theta\ \hat{a}_y$, and $\vec{k}_3 = k\hat{a}_x - k\Delta\theta\ \hat{a}_y$ we have

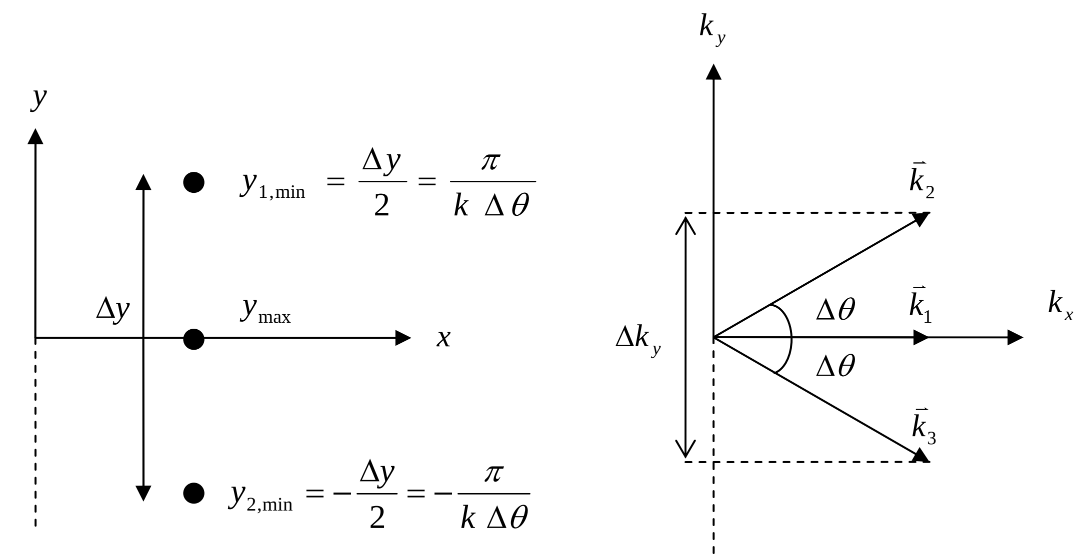
$$\psi(x, y, t=0) = g(\vec{k})\left[e^{ik\,x} + \frac{1}{2} e^{i[k\,x + k\,\Delta\theta\,y]} + \frac{1}{2} e^{i[k\,x - k\,\Delta\theta\,y]}\right] \Rightarrow$$

$$\psi(x, y, t=0) = g(\vec{k}) e^{ikx}[1 + \cos(k\ \Delta\theta\ y)]$$

• $\psi(x, y)$ is maximum when $y = 0$, i.e. $\cos(k\Delta\theta y) = 1$, therefore the maxima for $\psi(x, y)$ lies on x-axis. As we move away from x-axis (y increases) $\psi(x, y)$ decreases.

• let us find the values of y for which the wave function is zero
$\psi(x, y) \to 0 \Rightarrow \cos(k\ \Delta\theta\ y) = -1 \Rightarrow k\ \Delta\theta\ y_{\min,1,2} = \pm\pi \Rightarrow$
$y_{1,\min} = \dfrac{\pi}{k\ \Delta\theta}$, and $y_{2,\min} = \dfrac{-\pi}{k\ \Delta\theta}$. We define Δy (the spread in the y-direction) as

$$\Delta y = y_{1,\min} - y_{2,\min} = \frac{2\pi}{k\ \Delta\theta}$$

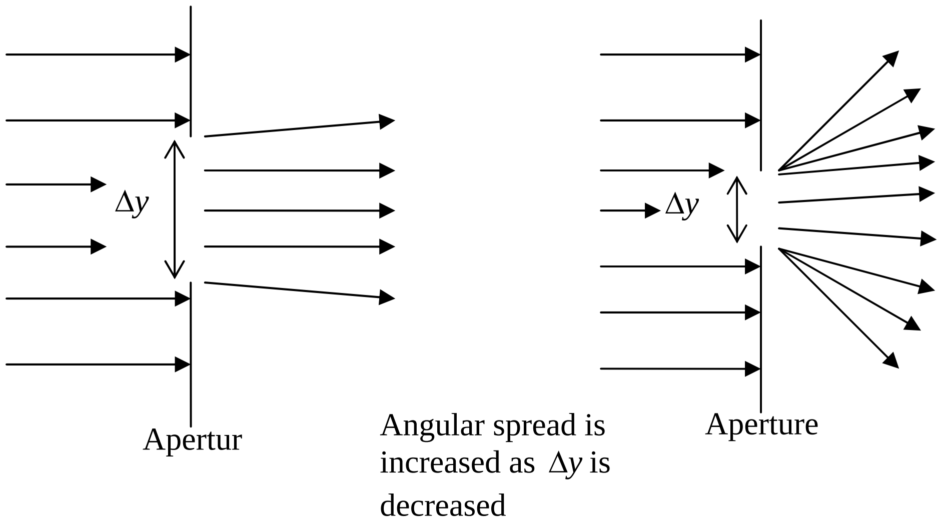


$$\Delta y = \frac{2\pi}{k\Delta\theta}$$

$$\Delta k_y = 2\Delta\theta\ k$$

• From above analysis we see that for our wave packet, composed of three plane waves $\vec{k}_1, \vec{k}_2$, and $\vec{k}_3$, the spread in y is given by $\Delta y = \dfrac{2\pi}{k\ \Delta\theta}$ and the spread in k_y is given by $\Delta k_y = 2\Delta\theta\ k \Rightarrow \Delta y\ \Delta k_y = 4\pi$ (compare this to our results for one-dimensional propagation)

• From
$\Delta y = \dfrac{2\pi}{k\ \Delta\theta} \Rightarrow \Delta y\ \Delta\theta = \dfrac{2\pi}{k} = \lambda$
This relation is well-known in diffraction theory. It says that if we try to confine the lateral extension of a beam, its angular spread must increase or visa versa.


Δy
Apertur
Angular spread is increased as Δy is decreased
Δy
Aperture