A. Duality Theorem:

• Consider the following equations as an example

$$\vec{H}_A = \frac{1}{\mu}\nabla\times\vec{A} \qquad\qquad \vec{E}_F = -\frac{1}{\varepsilon}\nabla\times\vec{F}$$

$$\vec{E}_A = -j\omega\vec{A} - j\frac{1}{\omega\mu\varepsilon}\nabla\left(\nabla\cdot\vec{A}\right) \qquad\qquad \vec{H}_F = -j\omega\,\vec{F} - j\frac{1}{\omega\mu\varepsilon}\nabla\left(\nabla\cdot\vec{F}\right)$$

$$\nabla^2\vec{A} + \beta^2\vec{A} = -\mu\,\vec{J} \qquad\qquad \nabla^2\vec{F} + \beta^2\vec{F} = -\varepsilon\,\vec{M}$$

$$\vec{A}(x,y,z) = \frac{\mu}{4\pi}\iiint_{v'}\vec{J}\left(x',y',z'\right)\frac{e^{-j\vec{\beta}\cdot\vec{R}}}{R}dv' \qquad\qquad \vec{F}(x,y,z) = \frac{\varepsilon}{4\pi}\iiint_{v'}\vec{M}\left(x',y',z'\right)\frac{e^{-j\vec{\beta}\cdot\vec{R}}}{R}dv'$$

• As we compare equations in the first column with those in the second column we note that these equations (and many others in EM theory) have similar mathematical constructs. For example, the vectors $\vec{E}_A$ and $\vec{H}_F$, or vectors $\vec{A}$ and $\vec{F}$ occupy similar position. In this sense we say $\vec{E}_A$ and $\vec{H}_F$ or $\vec{A}$ and $\vec{F}$ are dual of each other.

• Duality can be used to form a solution for a given variable if we know the solution for the dual variable.

Dual Quantities

Electric sources ($\vec{J}\neq 0,\ \vec{M}=0$)	**Magnetic Sources ($\vec{J}=0,\ \vec{M}\neq 0$)**
$\vec{H}_A$	$-\vec{E}_F$
$\vec{E}_A$	$\vec{H}_F$
$\vec{J}$	$\vec{M}$
$\vec{A}$	$\vec{F}$
ε	μ
μ	ε
β	β
η	$1/\eta$
$1/\eta$	η

B. Reciprocity Theorem:

• In circuit theory reciprocity states: in a physical linear network the position of an ideal voltage source and an ideal voltmeter can be interchanged without affecting their reading.

• Reciprocity in electromagnetic theory comes about from Maxwell's equations and has many useful applications. For example, it relates the receiving and transmitting properties of a radiating element (antenna.)

• Consider a linear, isotropic, but not necessarily homogeneous medium in which there exist two sets of sources $\left(\vec{J}_1, \vec{M}_1\right)$ and $\left(\vec{J}_2, \vec{M}_2\right)$. At a given frequency ω, $\left(\vec{J}_1, \vec{M}_1\right)$ produce the fields $\left(\vec{E}_1, \vec{H}_1\right)$ and $\left(\vec{J}_2, \vec{M}_2\right)$ produce the fields $\left(\vec{E}_2, \vec{H}_2\right)$.

$$\left.\begin{matrix}\vec{J}_1 \\ \vec{M}_1\end{matrix}\right\} \rightarrow \vec{E}_1, \vec{H}_1 \qquad\qquad \left.\begin{matrix}\vec{J}_2 \\ \vec{M}_2\end{matrix}\right\} \rightarrow \vec{E}_2, \vec{H}_2$$

• Each of the fields must satisfy the corresponding Maxwell's equations.

$$\nabla\times\vec{E}_1 = -\vec{M}_1 - j\omega\mu\,\vec{H}_1 \qquad (1) \qquad\qquad \nabla\times\vec{E}_2 = -\vec{M}_2 - j\omega\mu\,\vec{H}_2 \qquad (3)$$

$$\nabla\times\vec{H}_1 = \vec{J}_1 + j\omega\varepsilon\,\vec{E}_1 \qquad (2) \qquad\qquad \nabla\times\vec{H}_2 = \vec{J}_2 + j\omega\varepsilon\,\vec{E}_2 \qquad (4)$$

• Multiply (1) by $\vec{H}_2$ and (4) by $\vec{E}_1$, then

$$\vec{H}_2\cdot(\nabla\times\vec{E}_1) = -\vec{M}_1\cdot\vec{H}_2 - j\omega\mu\,\vec{H}_1\cdot\vec{H}_2 \qquad (5)$$

$$\vec{E}_1\cdot(\nabla\times\vec{H}_2) = \vec{J}_2\cdot\vec{E}_1 + j\omega\varepsilon\,\vec{E}_1\cdot\vec{E}_2 \qquad (6)$$

• Subtracting (5) from (6),we have

$$\vec{E}_1\cdot(\nabla\times\vec{H}_2) - \vec{H}_2\cdot(\nabla\times\vec{E}_1) = \vec{J}_2\cdot\vec{E}_1 + j\omega\varepsilon\,\vec{E}_1\cdot\vec{E}_2 + \vec{M}_1\cdot\vec{H}_2 + j\omega\mu\,\vec{H}_1\cdot\vec{H}_2\,. \qquad (7)$$

From the vector identity

$$\vec{B}\cdot(\nabla\times\vec{A}) - \vec{A}\cdot(\nabla\times\vec{B}) = \nabla\cdot(\vec{A}\times\vec{B}) = -\nabla\cdot(\vec{B}\times\vec{A})\,, \qquad (8)$$

(7) can be rewritten as

$$-\nabla\cdot(\vec{E}_1\times\vec{H}_2) = \vec{J}_2\cdot\vec{E}_1 + j\omega\varepsilon\,\vec{E}_1\cdot\vec{E}_2 + \vec{M}_1\cdot\vec{H}_2 + j\omega\mu\,\vec{H}_1\cdot\vec{H}_2\,. \qquad (9)$$

• In the above we have used Eqs. (1) and (4). If we now multiplying (2) with $\vec{E}_2$ and (3) with $\vec{H}_1$, we have

$$\vec{E}_2\cdot(\nabla\times\vec{H}_1) = \vec{E}_2\cdot\vec{J}_1 + j\omega\varepsilon\,\vec{E}_1\cdot\vec{E}_2 \qquad (10)$$

$$\vec{H}_1\cdot(\nabla\times\vec{E}_2) = -\vec{H}_1\cdot\vec{M}_2 - j\omega\mu\,\vec{H}_1\cdot\vec{H}_2 \qquad (11)$$

•Following steps similar to those described above we will have

$$-\nabla\cdot(\vec{E}_2\times\vec{H}_1) = \vec{J}_1\cdot\vec{E}_2 + j\omega\varepsilon\,\vec{E}_1\cdot\vec{E}_2 + \vec{M}_2\cdot\vec{H}_1 + j\omega\mu\,\vec{H}_1\cdot\vec{H}_2 \qquad (12)$$

• Finally we will subtract (12) from (9)

$$-[\nabla\cdot(\vec{E}_1\times\vec{H}_2)-\nabla\cdot(\vec{E}_2\times\vec{H}_1)]=\vec{J}_2\cdot\vec{E}_1-\vec{J}_1\cdot\vec{E}_2+\vec{M}_1\cdot\vec{H}_2-\vec{M}_2\cdot\vec{H}_1 \qquad (13)$$

• Equation (13) is the Lorentz reciprocity theorem in its differential form. Note that in (13) if we let $1\to 2$ and $2\to 1$ nothing will change.

• In integral form (13) can be written as

$$-\left[\oiint_s \nabla\cdot(\vec{E}_1\times\vec{H}_2)\cdot\vec{ds}-\oiint_s \nabla\cdot(\vec{E}_2\times\vec{H}_1)\cdot\vec{ds}\right]= \\ \iiint_v(\vec{J}_2\cdot\vec{E}_1-\vec{J}_1\cdot\vec{E}_2+\vec{M}_1\cdot\vec{H}_2-\vec{M}_2\cdot\vec{H})\,dv \qquad (14)$$

• Consider a region of space in which there are no sources $\vec{J}_1=\vec{J}_2=\vec{M}_1=\vec{M}_2=0$, then (13) and (14) will be given by

$$\nabla\cdot(\vec{E}_1\times\vec{H}_2)=\nabla\cdot(\vec{E}_2\times\vec{H}_1) \qquad (15)$$

$$\oiint_s \vec{E}_1\times\vec{H}_2\cdot\vec{ds}=\oiint_s \vec{E}_2\times\vec{H}_1\cdot\vec{ds} \qquad (16)$$

•As an example, consider inside a waveguide in which two modes co-exist, $(\vec{E}_1,\vec{H}_1)$, and $(\vec{E}_2,\vec{H}_2)$. For expressions of the two modes to be valid $\vec{E}_1,\vec{H}_1,\vec{E}_2,\vec{H}_2$ must satisfy (15) and/or (16).

C. Green's Function

• In EM theory we investigate the solutions to Maxwell's equations which are often expressed in terms of uncoupled differential equations subject to given boundary conditions and various excitations.

• The use of Green's function techniques is then to obtain solutions for these differential equations subject to Dirac delta (impulse) excitations. The solution to the actual excitation is then written as a superposition of these impulse responses (what we now call Green's functions) with the Dirac delta excitation at different locations. This superposition in the limit can be represented as an integral. In this sense we see that Green's function is nothing more than what in Engineering we call impulse response and in system theory is called transfer function.

• The Green's function can be represented as a finite function, an infinite series, or an integral. In the case of infinite series, the Green's function is presented as a sum of orthonormal functions. The coefficient of expansion for this series depends on the strength of the source and the eigenvalues associated with the functions (orthonormal functions forming a base) depend on the boundary conditions. In the case of integral representation of the Green's function, the spectrum of the associated eigenvalues is continuous.

• The above three presentations of the Green's function, though different in form, will produce the same final results in terms of the solutions to the underlying partial differential equations being considered. The choice of a particular presentation for the Green's function depends on the actual source of excitation.

• Usually there is the same amount of work in obtaining the Green's function for a differential equation as there is in obtaining direct solutions. However, the benefit of the Green's function is in the fact that once it is found, the solutions of the differential equations for any excitation (driving terms) can be easily obtained with the help of its Green's function.

Example of Green Function in Circuit Theory:

• To better understand the use of the Green's function in the circuits consider the simple RL circuit shown below

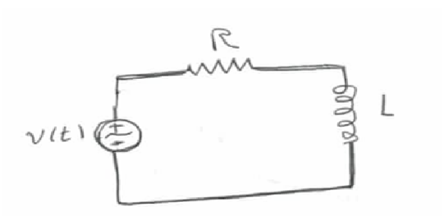

Fig. 1

• We assume that the circuit is at rest for $t < t'$. At $t = t'$ the voltage source is turned on for a very short period of time, $\Delta t'$, via an impulse of magnitude V_0. For times $t > t' + \Delta t'$, the excitation is again zero. The differential equation governing the behavior of the current , $i(t)$, in the circuit is given by the following:

For $\quad t' + \Delta t' > t > t' \qquad R\, i(t) + L\dfrac{di(t)}{dt} = v(t) \qquad$ (C-1)

For $\quad t > t' + \Delta t' \qquad R\, i(t) + L\dfrac{di(t)}{dt} = 0\,, \qquad$ (C-2)

Where as stated above $v(t)$ is an impulse function of magnitude V_0.

• Equation (C-1) represents the state of our circuit during the application of our impulse function (Dirac delta function), and Eq. (C-2) represents the state of the circuit after the application of our impulse function.

• Let us integrate (C-1) over the period of $t' + \Delta t' > t > t' \;\Rightarrow$

$$R \int_{t'}^{t'+\Delta t'} i(t)\, dt + L \int_{t'}^{t'+\Delta t'} \frac{di(t)}{dt}\, dt = \int_{t'}^{t'+\Delta t'} v(t)\, dt \qquad \text{(C-3)} \quad \Rightarrow$$

$$R\int_{t'}^{t'+\Delta t'} i(t)\,dt + L\,[i(t'+\Delta t') - i(t')] = V_0\,, \qquad \text{(C-4)}$$

Where the RHS of C-3 comes about because $v(t)$ represents a Dirac delta function of duration $\Delta t'$ and amplitude V_0.

• We assume that $\Delta t'$ is so short that during this excitation time $i(t)$ is not too large in a way that $R\int_{t'}^{t'+\Delta t'} i(t)\,dt \approx 0$. Figure below shows an exaggerated version of this scenario.

• Then (C-4) simplifies to

$$L\,[i(t'+\Delta t') - i(t')] = V_0 \qquad \text{(C-5)}$$

• We now consider the solution to (C-2), i.e. the solution to $R\,i(t) + L\frac{di(t)}{dt} = 0$ for $t > t' + \Delta t'$. The solution is given by

$$i(t) = I_0\, e^{-\frac{R}{L}t} = I_0\, e^{-\frac{t}{\tau_c}} \text{ for } t > t' + \Delta t' \text{ (C-6)}$$

• Since $i(t)$ is continuous then $i(t'+\Delta t') = I_0\, e^{-\frac{R}{L}(t'+\Delta t')}$. Also from Fig. 2 we see $i(t') = 0$. Using these results in (C-5) we have

$$L\,I_0\, e^{-\frac{R}{L}(t'+\Delta t')} = V_0 \Rightarrow I_0 = \frac{V_0}{L} e^{\frac{R}{L}(t'+\Delta t')} \qquad \text{(C-7)}$$

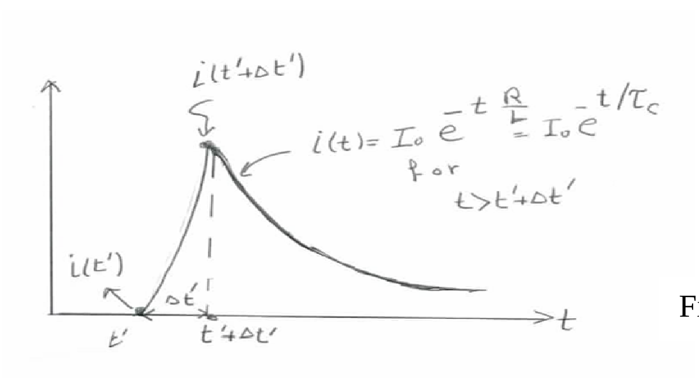

Fig. 2

• Then the solution $i(t)$ is given by

$$i(t) = \begin{cases} 0 & \textit{for} \quad t < t' \\ \frac{V_0}{L}\, e^{-\frac{R}{L}(t-t')} & \textit{for} \quad t \geq t' \end{cases} \qquad \text{(C-8)}$$

• Now let us assume that the circuit is subjected to a series of impulse functions (Dirac delta functions) each of duration $\Delta t'$ and magnitude $V_i(i = 0, 1, 2, ..., N)$ occurring at $t = t_i'(i = 0, 1, 2, ..., N)$. The total response of the circuit is given by

$$i(t) = \begin{cases} 0 & for \quad t < t_0' \\ \dfrac{V_0}{L} e^{-\frac{R}{L}(t-t_0')} & for \quad t_0' < t < t_1' \\ \dfrac{V_0}{L} e^{-\frac{R}{L}(t-t_0')} + \dfrac{V_1}{L} e^{-\frac{R}{L}(t-t_1')} & for \quad t_1' < t < t_2' \\ . & \\ . & \\ \sum_{i=0}^{N} \dfrac{V_i}{L} e^{-\frac{R}{L}(t-t_i')} & for \quad t_N' < t < t_{N+1}' \end{cases} \tag{C-9}$$

• If we now assume that $v(t)$ is a continuous arbitrary excitation, the excitation can be presented as a series of impulse functions of various amplitudes centered at different time intervals as described above. Then the results in (C-9) can easily be extended to this continuous arbitrary excitation by $\sum \to \int$, i.e., we have

$$i(t) = \int_{t'}^{t} V(t') \frac{e^{-\frac{R}{L}(t-t')}}{L} dt', \text{ or} \tag{C-10}$$

$$i(t) = \int_{t'}^{t} V(t')\, G(t, t')\, dt', \text{ where} \tag{C-11}$$

$$G(t, t') = \frac{e^{-\frac{R}{L}(t-t')}}{L} \text{ for } t > t' \text{ is the Green's function.} \tag{C-12}$$

• Equation (C-12) is our RL circuit Green's function which is the circuit response to an impulse function for $t > t'$. Knowing the Green's function as given by (C-12) the system response to any arbitrary input $v(t)$ can then be obtained from (C-11), i.e. via the convolution of the input with the Green's function.

• In the followings, since most of our formulations related to the Green's function deals with spatial coordinates we will rewrite (C-11) as

$$y(x) = \int_{a}^{b} f(x)\, G(x, x')\, dx' \tag{C-13}$$

where $y(x)$ is the output, $f(x)$ is the input, and $G(x, x')$ is the Green's function.

Properties of Green Function:

• We state the following properties associated with the Green's function without proving them.

1. $G(x, x')$ satisfies its corresponding differential equation except at $x = x'$.
2. $G(x, x')$ is symmetric with respect to x and x'.
3. $G(x, x')$ satisfies the corresponding differential equation boundary conditions.
4. $G(x, x')$ is continuous at $x = x'$.
5. $G(x, x')$ is discontinuous at $x = x'$.

Sturm-Liouville Differential Equation and Operator:

• Sturm-Liouville differential equation in one dimension is given by

$$\frac{d}{dx}\left[p(x)\,\frac{dy}{dx}\right] - q(x)\ y = f(x). \qquad \text{(C-14)}$$

• (C-14) can also be written in the language of operators as

$$\underline{\underline{L}}\ y = f(x), \qquad \text{(C-15)}$$

where $\underline{\underline{L}}$ is the Sturm-Liouville operator given by

$$\underline{\underline{L}} = \frac{d}{dx}\left[p(x)\,\frac{d}{dx}\right] - q(x) \qquad \text{(C-16)}$$

• The importance of Sturm-Liouville operator (or differential equation) can be appreciated from the following two facts.

• Any second order non-homogenous differential equation such as

$$A(x)\,\frac{d^2 y}{dx^2} + B(x)\,\frac{dy}{dx} + C(x)\ y = S(x) \qquad \text{(C-17)}$$

can be transformed into Sturm-Liouville differential equation by finding the relations between $p(x), q(x)$, and $f(x)$ with $A(x), B(x), C(x)$, and $S(x)$. (HW)

• Many of the governing differential equations in EM theory can be expressed in terms of Sturm-Liouville differential equation.

Green’s Function for Sturm-Liouville Differential Equation

• To start, we rewrite the Sturm-Liouville differential equation of (c-14) in a slightly different way, i.e.

$$\left\{\frac{d}{dx}\left[p(x)\,\frac{dy}{dx}\right]-q(x)\,y\right\}+\lambda\,r(x)\,y=f(x)\;\Rightarrow \qquad \text{(C-18)}$$

$$\left[\underline{\underline{L}}+\lambda\,r(x)\right]y=f(x) \qquad \text{(C-19)}$$

Where, λ is the eigenvalue (not to be confused with wavelength!)

• The Green’s function for Sturm-Liouville differential equation is a solution of the following equation

$$\frac{d}{dx}\left[p(x)\,\frac{dG}{dx}\right]-q(x)\,G+\lambda\,r(x)\,G=\delta(x-x') \qquad \text{(C-20)}$$

• The solutions to (C-20) over the domain (a,b) is given by

$$G(x,x')=\begin{cases}\dfrac{y_2(x')}{p(x')\,W(x')}\,y_1(x) & for \quad a\le x\le x' \\[2ex] \dfrac{y_1(x')}{p(x')\,W(x')}\,y_2(x) & for \quad x'\le x\le b\end{cases}$$

where $W(x')=y_1(x)\;y_2'(x')-y_2(x)\;y_1'(x')$ is known as the Wronskin and $y_1(x)$ and $y_2(x)$ are the nontrivial solutions of the homogenous equation (C-18). $y_1'(x')$ and $y_2'(x')$ are the derivatives of $y_1(x)$ and $y_2(x)$ with respect to their argument.

• Clearly in the above formulation we are assuming that solutions to the homogenous differential equation of (C-18) are known.