Homework #3 ECE 1228

1) The electric and magnetic responses of a medium are characterized by the Lorentz-Lorenz dispersion. For example, the electric permittivity is given by:

$$\varepsilon = 1 - \frac{\omega_{ep}^2 - \omega_{eo}^2}{\omega^2 - \omega_{eo}^2 - j\gamma_e \omega},$$

where ω_{ep} is the electric plasma frequencies, ω_{eo} is the electric resonance frequencies, and γ_e is the phenomenological electric damping constants.

a) By inspection write the magnetic response (permeability)

b) For the following parameters; $\omega_{eo} = 0$, $\omega_{mo} = 2\pi \times 21$, $\omega_{ep} = 2\pi \times 28$, $\omega_{mp} = 2\pi \times 24.5$ (GHz), $\gamma_e = 1.6 \times 10^9$, $\gamma_m = 4 \times 10^9$ (1/s), plot the real and imaginary parts of the permittivity and permeability for the frequency range of 18 to 26 GHz.

c) Plot the index of refraction for the same frequency range.

d) Can you identify the frequency region for which the index of refraction is negative?

2) Transmitter (T) of a time-harmonic wave of frequency ν moves with velocity $\vec{U}$ at an angle θ relative to the direct line to a stationary receiver R (see the figure 3.1).

a) Drive the expression for the frequency detected by the receiver R, assuming that the medium between T and R has a positive index of refraction n. (Apply the appropriate approximations.)

b) How is the expression obtained in part (a) is modified if the medium is metamaterial with negative index of refraction.

c) From the physical point of view, how is the situation in part (b) different from part (a)?

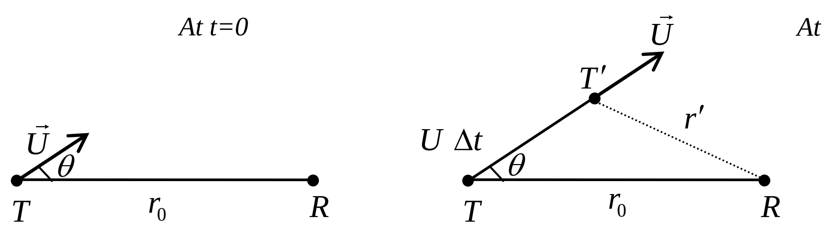

Fig. 3.1

3) Prove first Helmholtz's theorem, i.e. if vector $\vec{M}_1$ is defined by its divergence ($\nabla \cdot \vec{M}_1 = s$) and its curl ($\nabla \times \vec{M}_1 = \vec{C}$) within a region, and its normal component $\vec{M}_{1n}$ over the boundary, then $\vec{M}_1$ is uniquely specified.

Note: Assume there is a vector $\vec{M}_2$ with its divergence and curl equal to s and $\vec{C}$ respectively, then show that $\vec{M}_1 = \vec{M}_2$.

4) Prove that $-\nabla^2 \frac{1}{R} = 4\pi\, \delta^3(R)$ where $R = |\vec{R}|$ is the position vector.

5) Using Maxwell's equations given in the class notes, derive the Poynting theorem in both differential and integral form for instantaneous fields. Assume a linear, homogeneous medium with no temporal dispersion.

6) Figure shows a loop of wire located in *x*-*y* plane carrying current *I*. The loop's radius is R_l.
a) Calculate the magnetic field flux density, $\vec{B}$, along the loop axis at a distance *z* from its center.
b) Simplify the results in (a) for large distances along the *z*-axis ($z >> R_l$)
c) Express the results in (b) in terms of magnetic dipole moment. Make sure you write the expression in vector form.
d) In keeping with your understanding of magnetic bar's north and south poles, designate the north and south poles for the current carrying loop shown in the figure.

Hint: Use Biot-Savart law which states the following: A differential current element, $I\,\overrightarrow{dl'}$, produces a differential magnetic field, $\overrightarrow{dB}$, at a distance *R* from the current element given by

$$\overrightarrow{dB} = \frac{\mu_0 I \ \overrightarrow{dl'} \times \vec{R}}{4\pi \ R^3} \ \rightarrow \ \vec{B} = \int \frac{\mu_0 I \ \overrightarrow{dl'} \times \vec{R}}{4\pi \ R^3}.$$

Note that integration is carried over the source (current) and *R* points from the current elements to the point of observation.