Homework #7 ECE 1228

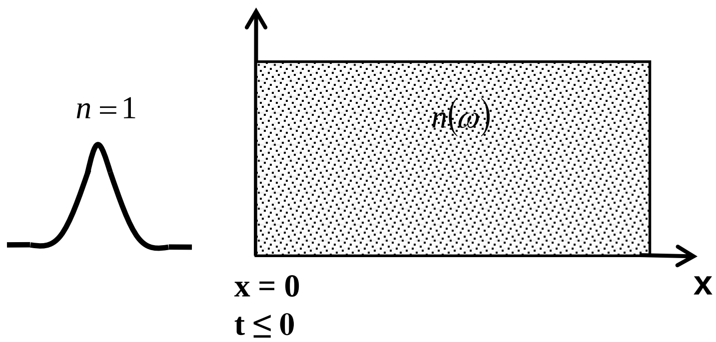
1) Prove the followings:

a) Show that for the orthonormal basis $\{u_i(r)\}$, the coefficients of expansions for $\psi(r)$ [$\psi(r)=\sum_{i=1} c_i\, u_i(r)$] are given by $c_i=(u_i,\psi)$.

b) Let $\psi_1(r)$ and $\psi_2(r)$ be both square integrable functions, show that $\lambda_1\,\psi_1(r)+\lambda_2\,\psi_2(r)$ is also square integrable, where λ_1 and λ_2 are in general constant complex numbers.

2) Given $\psi(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} dk\;\; \overline{\psi}(k)\, e^{ik\,x} dk$ show that $\int_{-\infty}^{+\infty} dx\;\; |\psi(x)|^2=\int_{-\infty}^{+\infty} dk\;\; |\overline{\psi}(k)|^2$. This is the Parseval theorem which implies that the norm of a wave function can be calculated in either x or k space.

3) In this problem we investigate the relation between causality and analyticity. Consider the case of one-dimensional wave propagation in a semi-infinite medium characterized with $n(\omega)$ and filling the $x>0$ region. Let the wave packet $U(x,t)$ be normally incident from vacuum on the medium and suppose $\frac{\partial U(x,t)}{\partial x}\Big|_{x=0}=0$.



a) Show that a general solution in the medium is given by

$$U(x,t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\frac{2}{1+n(\omega)}\,\overline{U}(\omega)\, e^{ik(\omega)x-i\omega\, t}d\omega$$

where $\overline{U}(\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}U(0,t)\, e^{i\omega\, t}dt$.

b) We require that $U(x,t)$ to be causal and bounded, i.e., $U(0,t)=0$ for $t\le 0$ and $|U(0,t)|\le M$, where $M\in\{R\}$. Show that this implies that $\overline{U}(\omega)$ is analytical in the upper half part (UHP) of the complex frequency plane ($\omega\to\eta+i\xi$).

c) Suppose $U(x,t)$ is not causal, what will be the conclusion of part (b). Justify your answer.

4) Show that the set $\{V_k(x)\}$ where $V_k(x)=\frac{1}{\sqrt{2\pi}}e^{i\,k\,x}$ are plane waves, form an orthonormal basis that also satisfies the closure relation.

Start by proving the following $\frac{1}{2\pi}\int_{-\infty}^{+\infty}e^{i\,\xi(\eta'-\eta)}d\xi=\delta(\eta'-\eta)$

5) Let the wave function $\psi(\vec{r})$ be expanded in the orthonormal bases $\{U_i(\vec{r})\}$ according to $\psi(\vec{r}) = \sum_{i=1} c_i\, U_i(\vec{r})$ where c_i are the coefficients of expansion. Show that the normalization requirement $\iiint |\psi(\vec{r})|^2\, dr^3 = 1$,

implies $\sum_{i=1} |c_i|^2 = 1$

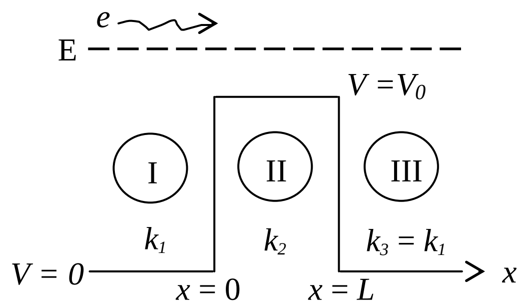


Fig. 4.1

6) An electron potential barrier is shown in Fig. 4.1. Suppose an electron with energy $E>V_0$ is propagating from left to right (normal incidence).
a) Write the wave functions $\phi(x)$ in each region in terms of k_1, k_2, and k_3.
b) Calculate the square of the transmission coefficient (T) and reflection coefficient (R) for the above case in terms of k_1, k_2, and k_3 and sin function.
c) Express the above transmission function T, in terms of energy E and potential V_0.
d) What are the minimum and maximum values for T?
e) For what values of k_2 and L the transmission is maximum (this is called resonance condition.)
f) Schematically plot T as a function of L for a given E and V_0. What optical system has a similar transmission function?
g) For the case of $E<V_0$, calculate the T as a function of E and V_0.