Electric Anisotropy, Magnetic Anisotropy, Uniaxial and Biaxial Materials, Bianisotropic Media (Definitions)

• A medium is called **electrically anisotropic** if $\vec{D} = \underline{\underline{\varepsilon}} \cdot \vec{E}$, where $\underline{\underline{\varepsilon}}$ is the permittivity tensor. **Note that $\vec{D}$ and $\vec{E}$ are no longer parallel**.

• A medium is **magnetically anisotropic** if $\vec{B} = \underline{\underline{\mu}} \cdot \vec{H}$, where $\underline{\underline{\mu}}$ is the permeability tensor. **Note that $\vec{B}$ and $\vec{H}$ are no longer parallel**.

• A medium can be **both electrically and magnetically anisotropic**.

• Consider the case of electrically **anisotropic medium** for which

$$\underline{\underline{\varepsilon}} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix}$$

• **Crystals**, in general, are described by a **symmetric permittivity tensor**. Then there always **exist a coordinate transformation** that transforms the symmetric matrix $\underline{\underline{\varepsilon}}$ to a **diagonal matrix** as given by

$$\underline{\underline{\varepsilon}} = \begin{bmatrix} \varepsilon_{xx} & 0 & 0 \\ 0 & \varepsilon_{yy} & 0 \\ 0 & 0 & \varepsilon_{zz} \end{bmatrix}.$$

This new coordinate system is called the **Principal System**, and the three coordinate axes are called the **Principal Axes**.

• For cubic crystal $\varepsilon_{xx} = \varepsilon_{yy} = \varepsilon_{zz} = \varepsilon$, and the crystal is **isotropic**.

• For tetragonal, hexagonal, and rhombohedral crystals two of the three ε are equal. Such crystal is called **uniaxial**

$$\underline{\underline{\varepsilon}} = \begin{bmatrix} \varepsilon & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & \varepsilon_{zz} \end{bmatrix}.$$

• The principal axis that is different (displays the anisotropy) is called the **optical axis**. For the above *z*-axis is the optical axis. For the above crystal, there is a two dimensional degeneracy.

• If $\varepsilon_{zz} > \varepsilon$ we say that the medium has **positive uniaxial behavior**, and if $\varepsilon_{zz} < \varepsilon$ we say that the medium has **negative uniaxial behavior**.

• If $\varepsilon_{xx} \neq \varepsilon_{yy} \neq \varepsilon_{zz}$ we say that the crystal is **biaxial**. Examples of biaxial crystals are orthorhombic, monoclinic, and triclinic.

• A **bianisotropic** medium provides a coupling between electric and magnetic fields. The constitutive relations for a bianisotropic medium is given by

$$\vec{D} = \underline{\underline{\varepsilon}} \cdot \vec{E} + \underline{\underline{\xi}} \cdot \vec{H}$$

$$\vec{B} = \underline{\underline{\varsigma}} \cdot \vec{E} + \underline{\underline{\mu}} \cdot \vec{H}$$

• A bianisotropic medium placed in an electric or magnetic field becomes both **polarized and magnetized**.

• Almost any media in **motion** becomes bianisotropic. The first cases of bianisotropic materials were indeed moving dielectrics and magnetic materials in the presence of electric or magnetic fields.

• In 1888 Roentgen discovered that **moving dielectrics** become **magnetized** when placed in an electric field. In 1905 Wilson showed that a **moving dielectric** becomes electrically **polarized** when placed in a uniform magnetic field.

• The topics of moving materials and their constitutive relations are the subject of the **relativistic electromagnetic theory**.

• Special relativity requires that all physical laws to be characterized by mathematical equations that are **form-invariant** from one observer to the other, independent of the relative motions of the two observers. That is to say that the physical laws remain **form-invariant under Lorentz transformation**.

• **Maxwell's** equations are form-invariant; however, **constitutive relations** are **only** form-invariant when they are **written in the bianisotropic form**.

Magnetoelectric Materials: Early History

• Magnetoelectric materials were first proposed by **Landau and Lifshitz** [1957] and Dzyaloshinskii [1959]. They were first observed by Astrov in 1960 in **anti-ferromagnetic chromium oxide**. The **constitutive relations proposed by Dzyaloshinskii** was of the form

$$\vec{D} = \begin{bmatrix} \varepsilon & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & \varepsilon_{zz} \end{bmatrix} \cdot \vec{E} + \begin{bmatrix} \xi & 0 & 0 \\ 0 & \xi & 0 \\ 0 & 0 & \xi_{zz} \end{bmatrix} \cdot \vec{H}$$, and

$$\vec{B} = \begin{bmatrix} \xi & 0 & 0 \\ 0 & \xi & 0 \\ 0 & 0 & \xi_{zz} \end{bmatrix} \cdot \vec{E} + \begin{bmatrix} \mu & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \mu_{zz} \end{bmatrix} \cdot \vec{H}$$

• Later it was shown by Indenbom [1960] and Birss [1963] that **58 magnetic crystal** classes can exhibit magnetoelectric effects.

• In 1948 **Tellegen** introduced a new element called **gyrator** which in addition to resistor, inductor, capacitor, and transformer was used to describe an electric network.

• To **realize** this new element, Tellegen had imagined **a new medium** for which the constitutive relations were given by

$\vec{D} = \underline{\underline{\varepsilon}} \cdot \vec{E} + \underline{\underline{\xi}} \cdot \vec{H}$ and

$\vec{B} = \underline{\underline{\xi}} \cdot \vec{E} + \underline{\underline{\mu}} \cdot \vec{H}$ where $\xi^2 / \mu\, \varepsilon \approx 1$.

• Tellegen had assumed that the medium had **permanent electric dipole (** $\vec{p}$ **)** and **magnetic dipole (** $\vec{m}$ **)** that were anti-parallel to each other, such that an applied $\vec{E}$ which aligned the $\vec{p}$ also aligned the $\vec{m}$ or similarly an applied $\vec{H}$ which aligned the $\vec{m}$ also aligned the $\vec{p}$.

• Tellegen also considered the **general constitutive relations**

$\vec{D} = \underline{\underline{\varepsilon}} \cdot \vec{E} + \underline{\underline{\xi}} \cdot \vec{H}$ and

$\vec{B} = \underline{\underline{\zeta}} \cdot \vec{E} + \underline{\underline{\mu}} \cdot \vec{H}$,

and studied the symmetry properties by considering the energy conservation.

Chiral Media

• For **chiral** materials the constitutive relations are given by

$$\vec{D} = \varepsilon\, \vec{E} - \chi \frac{\partial \vec{H}}{\partial t}$$

$$\vec{B} = \chi \frac{\partial \vec{E}}{\partial t} + \mu\, \vec{H}$$

where χ is called the **chiral parameter**. Examples of chiral materials are sugar solutions, amino acids, DNA, etc. Chiral materials are bi-isotropic.

Constitutive Matrices

• The constitutive relations in the **most general form** are written as

$c\,\vec{D} = \underline{\underline{P}} \cdot \vec{E} + \underline{\underline{L}} \cdot c\,\vec{B}$ and

$\vec{H} = \underline{\underline{M}} \cdot \vec{E} + \underline{\underline{Q}} \cdot c\,\vec{B}$ where c is the speed of light in vacuum and $\underline{\underline{P}}$, $\underline{\underline{L}}$, $\underline{\underline{M}}$, and $\underline{\underline{Q}}$ **are** 3×3 **matrices** which their elements are called the constitutive parameters. Note that $\underline{\underline{L}}$ and $\underline{\underline{M}}$ relate the electric and magnetic fields together. **When** $\underline{\underline{L}} \neq 0$ **and** $\underline{\underline{M}} \neq 0$ **the medium is called bianisotropic**.

• When there is **no coupling** between electric and magnetic fields, i.e. $\underline{\underline{L}} = 0$ and $\underline{\underline{M}} = 0$ we have $c\,\vec{D} = \underline{\underline{P}} \cdot \vec{E}$ and $\vec{H} = \underline{\underline{Q}} \cdot c\,\vec{B}$. In this case the medium is called **anisotropic**. If $\underline{\underline{P}} = c\,\varepsilon\,\underline{\underline{I}}$ and $\underline{\underline{Q}} = \dfrac{1}{c\,\mu}\underline{\underline{I}}$, where $\underline{\underline{I}}$ is the identity matrix, then medium is said to be **isotropic**.

• The relations $c\,\vec{D} = \underline{\underline{P}} \cdot \vec{E} + \underline{\underline{L}} \cdot c\,\vec{B}$ and $\vec{H} = \underline{\underline{M}} \cdot \vec{E} + \underline{\underline{Q}} \cdot c\,\vec{B}$ can be written as

$$\begin{bmatrix} c\,\vec{D} \\ \vec{H} \end{bmatrix} = \begin{bmatrix} \underline{\underline{P}} & \underline{\underline{L}} \\ \underline{\underline{M}} & \underline{\underline{Q}} \end{bmatrix} \cdot \begin{bmatrix} \vec{E} \\ c\,\vec{B} \end{bmatrix} = \underline{\underline{C_{EB}}} \cdot \begin{bmatrix} \vec{E} \\ c\,\vec{B} \end{bmatrix}. \qquad (1)$$

Here $\underline{\underline{C_{EB}}}$ **is a** 6×6 **constitutive matrix**. Above ($\underline{\underline{C_{EB}}}$) is called ***E-B* presentation**.

• The reason for choosing the above form is that constitutive relations written as (1) are **form invariant** under Lorentz transformation. They are so called **Lorentz-covariant**.

• $(\vec{E}, c\vec{B})$ and $(c\vec{D}, \vec{H})$ each form a single **tensor in four dimensional space**.

• Other representations are also possible. For example $\begin{bmatrix} \vec{D} \\ \vec{B} \end{bmatrix} = \underline{\underline{C_{EH}}} \cdot \begin{bmatrix} \vec{E} \\ \vec{H} \end{bmatrix}$, or $\begin{bmatrix} \vec{E} \\ \vec{H} \end{bmatrix} = \underline{\underline{C_{DB}}} \cdot \begin{bmatrix} \vec{D} \\ \vec{B} \end{bmatrix}$, or $\begin{bmatrix} \vec{E} \\ \vec{B} \end{bmatrix} = \underline{\underline{C_{DH}}} \cdot \begin{bmatrix} \vec{D} \\ \vec{H} \end{bmatrix}$, where they are called ***E-H*, *D-B*, or *D-H* presentation**, respectively. (Exercise: Find the matrix elements for $\underline{\underline{C_{DB}}}$ in terms of $\underline{\underline{P}}$, $\underline{\underline{L}}$, $\underline{\underline{M}}$, and $\underline{\underline{Q}}$).

Anisotropic Medium and *KDB* system

• We consider Maxwell's equations in a **source free region** $\vec{J}_i = \vec{M}_i = \vec{J}_c = 0, \rho_e = 0$.
The time harmonic Maxwell's equations are given by
$\nabla \times \vec{E} = -j\omega\,\vec{B}$, $\nabla \times \vec{H} = j\omega\,\vec{D}$, $\nabla \cdot \vec{B} = 0$, and $\nabla \cdot \vec{D} = 0$.

• The assumption that there are no sources within a given region of space does not mean that there are no sources anywhere else. In fact, if this was the case there will be no field anywhere. We **assume that fields are generated at a given point in space and now we are studying their dynamical evolution away from the source.**

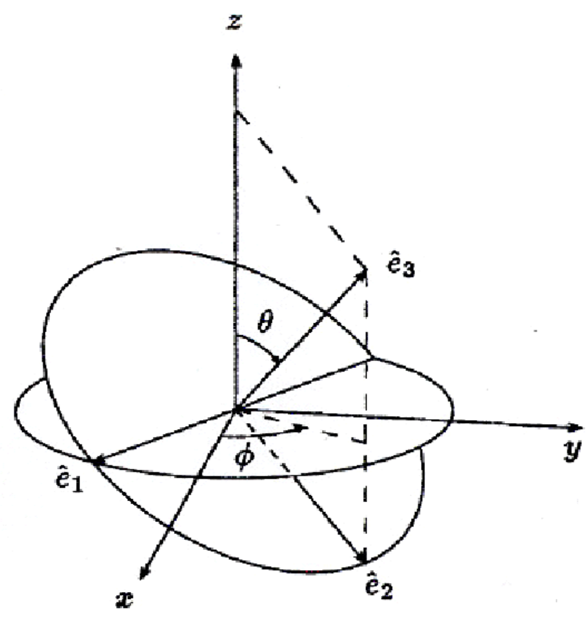
• We have seen that for a **plane wave** $\exp\left(-j\vec{k}\cdot\vec{r}\right)$ we have $\vec{k}\times\vec{E}=\omega\,\vec{B}$, $\vec{k}\times\vec{H}=-\omega\,\vec{D}$, $\vec{k}\cdot\vec{B}=0$, $\vec{k}\cdot\vec{D}=0$. From the last two equations we see that $\vec{k}$ is **perpendicular to the plane containing both $\vec{D}$ and $\vec{B}$**. Let us call this plane, the ***D-B-plane***. If $\vec{B}=\mu\,\vec{H}$ (μ, is a scalar function) then $\vec{H}$ also lies in the *D-B-plane*.

• For a medium with $\vec{D}=\underline{\underline{\varepsilon}}\cdot\vec{E}$ we see that $\vec{E}$ **may not lie on the *D-B-plane***. For this reason in anisotropic medium we define the **polarization in terms of $\vec{D}$ instead of $\vec{E}$**.

• Recall that **Poynting vector** and hence the power flow is along $\vec{\mathcal{E}}(\vec{r},t)\times\vec{\mathcal{H}}(\vec{r},t)$, which is **not necessarily in the same direction as the propagation vector $\vec{k}$** inside an anisotropic medium. In other words, **the direction of power flow** for a plane wave inside an **anisotropic** medium **is not necessarily the same as the direction of the wave vector**.

KDB **Coordinate System**

• To make our study of anisotropic medium easier we will **transform our *xyz* coordinate** system to the ***KDB*** **coordinate system**. Whereas the unit vector in *xyz* are $\hat{a}_x, \hat{a}_y, \hat{a}_z$, in the KDB we designate them by $\hat{e}_1, \hat{e}_2, \hat{e}_3$.



• We will take the $\vec{k}$ **to be along** $\hat{e}_3$, i.e. $\vec{k}=k\,\hat{e}_3$. From the figure we can see
$\hat{e}_3=\hat{a}_x\,\sin\theta\cos\phi+\hat{a}_y\,\sin\theta\sin\phi+\hat{a}_z\,\cos\theta$.

• $\hat{e}_1$ **lies in the *x-y-plane*** and is perpendicular to the projection of $\vec{k}=k\,\hat{e}_3$ to the *x-y-plane*. It is given by

$$\hat{e}_1=\cos\left(\frac{\pi}{2}-\phi\right)\hat{a}_x-\sin\left(\frac{\pi}{2}-\phi\right)\hat{a}_y \Rightarrow$$
$$\hat{e}_1=\sin\phi\,\hat{a}_x-\cos\phi\,\hat{a}_y$$

• $\hat{e}_2$ can be calculated from $\hat{e}_2=\hat{e}_3\times\hat{e}_1 \Rightarrow$

$$\hat{e}_2=\hat{a}_x\,\cos\theta\cos\phi+\hat{a}_y\,\cos\theta\sin\phi-\hat{a}_z\,\sin\theta$$

• *KDB* system can be obtained from the *xyz* system by **multiple rotations**.

Transforming a Vector From *xyz* to *KDB* and Vice Versa

• Let the vector $\vec{A}$ in ***xyz* system** to be given by

$\vec{A} = \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$ and in ***KDB* system** by $\vec{A} = \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix}$ then $\vec{A}_k = \underline{\underline{T}}\,\vec{A}$ and $\vec{A} = \underline{\underline{T}}^{-1}\,\vec{A}_k$, where

$$\underline{\underline{T}} = \begin{bmatrix} \sin\phi & -\cos\phi & 0 \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \end{bmatrix}.$$ **Note that $\underline{\underline{T}}$ is unitary**.

• Exercise: Show that indeed $\underline{\underline{T}}$ is given by above and calculate $\underline{\underline{T}}^{-1}$.

Constitutive Relations in the *KD*B system: The $\underline{\underline{C}}_{DB}$ Formulation

• Recall that in *xyz* system the $\underline{\underline{C}}_{DB}$ **formulation** of constitutive relations was given by

$$\begin{bmatrix} \vec{E} \\ \vec{H} \end{bmatrix} = \underline{\underline{C}}_{DB} \cdot \begin{bmatrix} \vec{D} \\ \vec{B} \end{bmatrix} = \begin{bmatrix} \underline{\underline{\kappa}} & \underline{\underline{\chi}} \\ \underline{\underline{\gamma}} & \underline{\underline{\nu}} \end{bmatrix} \cdot \begin{bmatrix} \vec{D} \\ \vec{B} \end{bmatrix} \tag{1}$$

With the help of transformation $\vec{A} = \underline{\underline{T}}^{-1}\,\vec{A}_k$, $\underline{\underline{T}}$, and $\underline{\underline{T}}^{-1}$ we will find the **equivalent relations in the *KDB* system**.

• Note that in the *KDB* system the $\vec{D}$ and $\vec{B}$ will take a **simpler form** than $\vec{E}$ and $\vec{H}$ since $D_3 = B_3 = 0$ [recall $\vec{k} = k\,\hat{e}_3$].

• In long hand (1) can be written as

$$\vec{E} = \underline{\underline{\kappa}} \cdot \vec{D} + \underline{\underline{\chi}} \cdot \vec{B} \tag{2}$$

$$\vec{H} = \underline{\underline{\gamma}} \cdot \vec{D} + \underline{\underline{\nu}} \cdot \vec{B} \tag{3}$$

• Using the fact that $\vec{E} = \underline{\underline{T}}^{-1} \cdot \vec{E}_k$, $\vec{D} = \underline{\underline{T}}^{-1} \cdot \vec{D}_k$, $\vec{B} = \underline{\underline{T}}^{-1} \cdot B_k$, and $\vec{H} = \underline{\underline{T}}^{-1} \cdot \vec{H}_k$ **(2) and (3) can be written as**

$$\underline{\underline{T}}^{-1} \cdot \vec{E}_k = \underline{\underline{\kappa}} \cdot (\underline{\underline{T}}^{-1} \cdot \vec{D}_k) + \underline{\underline{\chi}} \cdot (\underline{\underline{T}}^{-1} \cdot \vec{B}_k) \tag{4}$$

$$\underline{\underline{T}}^{-1} \cdot \vec{H}_k = \underline{\underline{\gamma}} \cdot (\underline{\underline{T}}^{-1} \cdot \vec{D}_k) + \underline{\underline{\nu}} \cdot (\underline{\underline{T}}^{-1} \cdot \vec{B}_k)\,. \tag{5}$$

Multiplying (4) and (5) with $\underline{\underline{T}}$ and rearranging terms we have

$$\vec{E}_k = (\underline{\underline{T}} \cdot \underline{\underline{\kappa}} \cdot \underline{\underline{T}}^{-1}) \cdot \vec{D}_k + (\underline{\underline{T}} \cdot \underline{\underline{\chi}} \cdot \underline{\underline{T}}^{-1}) \cdot \vec{B}_k \tag{6}$$

$$\vec{H}_k = (\underline{\underline{T}} \cdot \underline{\underline{\gamma}} \cdot \underline{\underline{T}}^{-1}) \cdot \vec{D}_k + (\underline{\underline{T}} \cdot \underline{\underline{\nu}} \cdot \underline{\underline{T}}^{-1}) \cdot \vec{B}_k \qquad (7)$$

• The last two equations can be written as

$$\vec{E}_k = \underline{\underline{\kappa_k}} \cdot \vec{D}_k + \underline{\underline{\chi_k}} \cdot \vec{B}_k$$

$$\vec{H}_k = \underline{\underline{\gamma_k}} \cdot \vec{D}_k + \underline{\underline{\nu_k}} \cdot \vec{B}_k$$

where the **definition of** $\underline{\underline{\kappa_k}}$ **,** $\underline{\underline{\chi_k}}$ **,** $\underline{\underline{\gamma_k}}$ **, and** $\underline{\underline{\nu_k}}$ are evident.

$$\underline{\underline{\kappa_k}} = \underline{\underline{T}} \cdot \underline{\underline{\kappa}} \cdot \underline{\underline{T}}^{-1} = \begin{bmatrix} \kappa_{11} & \kappa_{12} & \kappa_{13} \\ \kappa_{21} & \kappa_{22} & \kappa_{23} \\ \kappa_{31} & \kappa_{32} & \kappa_{33} \end{bmatrix}, \ \underline{\underline{\chi_k}} = \underline{\underline{T}} \cdot \underline{\underline{\chi}} \cdot \underline{\underline{T}}^{-1} = \begin{bmatrix} \chi_{11} & \chi_{12} & \chi_{13} \\ \chi_{21} & \chi_{22} & \chi_{23} \\ \chi_{31} & \chi_{32} & \chi_{33} \end{bmatrix}$$

$$\underline{\underline{\gamma_k}} = \underline{\underline{T}} \cdot \underline{\underline{\gamma}} \cdot \underline{\underline{T}}^{-1} = \begin{bmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} \\ \gamma_{21} & \gamma_{22} & \gamma_{23} \\ \gamma_{31} & \gamma_{32} & \gamma_{33} \end{bmatrix}, \ \underline{\underline{\nu_k}} = \underline{\underline{T}} \cdot \underline{\underline{\nu}} \cdot \underline{\underline{T}}^{-1} = \begin{bmatrix} \nu_{11} & \nu_{12} & \nu_{13} \\ \nu_{21} & \nu_{22} & \nu_{23} \\ \nu_{31} & \nu_{32} & \nu_{33} \end{bmatrix}$$

Dispersion Relation for Bianisotropic Medium

• In the ***KDB*** **system similar to** ***xyz*** **system** we have

$$\vec{k} \times \vec{E}_k = \omega \vec{B}_k , \qquad (1)$$

$$\vec{k} \times \vec{H}_k = -\omega \vec{D}_k , \qquad (2)$$

$$\vec{k} \cdot \vec{B}_k = 0 , \qquad (3)$$

$$\vec{k} \cdot \vec{D}_k = 0 \qquad (4)$$

• From **(3) and (4) and since in** ***KDB*,** $\vec{k} = k \, \hat{e}_3$ then

$$\vec{D}_k = \begin{bmatrix} D_1 \\ D_2 \\ 0 \end{bmatrix} \text{ and } \vec{B}_k = \begin{bmatrix} B_1 \\ B_2 \\ 0 \end{bmatrix}$$

• **From (1)**

$$\vec{k} \times \vec{E}_k = \omega \vec{B}_k \Rightarrow \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ 0 & 0 & k \\ E_1 & E_2 & E_2 \end{vmatrix} = \omega B_1 \hat{e}_1 + \omega B_2 \hat{e}_2 \Rightarrow$$

$$-k E_2 \hat{e}_1 + k E_1 \hat{e}_2 = \omega B_1 \hat{e}_1 + \omega B_2 \hat{e}_2 \Rightarrow$$

$$k E_1 = \omega B_2 \quad \text{and}$$

$$-k E_2 = \omega B_1$$

• Now recall that

$$\vec{E}_k = \underline{\underline{\kappa_k}} \cdot \vec{D}_k + \underline{\underline{\chi_k}} \cdot \vec{B}_k \Rightarrow$$

$$\begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix} = \begin{bmatrix} \kappa_{11} & \kappa_{12} & \kappa_{13} \\ \kappa_{21} & \kappa_{22} & \kappa_{23} \\ \kappa_{31} & \kappa_{32} & \kappa_{33} \end{bmatrix} \cdot \begin{bmatrix} D_1 \\ D_2 \\ 0 \end{bmatrix} + \begin{bmatrix} \chi_{11} & \chi_{12} & \chi_{13} \\ \chi_{21} & \chi_{22} & \chi_{23} \\ \chi_{31} & \chi_{32} & \chi_{33} \end{bmatrix} \cdot \begin{bmatrix} B_1 \\ B_2 \\ 0 \end{bmatrix} \Rightarrow$$

$$E_1 = \kappa_{11}\, D_1 + \kappa_{12}\, D_2 + \chi_{11}\, B_1 + \chi_{12}\, B_2 \quad \text{and} \tag{1}$$

$$E_2 = \kappa_{21}\, D_1 + \kappa_{22}\, D_2 + \chi_{21}\, B_1 + \chi_{22}\, B_2 \tag{2}$$

• **Using (1) and (2) in** $k\ E_1 = \omega\ B_2$ **and** $-k\ E_2 = \omega\ B_1$ we have

$$k\ E_1 = k[\kappa_{11}\, D_1 + \kappa_{12}\, D_2 + \chi_{11}\, B_1 + \chi_{12}\, B_2] = \omega\ B_2 \quad \text{and} \tag{3}$$

$$-k\ E_2 = -k[\kappa_{21}\, D_1 + \kappa_{22}\, D_2 + \chi_{21}\, B_1 + \chi_{22}\, B_2] = \omega\ B_1 \tag{4}$$

• (3) and (4) can further be written as

$$k\ \kappa_{11}\ D_1 + k\ \kappa_{12}\ D_2 = -k\ \chi_{11}\ B_1 + (\omega - k\ \chi_{12})\ B_2 \ \text{ and} \tag{5}$$

$$k\ \kappa_{21}\ D_1 + k\ \kappa_{22}\ D_2 = -(\omega + k\ \chi_{21})\ B_1 - k\ \chi_{22}\ B_2 \tag{7}$$

• **(5) and (7)** can be given in matrix form according to

$$\begin{bmatrix} k\ \kappa_{11} & k\ \kappa_{12} \\ k\ \kappa_{21} & k\ \kappa_{22} \end{bmatrix} \cdot \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} = \begin{bmatrix} -k\ \chi_{11} & (\omega - k\ \chi_{12}) \\ -(\omega + k\ \chi_{21}) & -k\ \chi_{22} \end{bmatrix} \cdot \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \tag{8}$$

• **Dived both side of (8) by** k **and let** $u = \dfrac{\omega}{k}$, we have

$$\begin{bmatrix} \kappa_{11} & \kappa_{12} \\ \kappa_{21} & \kappa_{22} \end{bmatrix} \cdot \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} = -\begin{bmatrix} \chi_{11} & \chi_{12} - u \\ u + \chi_{21} & \chi_{22} \end{bmatrix} \cdot \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \tag{9}$$

• We arrived at above by using $\vec{k} \times \vec{E}_k = \omega\ \vec{B}_k$ and $\vec{E}_k = \underline{\underline{\kappa_k}} \cdot \vec{D}_k + \underline{\underline{\chi_k}} \cdot \vec{B}_k$. We **do the same steps but now with** $\vec{k} \times \vec{H}_k = -\omega\ \vec{D}_k$ **and** $\vec{H}_k = \underline{\underline{\nu_k}} \cdot \vec{B}_k + \underline{\underline{\gamma_k}} \cdot \vec{D}_k$ to arrive at

$$\begin{bmatrix} \nu_{11} & \nu_{12} \\ \nu_{21} & \nu_{22} \end{bmatrix} \cdot \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = -\begin{bmatrix} \gamma_{11} & \gamma_{12} + u \\ \gamma_{21} - u & \gamma_{22} \end{bmatrix} \cdot \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} \tag{10}$$

• Now, **(9) and (10) can be used to eliminate** the $\vec{D}$ or $\vec{B}$. For example, let us eliminate the $\vec{B}$ by using (10). We have

$$\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = -\begin{bmatrix} \nu_{11} & \nu_{12} \\ \nu_{21} & \nu_{22} \end{bmatrix}^{-1} \cdot \begin{bmatrix} \gamma_{11} & \gamma_{12} + u \\ \gamma_{21} - u & \gamma_{22} \end{bmatrix} \cdot \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} \Rightarrow \tag{11}$$

$$\begin{bmatrix} \kappa_{11} & \kappa_{12} \\ \kappa_{21} & \kappa_{22} \end{bmatrix} \cdot \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} = \begin{bmatrix} \chi_{11} & \chi_{12} - u \\ u + \chi_{21} & \chi_{22} \end{bmatrix} \cdot \begin{bmatrix} \nu_{11} & \nu_{12} \\ \nu_{21} & \nu_{22} \end{bmatrix}^{-1} \cdot \begin{bmatrix} \gamma_{11} & \gamma_{12} + u \\ \gamma_{21} - u & \gamma_{22} \end{bmatrix} \begin{bmatrix} D_1 \\ D_2 \end{bmatrix}$$

• The above can be finally written as

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \left\{ \frac{1}{\nu_{11}\nu_{22} - \nu_{12}\nu_{21}} \begin{bmatrix} \chi_{11} & \chi_{12} - u \\ u + \chi_{21} & \chi_{22} \end{bmatrix} \cdot \begin{bmatrix} \nu_{22} & -\nu_{12} \\ -\nu_{21} & \nu_{11} \end{bmatrix} \cdot \begin{bmatrix} \gamma_{11} & \gamma_{12} + u \\ \gamma_{21} - u & \gamma_{22} \end{bmatrix} - \begin{bmatrix} \kappa_{11} & \kappa_{12} \\ \kappa_{21} & \kappa_{22} \end{bmatrix} \right\} \begin{bmatrix} D_1 \\ D_2 \end{bmatrix}$$

• For above equation to have **nontrivial solutions the determinant of the matrix multiplying** $\begin{bmatrix} D_1 \\ D_2 \end{bmatrix}$ **must be zero**. This condition will provide us with the **required dispersion relation**.

• Exercise: Find (recover) the dispersion relation for an isotropic, homogeneous medium characterized by $\vec{E}_k = \underline{\underline{\kappa_k}} \cdot \vec{D}_k$ and $\vec{H}_k = \underline{\underline{\nu_k}} \cdot \vec{B}_k$ from our previous discussion.