# Dispersive Effects: Velocities and Indices



- **Equivalent representations of dispersive effects:**

$$\omega(k) = \omega(k_0) + \left.\frac{d\omega}{dk}\right|_{k_0}(k-k_0) + \frac{1}{2}\left.\frac{d^2\omega}{dk^2}\right|_{k_0}(k-k_0)^2 + \ldots$$

$$= v_p\, k_0 + v_g(k-k_0) + (1/2)\,\beta(k-k_0)^2 + \ldots$$

$$k(\omega) = k(\omega_0) + \left.\frac{dk}{d\omega}\right|_{\omega_0}(\omega-\omega_0) + \frac{1}{2}\left.\frac{d^2k}{d\omega^2}\right|_{\omega_0}(\omega-\omega_0)^2 + \ldots$$

$$= v_p^{-1}\omega_0 + v_g^{-1}(\omega-\omega_0) + (1/2)\,\psi(\omega-\omega_0)^2 + \ldots$$

$$k(\omega) = \frac{\omega}{c}\, n_p(\omega) = \frac{1}{c}\left\{\omega_o\, n_p\big|_{\omega_0} + n_g\big|_{\omega_0}(\omega-\omega_0) + \frac{1}{2}\left[2\frac{dn_p}{d\omega} + \omega\frac{d^2 n_p}{d\omega^2}\right](\omega-\omega_0)^2 + \ldots\right\}$$

$$v_p = \omega/k = c/n_p$$

$$v_g = \frac{d\omega}{dk} = \frac{c}{n_g(\omega)} = \frac{c}{n_p + \omega\, dn_p/d\omega}$$

$$\psi = \frac{d^2k}{d\omega^2} = \frac{-\beta}{v_g^3} = -\frac{1}{v_g^2}\left(\frac{d}{d\omega}v_g\right) = GVD$$

$$n_g = n_p + \omega\, dn_p(\omega)/d\omega$$

# Dispersive Effects: Delays

- **More general representation using transfer function** $T(\omega)=|T(\omega)|\, e^{j\phi(\omega)}$
  - **Applicable to both spatially extended and spatially negligible systems**

- **Expanding transmission phase as a function of frequency: phase delay, group delay, group delay dispersion, …**

$$\phi(\omega)=\phi(\omega_0)+\left.\frac{\partial\phi}{\partial\omega}\right|_{\omega_0}(\omega-\omega_0)+\frac{1}{2}\left.\frac{\partial^2\phi}{\partial\omega^2}\right|_{\omega_0}(\omega-\omega_0)^2+\ldots$$

$$\phi(\omega)=-\left\{\tau_p\omega_0+\tau_g(\omega-\omega_0)+(1/2)\,GDD\,(\omega-\omega_0)^2+\ldots\right\}$$

$$\tau_p=-\phi/\omega\big|_{\omega_0} \qquad \tau_g=-\partial\phi/\partial\omega\big|_{\omega_0} \qquad GDD=-\partial^2\phi/\partial\omega^2\big|_{\omega_0}$$

- **Relations between phase and group delays and phase and group velocities for spatially extended system of the physical length *L*:**

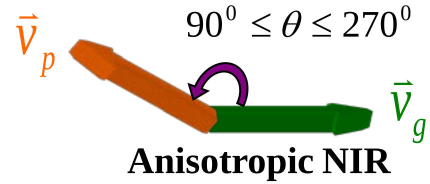
$$v_p=\frac{L}{\tau_p}=\frac{c}{n_p(\omega)} \qquad v_g=\frac{L}{\tau_g}=\frac{c}{n_g(\omega)}$$

# Signs Associated with Various Delays (Dispersion Engineering)

| $\tau_p(v_p, n_p)$ | $\tau_g(v_g, n_g)$ | *GDD (GVD, GID)* |
|---|---|---|
| + | + | + |
| + | + | - |
| + | - | + |
| + | - | - |
| - | + | + |
| - | + | - |
| - | - | + |
| - | - | - |

# Negative Index of Refraction (Negative Phase Delay)

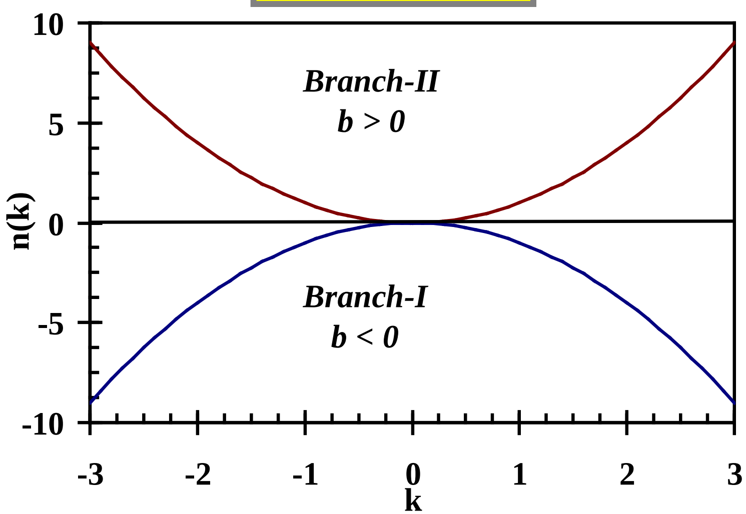
- Anti-parallel phase and group velocities will lead to a unique negative index of refraction (NIR)

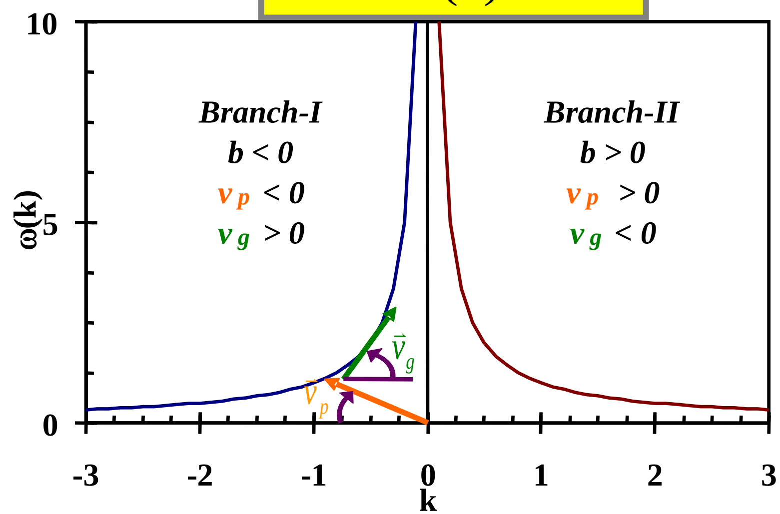




$$\frac{\hat{k}_x}{n(k)} + \frac{\hat{k}_y}{n(k)} + \frac{\hat{k}_z}{n(k)} - \frac{k\nabla_k n(k)}{n(k)^2} = d_x \frac{\hat{k}_x}{n(k)} + d_y \frac{\hat{k}_y}{n(k)} + d_z \frac{\hat{k}_z}{n(k)}$$

$$n(k) = b\,k^2$$

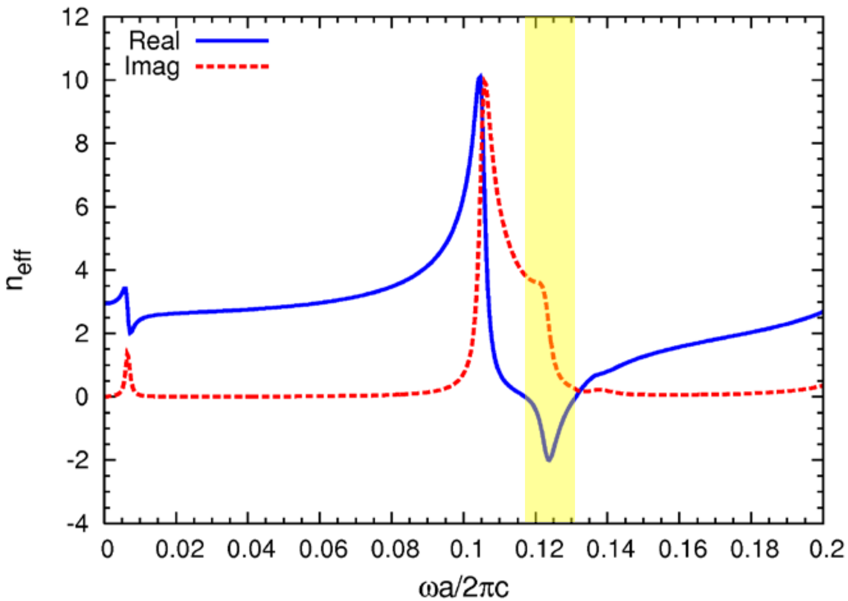
$$\omega(k) = \frac{c_o k}{n(k)} = \frac{c_o}{b\,k}$$

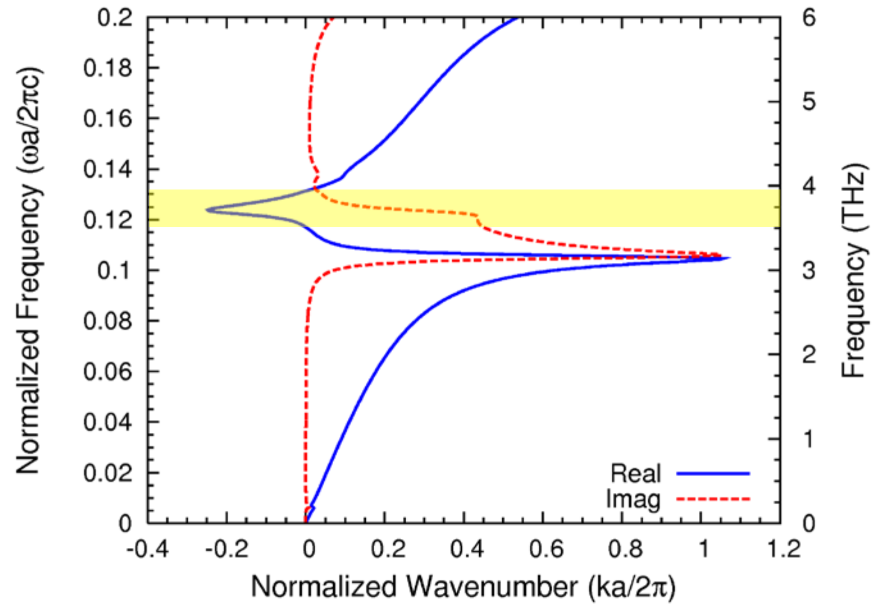


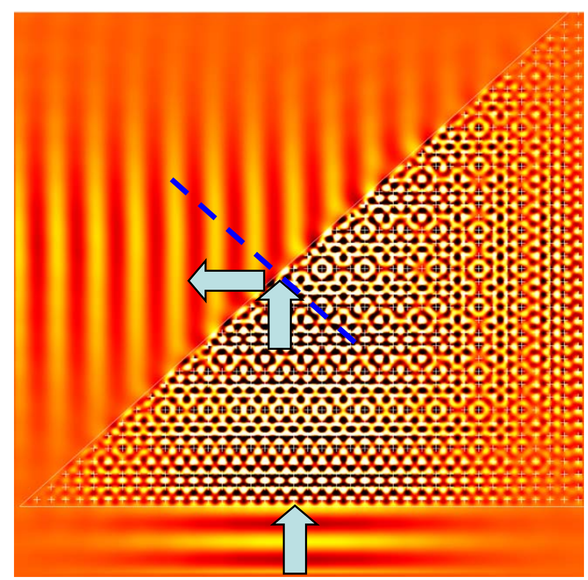

Mojahedi et al. Journal of Optical Society of America -B

# Results: Negative Index with Coated Spheres

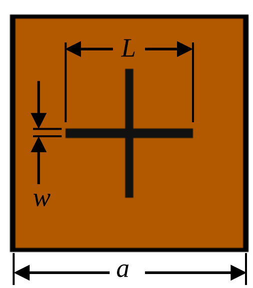



- FCC lattice, $a$ = 10 μm, $LiTaO_3$ core $r_{inner} = 0.33a$, Drude shell thickness = $0.02a$, $\omega_p$ = 3.98 THz, $\gamma = \omega_p/100$
- Negative refraction (not necessarily NIR) can be achieved with photonic crystals



- The scattering elements are metallic crosses



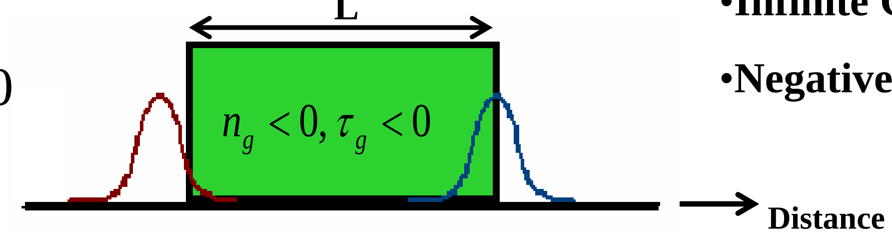
# Negative or Superluminal Group Velocity and Group Delay

## Group Velocity (GV)

- **Normal GV,** $n_g > 1$
- **Superluminal GV,** $0 < n_g < 1$
- **Infinite GV,** $n_g = 0$
- **Negative GV,** $n_g < 0$

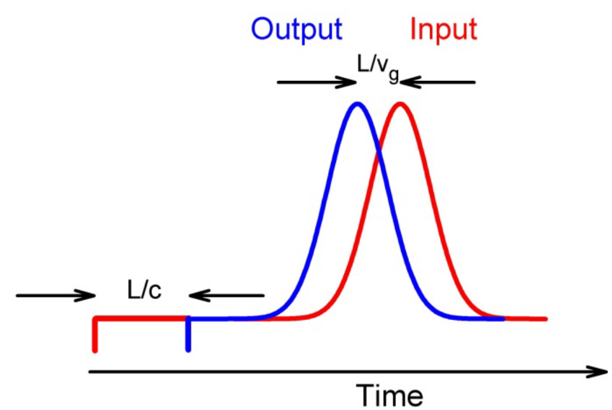
## Group Delay (GD)

- **Normal GD,** $\tau_g > L/c$
- **Superluminal GD,** $\tau_g < L/c$
- **Infinite GV,** $\tau_g = 0$
- **Negative GV,** $\tau_g < 0$




$$v_g = c/n_g(\omega) = L/\tau_g$$

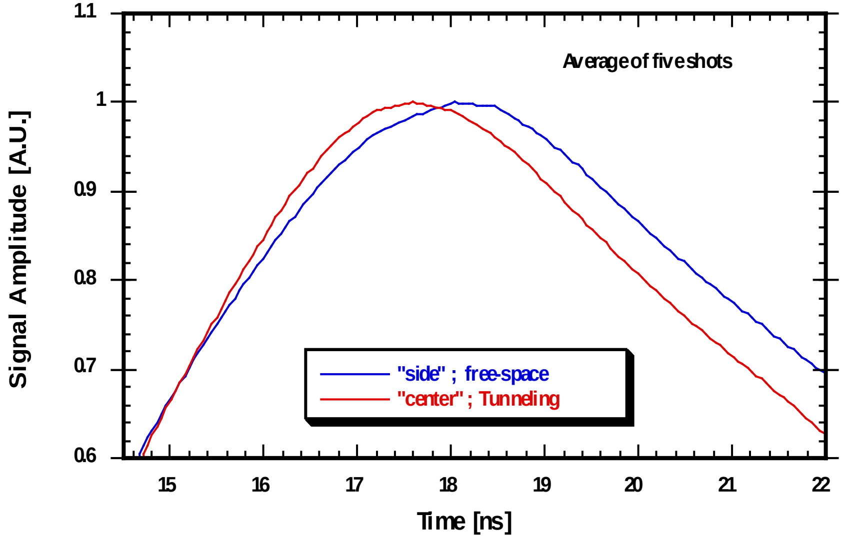
- **From purely theoretical point of view information velocity and group velocity are not the same under all circumstances**
- **Input and output peaks are not causally connected**
- **The fact that output peak precedes the input peak is due to pulse reshaping**
- **Abnormal group delay (group velocity) do not violate relativistic causality since the "front" delay is always luminal (*L/c*)**



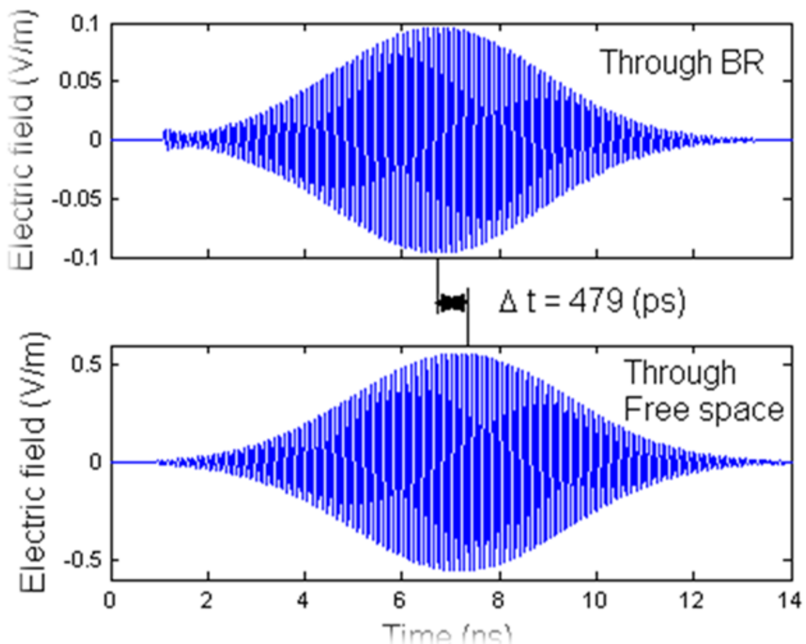
# Superluminal Group Velocity

Large number of experiments have shown that group velocity, similar to phase velocity, can exceed the speed of light in vacuum (*c*), i.e. group velocity can be superluminal.

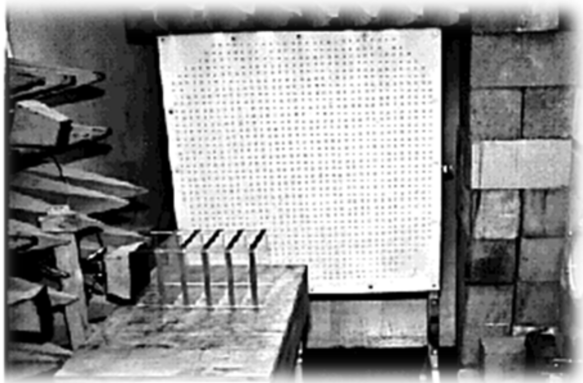
## Experimental Results




## Simulated Results




## Experimental Setup



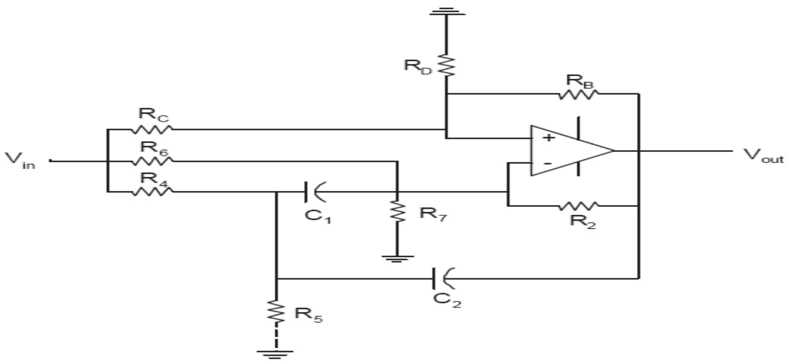


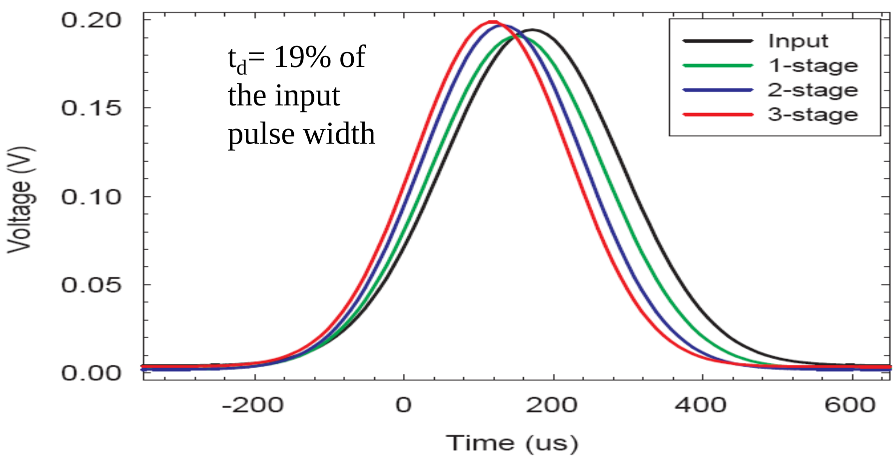
**Mojahedi, et al., Phys. Rev.**

# Negative Group Velocity (Negative Group Delay) in Active Medium

Group velocity can also be negative: The peak of the output pulse leaves the medium before the peak of the input pulse enters the medium

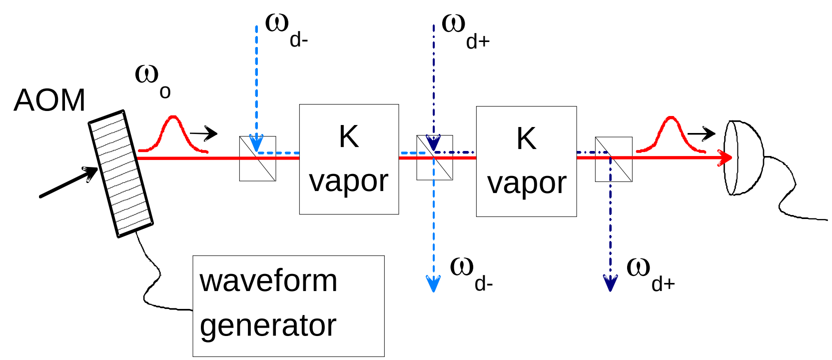
## Negative Group Delay in Electronics

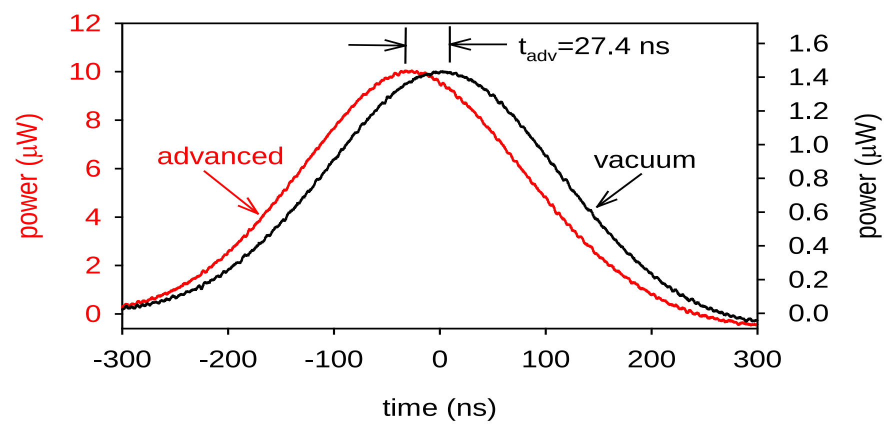



***Mojahedi*** et al. IEEE AP-S Int. Symp.,

## Negative Group Velocity in Optics




***Dogariu et al.*** IEEE J. Sel. Top. Quantum Electron.

# Example: Single Resonance Lorentzian

Material parameters:

$$n(\omega) = \left[ 1 + \frac{\omega_p^2}{\omega_0^2 - \omega^2 - 2j\delta\omega} \right]^{1/2}$$

$\omega_0$ : Resonance frequency

$\delta$ : Damping factor

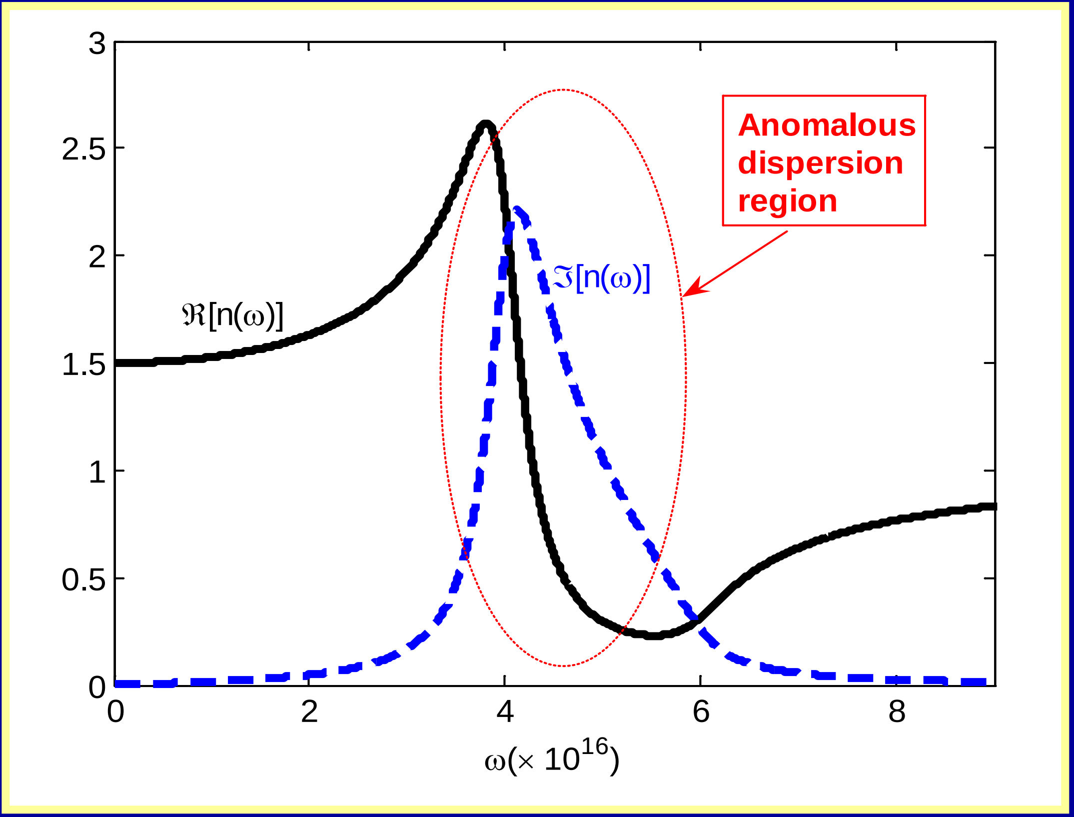
$\omega_p$ : Plasma frequency

Brillouin's choice of the material parameters:

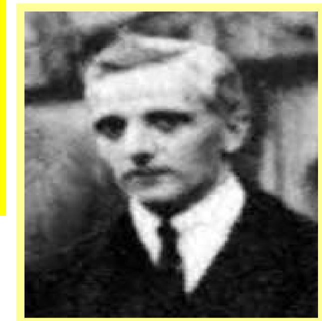
$$\omega_0 = 4 \times 10^{16}$$

$$\delta = 0.28 \times 10^{16}$$

$$\omega_p = \sqrt{20} \times 10^{16}$$

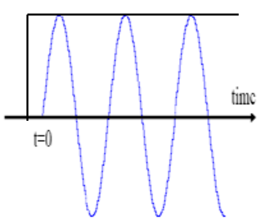


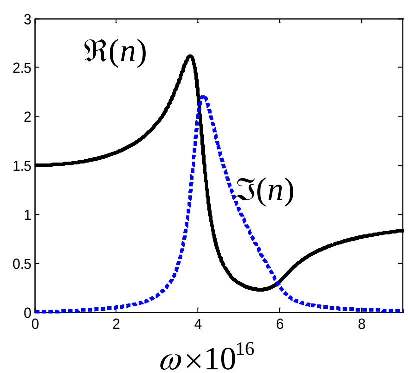
# What Are Precursors

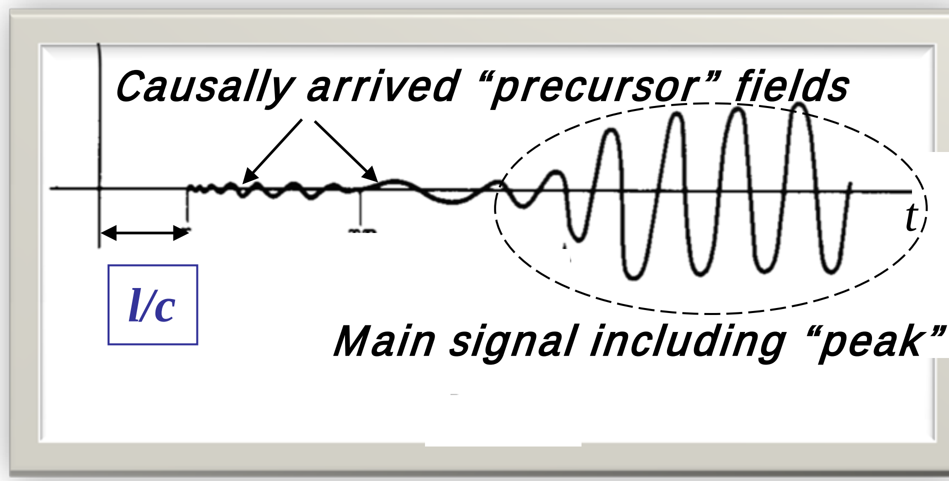


Precursors or forerunners are transient fields in space and time, first studied by Sommerfeld and Brillouin.

- They considered the propagation of a step function in a Lorentzian medium using the steepest descent method as an asymptotic technique to calculate the field inside the dispersive medium.




**A.Sommerfeld, Physik. Z. 8, 841, 1907.**
**L. Brillouin, Wave Propagation and Group Velocity, Academic,**

# Analytical Calculation of Precursors: Asymptotic Techniques

- The general formulation describing the propagation of an arbitrary plane-wave pulse through a dispersive medium:

$$A(z,t) = \frac{1}{2\pi}\int_C f(\omega)\exp(ik(\omega)z - \omega t)d\omega,$$

$$f(\omega) = \int_C f(t)\exp(-i\omega t)d\omega$$

- A(z,t) can be written in form that is more suitable for asymptotic techniques:

$$A(z,t) = \frac{1}{2\pi}\int_C f(\omega)\exp(\frac{z}{c}\phi(\omega,\theta))d\omega$$

Phase function: $\phi(\omega,\theta) = i\omega[n(\omega) - \theta],$ $\theta = \frac{ct}{z}$

$\theta > 1$ : Subluminal propagation

$\theta < 1$ : Superluminal propagation

# Passive and Active Lorentzian Media

Passive Lorentzian

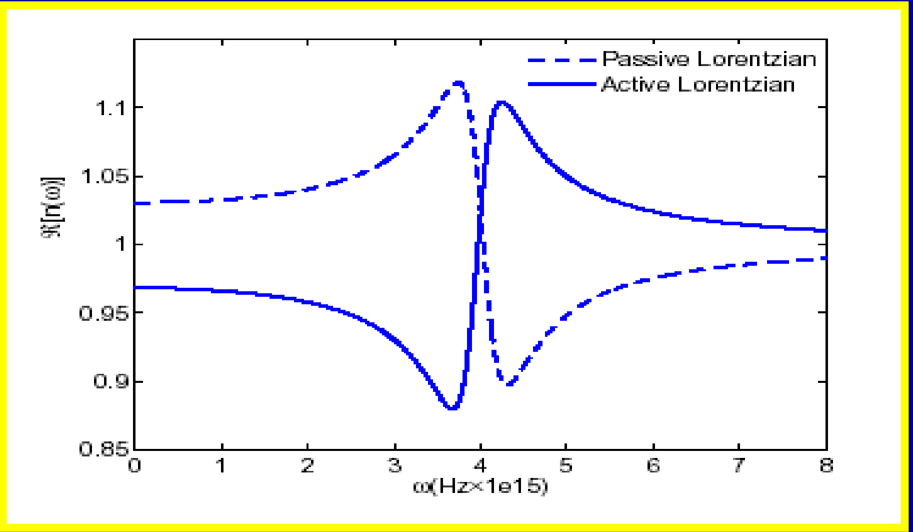
$$n(\omega) = \left(1 + \frac{\omega_p^2}{\omega_0^2 - \omega^2 - j\omega\delta}\right)^{1/2}$$

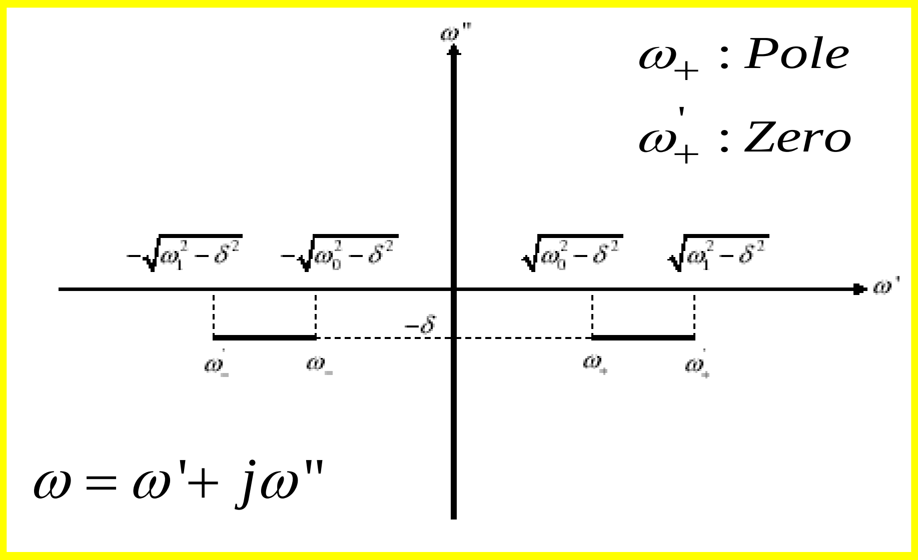
$$\omega_p \rightarrow j\omega_p$$

Active Lorentzian

$$n(\omega) = \left(1 - \frac{\omega_p^2}{\omega_0^2 - \omega^2 - j\omega\delta}\right)^{1/2}$$

$$\delta < \omega_p < \omega_0$$



# Steepest Descent Method

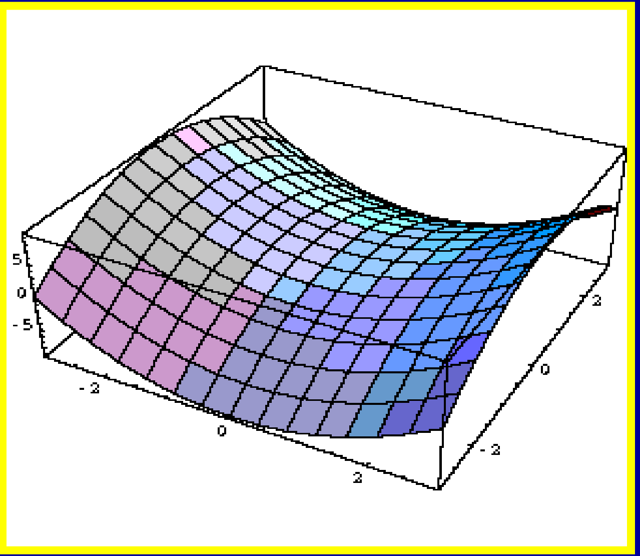
• The steepest descent method is a powerful method for studying the large z asymptotic of the integrals of the form

$$I(\omega)=\int_C f(\omega)e^{z\psi(\omega)}d\omega,$$

$$\psi(\omega)=\phi(\omega)/c,\quad \phi(\omega)=X(\omega',\omega'')+iY(\omega',\omega'')$$

• The basic idea is to utilize the analyticity of the integrand to justify deforming the contour C to a new contour C' on which phase function has a constant imaginary part.

$$I(\omega)=e^{izY}\int_{C'} f(\omega)e^{zX(\omega',\omega'')}d\omega$$

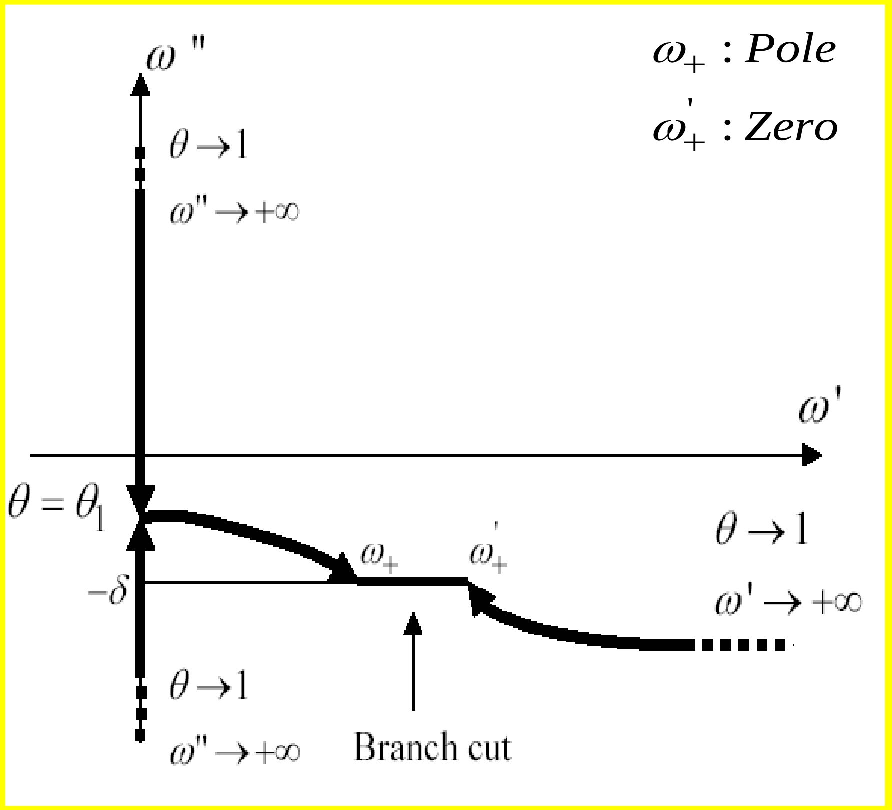



• The paths of steepest descent will go through a point $z_0$ for which the derivative of the phase function is zero. Such a point is called saddle point.

$$\psi'(\omega,\theta)=n(\omega)-\omega n'(\omega)-\theta=0$$

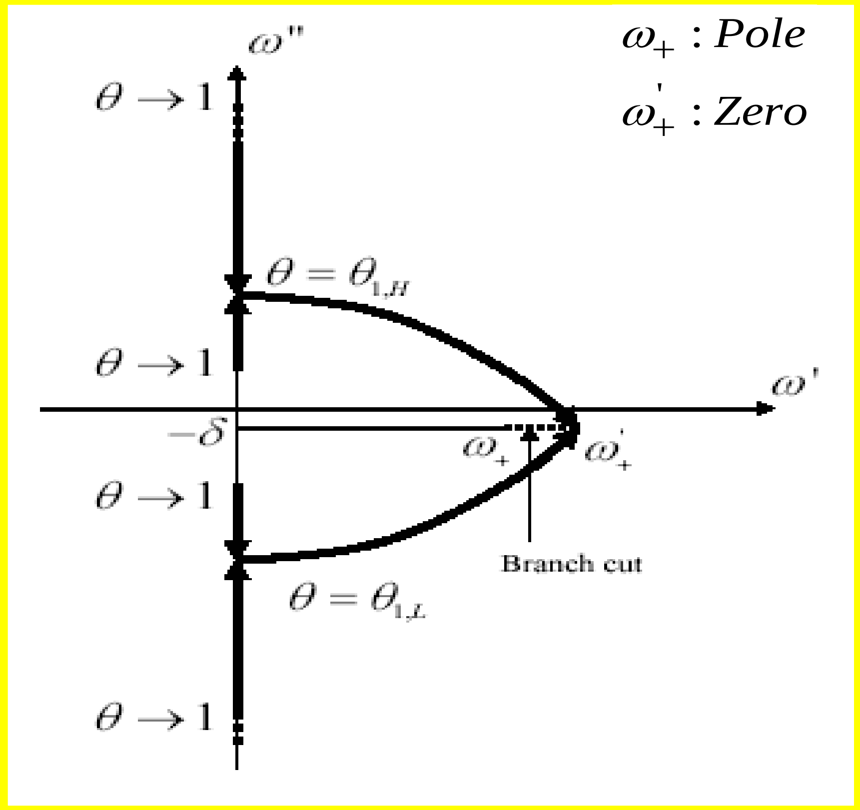
# Lorentzian Media: Saddle Point Evolution

**Passive Lorentzian**




G.C. Sherman and K.E. Oughstun, *Phys. Rev. Lett.*, 1981.

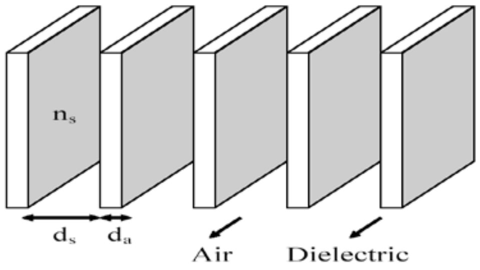
**Active Lorentzian**



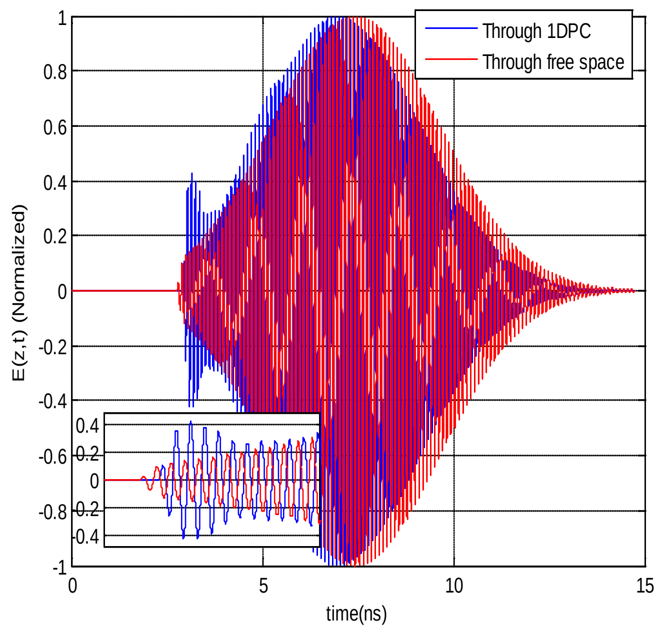

**R. Safian,, C.D. Sarris, and M. Mojahedi**
*Physical Review E*, 2007.

# Precursor In Distributed Bragg Reflector

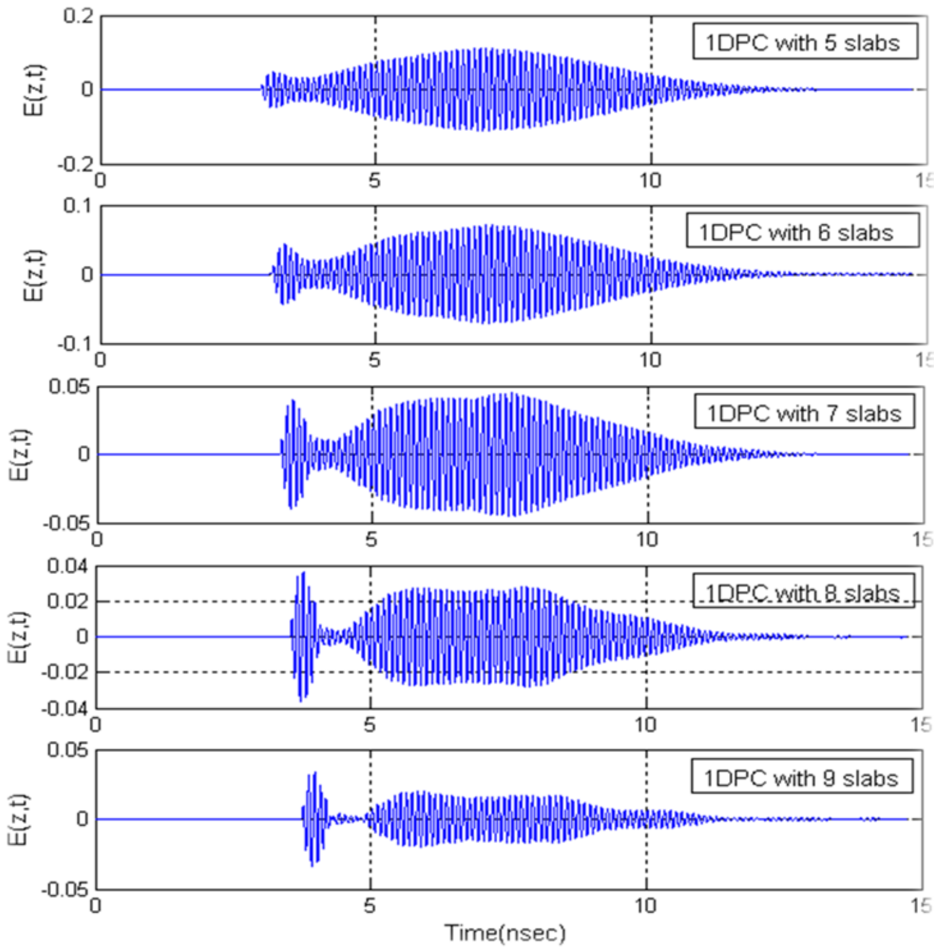
## Distributed Bragg Reflector (DBR)




**Group velocity is superluminal but precursor is subluminal**




## Causal Gaussian Pulse in DBR




**Mojahedi, et al., Phys. Rev.**

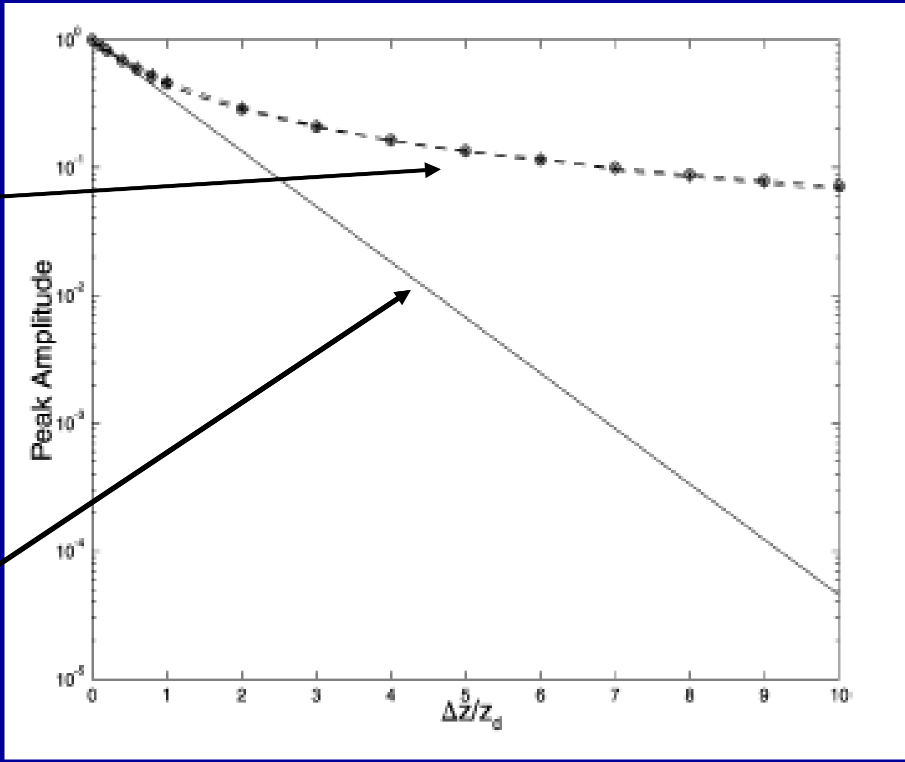
# Precursor Field : Attenuation Rate

Precursors

Attenuation rate ~ $\frac{1}{\sqrt{z}}$

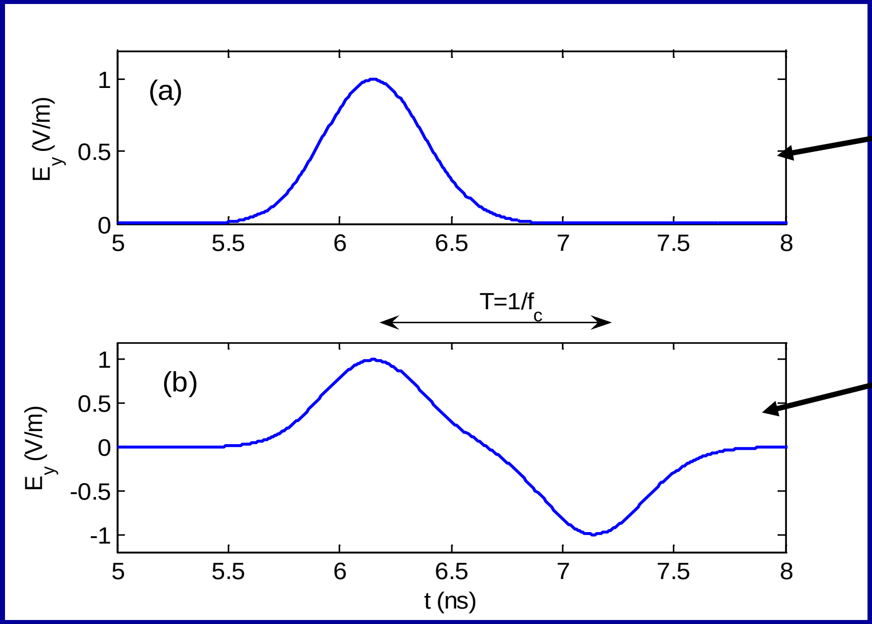
Main part of the pulse

Attenuation rate ~ $\exp(-z)$




K.E. Oughstun, *IEEE Transactions* on *Antennas and Propagation*

# Long Range Propagation: Double Brillouin Pulse

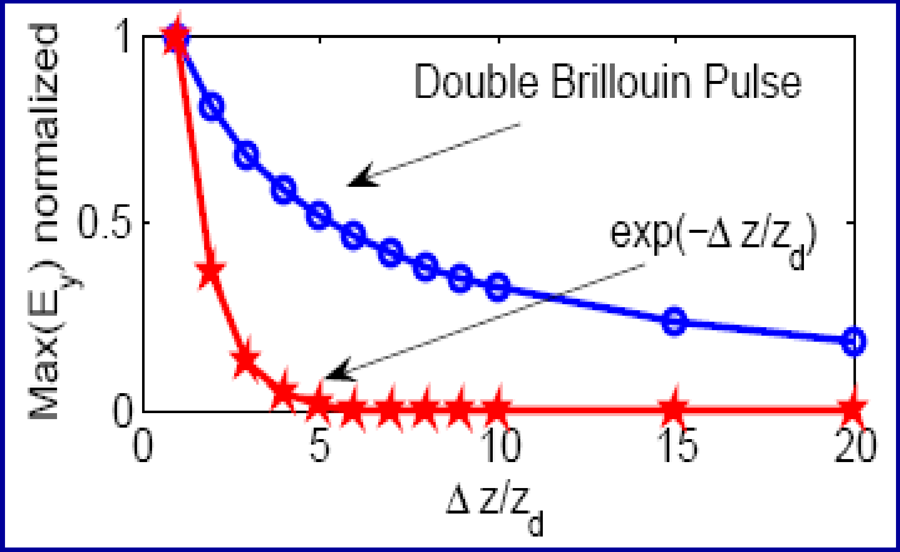



Single Brillouin pulse

Double Brillouin pulse

$$z_d = 1/\alpha$$

$\alpha$: the real part of the propagation constant.




**K.E. Oughstun,**
***IEEE Transactions* on *Antennas and Propagation.***