Optical electron Analogies

• We begin our discussion by revisiting the Schrödinger and Helmholtz's wave equations in their stationary state form. Recall that for the Schrödinger equation we had

$$\underbrace{-\frac{\hbar^2}{2m}\nabla^2\phi(\vec{r})}_{\substack{\text{measure of kinetic} \\ \text{energy (K.E.)}}} + \underbrace{V(\vec{r})\phi(\vec{r})}_{\substack{\text{measure of} \\ \text{potential energy} \\ \text{(P.E.)}}} = \underbrace{E\phi(\vec{r})}_{\substack{\text{measure of} \\ \text{total energy} \\ (E_T)}},$$

and that in the stationary state form the probability density $|\Psi(\vec{r},t)|^2 = |\phi(\vec{r})|^2$ is time independent and we have a well defined energy $E = \hbar\omega$.

• The $-\frac{\hbar^2}{2m}\nabla^2 + V(\vec{r}) = H$ is called the Hamiltonian. With this the Schrödinger can be written as

$$H\phi(\vec{r}) = E\phi(\vec{r}) \tag{1}$$

• For a source free simple medium Helmholtz's equation is given by

$$\nabla^2\vec{E} = -\Omega^2\mu\varepsilon\,\vec{E} \tag{2}$$

• (1) and (2) are eigenvalue equation.

• Note that ∇^2 and H are linear operator, i.e. for $\lambda_1, \lambda_2 \in \underline{C}$

$$H[\lambda_1\phi_1(\vec{r}) + \lambda_2\phi(\vec{r})] = \lambda_1 H\phi_1(\vec{r}) + \lambda_2 H\phi_2(\vec{r})$$

• $i\hbar\frac{\partial}{\partial t}\Psi(\vec{r},t) = \frac{-\hbar^2}{2m}\nabla^2\Psi(\vec{r},t) + V(\vec{r})\Psi(\vec{r},t)$ and $\nabla^2\vec{E}(\vec{r},t) - \mu\varepsilon\frac{\partial\vec{E}(\vec{r},t)}{\partial^2 t} = 0$ give the evolution of $\Psi(\vec{r},t)$ and $\vec{E}(\vec{r},t)$, whatever the state of system. $H\phi(\vec{r}) = E\phi(\vec{r})$ and $\nabla^2\vec{E} = -\Omega^2\mu\varepsilon\,\vec{E}$ give among all the possible states those that are stationary

• Comparing Schrödinger and Helmholtz equations in 1D we have

$$-\frac{\hbar^2}{2m}\frac{d^2}{dz^2}\psi(z) + V(z)\psi(z) = E\psi(z) \Rightarrow \left[\frac{d^2}{dz^2} + \frac{2m}{\hbar^2}(E - V(z))\right]\psi(z) = 0$$

$$\left[\frac{d^2}{dz^2} + \frac{\Omega^2 n^2}{c^2}\right]u(z) = 0$$ (u is any scalar component of $\vec{E}$ or $\vec{H}$)

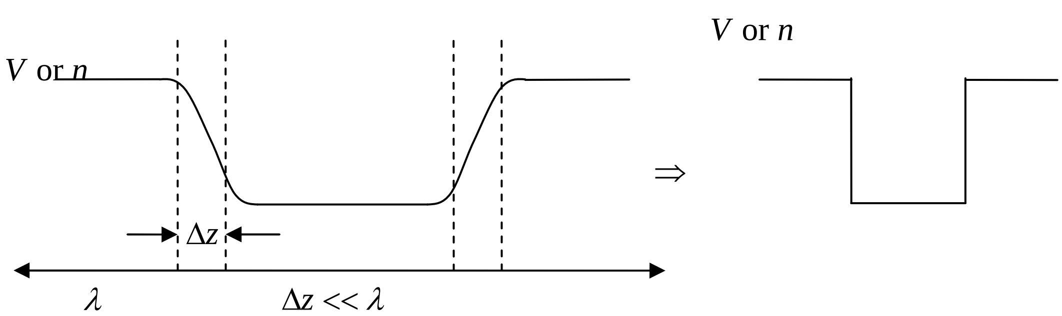
• Comparing the two equations we can make the following correspondence

$$\frac{\Omega^2 n^2}{C^2} = \frac{2m}{\hbar^2}(E - V) \Rightarrow n = \frac{\sqrt{2mc^2(E - V)}}{\hbar\Omega},$$

with $k_e = \frac{\sqrt{2m(E-V)}}{\hbar}$ representing the electron wave vector.

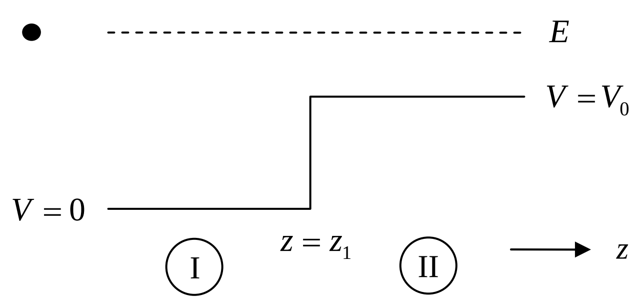
Index and Potential Profiles

• If dimensions over which potential or index of refraction vary rapidly is much smaller than the wavelength of consideration, the potential or index can be considered approximately discontinuous



Classical, Quantum Mechanical and Optical Analogies

Potential step:



Case 1: $E > V_0$

Classical Consideration:
Total energy is conserved
(total energy $= E$)

• In region I: $\text{P.E.} + \text{K.E.} = E \Rightarrow \text{K.E.} = E > 0$. This is classically allowed

• In region II: $\text{P.E.} + \text{K.E.} = E \Rightarrow \text{K.E.} = E - V_0 > 0$, since $E > V_0$. This also is classically allowed

• Our experience tells us that as the particle moves from region I to region II, it does not turn around (reflect) or crash.

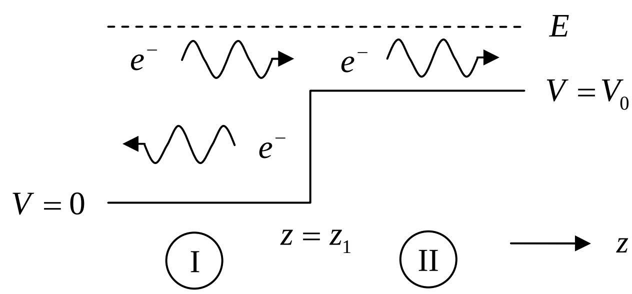
Quantum Mechanical Consideration

• In quantum mechanics even though $E > V_0$, there is a finite probability that the electron will be reflected and a finite probability that it will clear off the barrier at $z = z_1$

• In region I: $k_1 = \sqrt{\dfrac{2mE}{\hbar^2}} \in \Re$ since $E > 0$

• In region II: $k_2 = \sqrt{\dfrac{2m(E - V_0)}{\hbar^2}} \in \Re$ since $E > V_0$

• The wave functions in regions I and II are then given by $\Psi_1 \propto e^{ik_1 z} e^{-i\omega t}$, and $\Psi_2 \propto e^{ik_2 z} e^{-i\omega t}$, respectively. They are traveling waves.



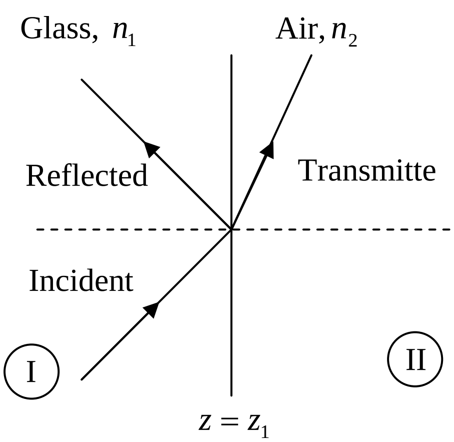
Optical Consideration

• Recall $n = \dfrac{1}{\hbar\Omega}\sqrt{2mc^2(E - V)}$

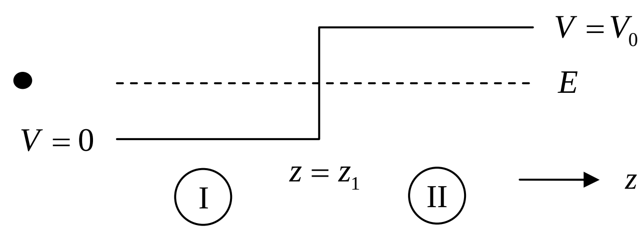
• In region I: $V = 0 \Rightarrow n_1 = \dfrac{1}{\hbar\Omega}\sqrt{2mc^2 E} \in \Re$

• In region II: $V = V_0$ and

$E > V_0 \Rightarrow n_2 = \dfrac{1}{\hbar\Omega}\sqrt{2mc^2(E - V_0)} \in \Re$



• From the above we conclude $n_2 < n_1$. The situation is depicted in the figure. Note the similarity between optical and quantum mechanical considerations. We have incident, reflected, and transmitted waves.



Case 2: $E < V_0$

Classical Consideration:

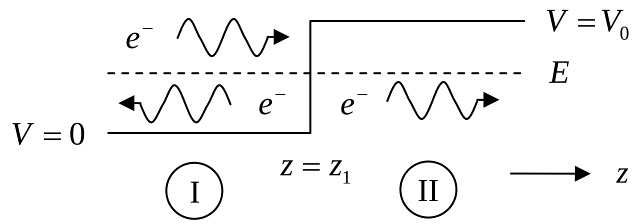
• As our particle encounters the potential barrier it will reflect.

• In Region I: $\text{K.E.} + \underbrace{\text{P.E.}}_{0} = E \Rightarrow \text{K.E.} = E > 0$. This is classically allowed.

• In Region II: $\text{K.E.} + \underbrace{\text{P.E.}}_{V_0} = E \Rightarrow \text{K.E.} = E - V_0 < 0$ since $E < V_0$. This is not classically allowed. Recall $K.E. = \frac{1}{2}mv^2$ and hence always positive.

Quantum Mechanical Consideration

• In quantum mechanics there is a finite probability for the electron to be reflected and there is a finite probability that it will cross the barrier at $z = z_1$ and can be found at some $z > z_1$



• Note that classical physics will not allow the electron to be found in $z > z_1$.

• In region I: $k_1 = \sqrt{\frac{2mE}{\hbar^2}} \in \Re$ since $E > 0$

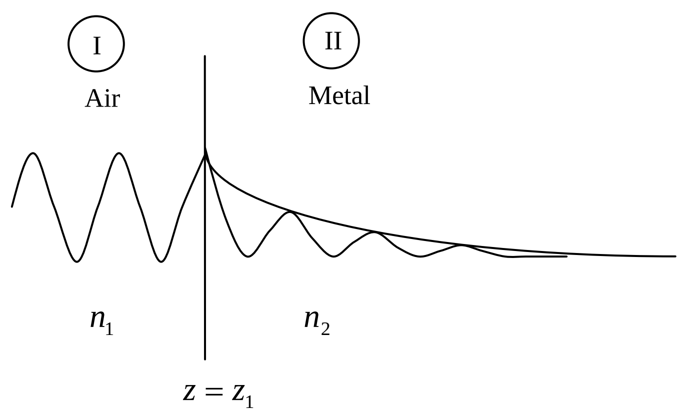
• In region II: $k_2 = \sqrt{\frac{2m(E - V_0)}{\hbar^2}} = i\sqrt{\frac{2m(V_0 - E)}{\hbar^2}} = i\alpha$ where $\alpha = \sqrt{\frac{2m(V_0 - E)}{\hbar^2}} \in \Re$.
Note that in region II $V_0 > E$.

• The wave functions in regions I and II are then given by $\Psi_1 = e^{ik_1 z} e^{-i\omega t}$ and

$$\Psi_2 \propto e^{i(i\alpha z)} e^{-i\omega t} = \underbrace{e^{-\alpha z}}_{\substack{\text{evanescent} \\ \text{wave}}} e^{-i\omega t}$$

Optical Consideration

Recall $n = \frac{1}{\hbar\Omega}\sqrt{2mc^2(E - V)} \Rightarrow$
$n_1 = \frac{1}{\hbar\Omega}\sqrt{2mc^2 E} \in \Re$ since $E > 0$ and
$n_2 = \frac{1}{\hbar\Omega}\sqrt{2mc^2(E - V_0)} = \frac{i}{\hbar\Omega}\sqrt{2mc^2(V_0 - E)}$.
n_2 is purely imaginary



• Medium I can be thought of as a good dielectric, e.g. air. Medium II can be thought of as an ideal metal

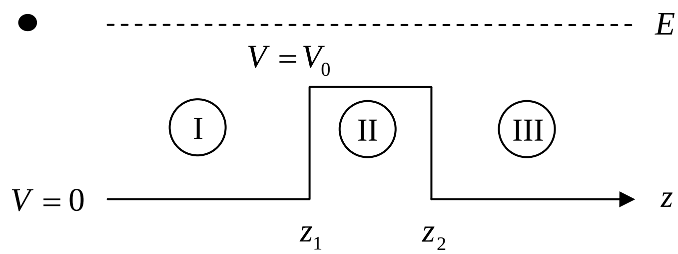
• This analogy should not be taken too far, since real metal has both real and imaginary parts of the index. Hence, we have a traveling attenuated wave and not a purely evanescent wave.

Potential Barrier

• Here we replace our previous step potential with a potential barrier (finite in extent). The effects of evanescent waves are more pronounced in the case of potential barrier

Case 1: $E > V_0$

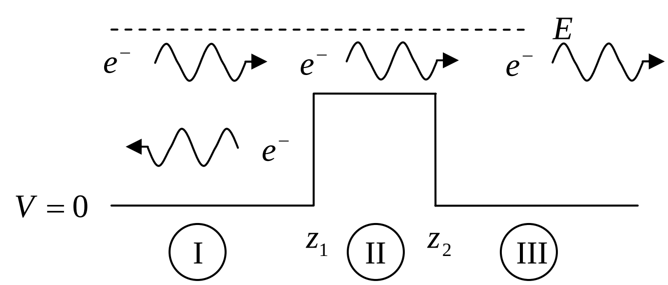
Classical Consideration:



• The treatment is similar to the case of $E > V_0$ for potential step. Particle moves from I to II to III without being reflected (crashing)

• K.E. is positive in all the three regions and is reduced in region II since $\text{K.E.} = E - V_0$

Quantum Mechanical Consideration:



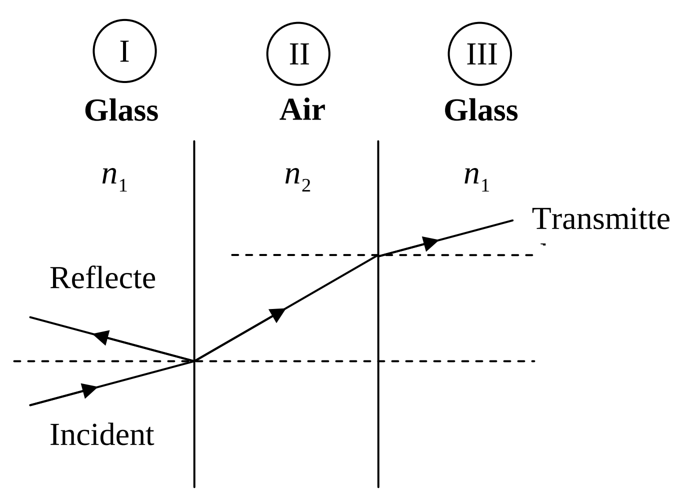
• Treatment is similar to the case of potential step. Particle moves from I to II to III but there is a finite probability for it to reflect at z_1 or z_2, and a finite probability to be transmitted through.

• $k_1 = \sqrt{\dfrac{2mE}{\hbar^2}} \in \Re$ and

$k_2 = \sqrt{\dfrac{2m(E - V_0)}{\hbar^2}} \in \Re$

Ψ_1 and $\Psi_3 \propto e^{ik_1 z} e^{-i\omega t}$ $\quad \Psi_2 \propto e^{ik_2 z} e^{-i\omega t}$

Optical consideration



• $n_1 = \dfrac{1}{\hbar\Omega}\sqrt{2mc^2 E} \in \Re$ since $E > 0$

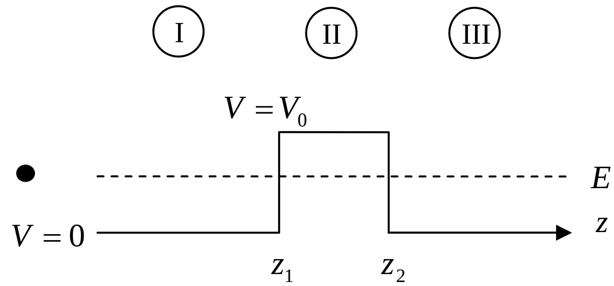
$n_2 = \dfrac{1}{\hbar\Omega}\sqrt{2mc^2(E - V_0)} \in \Re$ since $E > V_0$

$n_1 > n_2$

• For glass-air-glass some of the wave will be reflected and some will be transmitted

Case 2: $E < V_0$

Classical Consideration

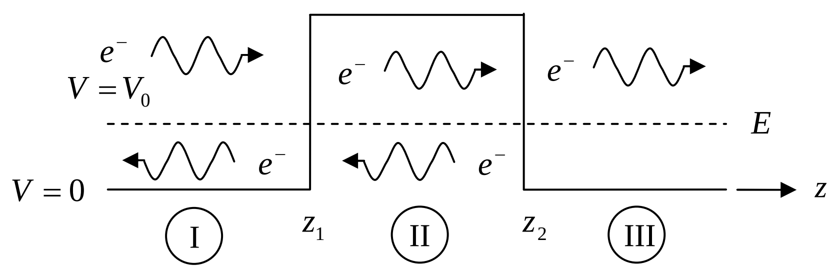


• Particle will never get past $z = z_1$

• In region I: $\text{K.E.} = E$

• In region II: $K.E. = E - V_0 < 0$ because $E < V_0$. This classically is not allowed

Quantum Mechanical Consideration



• There is finite probability that particle will be reflected and a finite probability that it gets through

• In region I: $k_1 = \sqrt{\frac{2mE}{\hbar^2}} \in \Re$ since $E > 0$

• In region II: $k_2 = \sqrt{\frac{2m(E - V_0)}{\hbar^2}} = i\sqrt{\frac{2m(V_0 - E)}{\hbar^2}} = i\alpha$ since $E < V_0$

$\Psi_1 \propto e^{ik_1 z} e^{-i\omega t}$, $\Psi_2 \propto e^{-\alpha z} e^{-i\omega t}$, $\Psi_3 \propto e^{ik_1 z} e^{-i\omega t}$ with $k_1 = k_3$

• If $z_2 - z_1 \approx \frac{1}{\alpha}$ then the particle can be detected at $z > z_2$. This phenomenon is called tunneling effect and corresponds to the evanescent wave being detected on the other side of the barrier.

Optical Consideration

• $n_1 = \frac{1}{\hbar\Omega}\sqrt{2mc^2 E} \in \Re \qquad E > 0$

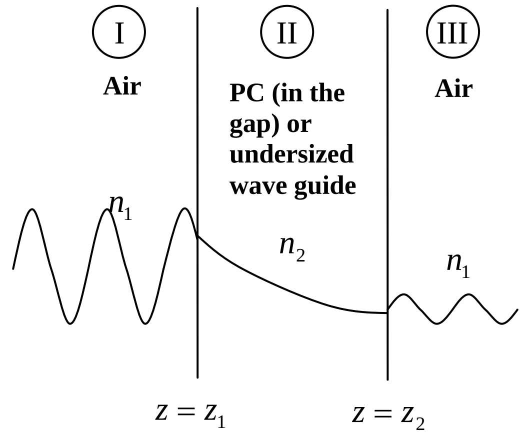
$n_2 = \frac{1}{\hbar\Omega}\sqrt{2mc^2(E - V_0)} = \frac{i}{\hbar\Omega}\sqrt{2mc^2(V_0 - E)}$, where n_2 is purely imaginary and $n_3 = n_1$

• Note that in the case of purely imaginary index, $n_2 = in_2'$ where $n_2' \in \Re$. Then the wave vector in region II is purely imaginary

$k_2 = \frac{\Omega}{c} n_2 = i\frac{\Omega}{c} n_2'$. This wave acquires no phase and will exponentially decay

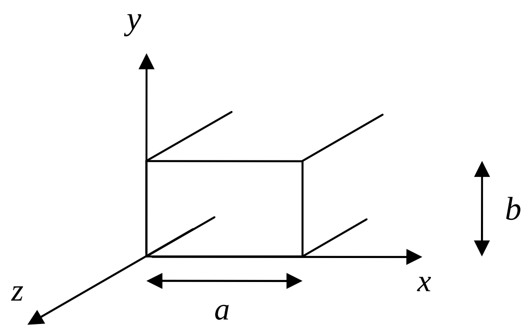
$$e^{i\,k_2\,z} e^{-i\Omega t} = e^{i\frac{\Omega}{c} i\,n_2'\,z} e^{-i\Omega t} = e^{-\frac{\Omega}{c} n_2'\,z} e^{-i\Omega\,t},$$

which implies an evanescent wave similar to an electron tunneling through a potential barrier.



• A good optical analogy is the case of waveguide operated below cutoff or an optical multilayer (photonic band gap material) excited in the gap.

Analogy with Cut off waveguide:



• For wave guide we have $k^2 = k_x{}^2 + k_y{}^2 + k_z{}^2$ where $k^2 = \dfrac{\omega^2}{c^2}$ (the waveguide is filled with air).

• From boundary conditions, $k_x{}^2 = \left(\dfrac{N\pi}{a}\right)^2$, $k_y^2 = \left(\dfrac{M\pi}{b}\right)^2$ with $M, N = 0, 1, 2\ldots$ but not zero simultaneously.

• Define $k_c{}^2 = k^2 - k_z{}^2 = k_x{}^2 + k_y{}^2 = \left(\dfrac{N\pi}{a}\right)^2 + \left(\dfrac{M\pi}{b}\right)^2$

• Then $k_z{}^2 = k^2 - \left(k_x{}^2 + k_y{}^2\right) = k^2 - k_c{}^2$

$k_c \equiv$ cutoff wave number

• We can write $k_c{}^2 = \dfrac{\omega_c{}^2}{c^2} = \left(\dfrac{2\pi\, \nu_c}{c}\right)^2$ and $k^2 = \dfrac{\omega^2}{c^2} = \left(\dfrac{2\pi\, \nu}{c}\right)^2$ where ω_c or ν_c are cutoff frequency. Then

$k_z{}^2 = k^2 - k_c{}^2 = \dfrac{\omega^2}{c^2} - \dfrac{\omega_c{}^2}{c^2} = \left(\dfrac{2\pi\, \nu}{c}\right)^2 - \left(\dfrac{2\pi\, \nu_c}{c}\right)^2 \Rightarrow k_z = \dfrac{2\pi}{c}\sqrt{\nu - \nu_c}$. If $\nu > \nu_c$ then k_z is purely real.

• Remark: In this section I have used k and ω to designate the electromagnetic waves propagation vector and frequency.

• If $\nu_c > \nu$ then k_z is purely imaginary

$k_z = \dfrac{2\pi}{c}\sqrt{-1(\nu_c - \nu)} = i\dfrac{2\pi}{c}\sqrt{\nu_c - \nu} = i\alpha'$ where $\alpha' = \dfrac{2\pi}{c}\sqrt{\nu_c - \nu}$