Conservation of Energy & Poynting Theorem

• From **Maxwell's equations** we have

$$\nabla\times\vec{E}=-\vec{M}_i-\frac{\partial\vec{B}}{\partial t}=-\vec{M}_i-\frac{\partial}{\partial t}\mu\,\vec{H}=-\vec{M}_i-\vec{M}_d$$

$$\nabla\times\vec{H}=\vec{J}_i+\sigma\,\vec{E}+\frac{\partial\vec{D}}{\partial t}=\vec{J}_i+\sigma\,\vec{E}+\frac{\partial}{\partial t}\varepsilon\,\vec{E}=\vec{J}_i+\vec{J}_c+\vec{J}_d$$

• From above it **can be shown** (HW)

$$\nabla\cdot\vec{E}\times\vec{H}+\vec{H}\cdot\left(\vec{M}_i+\vec{M}_d\right)+\vec{E}\cdot\left(\vec{J}_i+\vec{J}_c+\vec{J}_d\right)=0\ \text{ or}$$

$$\oiint_S\vec{E}\times\vec{H}\cdot d\vec{s}+\iiint_v\vec{H}\cdot\left(\vec{M}_i+\vec{M}_d\right)dv+\iiint_v\vec{E}\cdot\left(\vec{J}_i+\vec{J}_c+\vec{J}_d\right)dv=0$$

• We **rewrite** the above according to

$$\oiint_S\vec{E}\times\vec{H}\cdot d\vec{s}+\iiint_v\left(\vec{H}\cdot\vec{M}_i+\vec{E}\cdot\vec{J}_i\right)dv+\iiint_v\left(\vec{H}\cdot\vec{M}_d\right)dv+\iiint_v\left(\vec{E}\cdot\vec{J}_d\right)dv+\iiint_v\vec{E}\cdot\vec{J}_c\ dv=0$$

• let us **define**

$$\vec{H}\cdot\vec{M}_i+\vec{E}\cdot\vec{J}_i=-\rho_{\text{supp}}$$

$\rho_{\text{supp}}\equiv$ **Supplied power density** [Watt/m^3]

$$\iiint_v\left(\vec{H}\cdot\vec{M}_i+\vec{E}\cdot\vec{J}_i\right)dv=-\iiint_v\rho_{\text{supp}}\ dv=-P_{\text{supp}}\ \text{[Watt]}$$

• $\displaystyle\iiint_v\vec{H}\cdot\vec{M}_d\ dv=\iiint_v\vec{H}\cdot\frac{\partial}{\partial t}\vec{B}\ dv=\iiint_v\vec{H}\cdot\frac{\partial}{\partial t}\mu\vec{H}\ dv=$

$$\iiint_v\frac{1}{2}\frac{\partial}{\partial t}\mu\,\vec{H}\cdot\vec{H}dv=\frac{\partial}{\partial t}\underbrace{\iiint_v\frac{1}{2}\mu\left|\vec{H}\right|^2dv}_{W_m}=\frac{\partial}{\partial t}W_m$$

$$W_m:\left[\frac{\text{H}}{\text{m}}\cdot\frac{\text{A}^2}{\text{m}^2}\text{m}^3=\text{HA}^2=\frac{\text{V}\cdot\text{s}}{\text{A}}\text{A}^2=\text{Watt}\cdot\text{s}=\text{J}\right]$$

$\dfrac{\partial}{\partial t}W_m\equiv$ **Rate of change of stored magnetic energy**: [J/s = Watt]

• $\displaystyle\iiint_v\vec{E}\cdot\vec{J}_d\ dv=\iiint_v\vec{E}\cdot\frac{\partial}{\partial t}\varepsilon\,\vec{E}\ dv=\iiint_v\frac{1}{2}\frac{\partial}{\partial t}\varepsilon\left|\vec{E}\right|^2\ dv=\frac{\partial}{\partial t}\underbrace{\iiint_v\frac{1}{2}\varepsilon\left|\vec{E}\right|^2\ dv}_{W_e}=\frac{\partial}{\partial t}W_e$

$$W_e : \left[\frac{\text{F}}{\text{m}} \cdot \frac{\text{V}^2}{\text{m}^2} \text{m}^3 = \text{J} \right]$$

$\frac{\partial}{\partial t} W_e \equiv$ **Rate of change of stored electric energy** [J/s = Watt]

• $\iiint_v \vec{E} \cdot \vec{J}_c \, dv = \iiint_v \vec{E} \cdot \sigma \vec{E} \, dv = \iiint_v \sigma \left| \vec{E} \right|^2 dv = P_{\text{disp}}$

$P_{\text{disp}} \equiv$ **Dissipated power** (ohmic loss): $\left[\frac{1}{\Omega \cdot \text{m}} \cdot \frac{\text{V}^2}{\text{m}^2} \cdot \text{m}^3 = \frac{\text{V}^2}{\Omega} = \text{Watt} \right]$

• $\oiint \vec{E} \times \vec{H} \cdot d\vec{s} = P_{\text{exit}}$

$P_{\text{exit}} \equiv$ **Power exiting the volume enclosed by surface** S : $\left[\frac{\text{A}}{\text{m}} \cdot \frac{\text{V}}{\text{m}} \text{m}^2 = \text{Watt} \right]$

• We can **rewrite Poynting Equation**

$$\oiint_S \vec{E} \times \vec{H} \cdot d\vec{s} + \iiint_v \left(\vec{H} \cdot \vec{M}_d \right) dv + \iiint_v \left(\vec{E} \cdot \vec{J}_d \right) dv + \iiint_v \vec{E} \cdot \vec{J}_c \, dv = -\iiint \left(\vec{H} \cdot \vec{M}_i + \vec{E} \cdot \vec{J}_i \right) dv$$

$P_{\text{exit}} + \frac{\partial}{\partial t} W_m + \frac{\partial}{\partial t} W_e + P_{\text{disp}} = P_{\text{sup}}$ This is **Conservation of Energy**

Time Harmonic or Sinusoidal Steady State Electromagnetic Fields

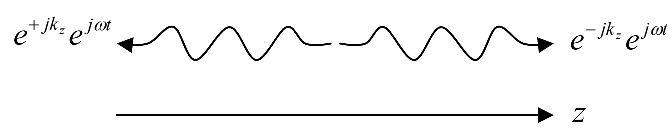
• In time harmonic picture the **instantaneous field** $\vec{E}(x, y, z, t)$ and the **complex spatial field** $\vec{E}(x, y, z)$ are related by

$$E(x, y, z, t) = \text{Re}\left[\vec{E}(x, y, z) e^{j\omega t} \right]$$
$$H(x, y, z, t) = \text{Re}\left[\vec{H}(x, y, z) e^{j\omega t} \right]$$
$$\vdots$$

• **Remark 1:** Fields can also be described as **imaginary parts**
$E(x, y, z, t) = \text{Im}\left[E(x, y, z) e^{j\omega t} \right]$

• **Remark 2:** Most **engineering books (not all)** use time dependency of $e^{j\omega t}$, most **physics books (not all)** use $e^{-i\omega t}$, $i \leftrightarrow -j$

• **Remark 3: We will see that for** $e^{j\omega t}$ the wave $e^{-jkz} e^{j\omega t}$ and for $e^{-i\omega t}$ the wave $e^{ikz} e^{-i\omega t}$ are **positively traveling waves**



• With help of $e^{j\omega t}$ **time dependency** $\frac{\partial}{\partial t} \Leftrightarrow j\omega$

• This is similar to **circuit analysis** for which $\frac{\partial}{\partial t} \leftrightarrow s = \sigma + j\omega \leftrightarrow j\omega$

• **Ex:** $\nabla \times \vec{E} = -\frac{\partial B}{\partial t}$

$$\nabla \times \vec{E}(\vec{r})e^{j\omega t} = -\frac{\partial}{\partial t}\mu \vec{H}(\vec{r})\, e^{j\omega t}$$

$$e^{j\omega t}\ \nabla \times \vec{E}(\vec{r}) = -\mu j\omega\ \vec{H}(\vec{r})\, e^{j\omega t} \Rightarrow \nabla \times \vec{E}(x,y,z) = -j\omega\mu\ \vec{H}(x,y,z)$$

• Or in **integral form**

$$\oint_l \vec{E}\cdot d\vec{l} = -j\omega \iint_S \mu \vec{H}\cdot d\vec{s}$$

Poynting theorem for time harmonic fields

• $\nabla \times \vec{E} = -\vec{M}_i - j\omega\mu\ \vec{H}$ and $\nabla \times \vec{H} = \vec{J}_i + j\omega\varepsilon\ \vec{E} + \sigma\ \vec{E} \Rightarrow$ (1)

• $\nabla \times \vec{H}^* = \vec{J}_i^{\,*} - j\omega\varepsilon\ \vec{E}^* + \sigma\ \vec{E}^*$ (2)

• **From (1) and (2)** we have (HW)

$$-\frac{1}{2}\left(\vec{H}^*\cdot\vec{M}_i + \vec{E}\cdot\vec{J}_i^{\,*}\right) = \nabla\cdot\frac{1}{2}\vec{E}\times\vec{H}^* + \frac{1}{2}\sigma\left|\vec{E}\right|^2 + j\omega\left[\frac{1}{2}\mu\left|\vec{H}\right|^2 - \frac{1}{2}\varepsilon\left|\vec{E}^2\right|\right]$$

Or

$$-\frac{1}{2}\iiint_v\left(\vec{H}^*\cdot\vec{M}_i + \vec{E}\cdot\vec{J}_i^{\,*}\right)dv = \oiint_S \frac{1}{2}\vec{E}\times\vec{H}^*\cdot d\vec{s} + \iiint_v \frac{1}{2}\sigma\left|\vec{E}\right|^2 dv + j\omega\left[\iiint_v \frac{1}{2}\mu\left|\vec{H}\right|^2 dv - \iiint_v \frac{1}{2}\varepsilon\left|\vec{E}\right|^2 dv\right]$$

• **If** ε **and** μ **are complex** ($\varepsilon \to \varepsilon' - j\varepsilon''$ and $\mu \to \mu' - j\mu''$) then their imaginary parts **contribution to the dissipated power must be added to** $P_d = \iiint \frac{1}{2}\sigma\left|\vec{E}\right|^2$. In other words the term $j\omega\left[\iiint_v \frac{1}{2}\mu\left|\vec{H}\right|^2 dv - \iiint_v \frac{1}{2}\varepsilon\left|\vec{E}\right|^2 dv\right]$ is **considered as reactive (purely imaginary)**.

Poynting Vector

• **Instantaneous Poynting Vector** is defined as $\vec{\mathcal{S}} = \vec{\mathcal{E}}(\vec{r},t) \times \vec{\mathcal{H}}(\vec{r},t)$

• Note: in the followings I use the **scripted letters** $\mathcal{E}, \vec{\mathcal{H}}, \ldots$ to designate instantaneous fields, i.e. $\mathcal{E}(\vec{r},t)$ and $\vec{\mathcal{H}}(\vec{r},t)$, and **regular letters** $\vec{E}(\vec{r})$, $\vec{H}(\vec{r})$, to designate the time harmonic fields, i.e., only the spatial dependency

• We are to write the $\mathcal{S}(\vec{r},t)$ in **terms of time harmonic fields** $\vec{E}(\vec{r}), \vec{H}(\vec{r})$

$$\vec{\mathcal{S}}(\vec{r},t) = \mathrm{Re}\left[\vec{E}(\vec{r})e^{j\omega t}\right] \times \mathrm{Re}\left[\vec{H}(\vec{r},t)e^{j\omega t}\right]$$

• Note that: $\mathrm{Re}\left[\vec{A}\right] \times \mathrm{Re}\left[\vec{B}\right] \neq \mathrm{Re}\left[\vec{A} \times \vec{B}\right]$

• $\vec{\mathcal{S}}(\vec{r},t) = \vec{\mathcal{E}}(\vec{r},t) \times \vec{\mathcal{H}}(\vec{r},t) = \left[\dfrac{\vec{E}e^{j\omega t} + \vec{E}^* e^{-j\omega t}}{2}\right] \times \left[\dfrac{\vec{H}e^{j\omega t} + \vec{H}^* e^{-j\omega t}}{2}\right] \Rightarrow$

$$\vec{\mathcal{S}}(\vec{r},t) = \vec{\mathcal{E}}(\vec{r},t) \times \vec{\mathcal{H}}(\vec{r},t) = \frac{1}{2}\mathrm{Re}\left[\vec{E} \times \vec{H}^* + \vec{E} \times \vec{H}e^{j2\omega t}\right]$$

• Now let's calculate the **time average of** $\mathcal{S}$

$$\vec{\mathcal{S}}_{\text{ave}} = \left\langle \vec{\mathcal{S}} \right\rangle = \frac{1}{T}\int_0^T \vec{\mathcal{S}}\, dt$$

then

$$\left\langle \vec{\mathcal{S}} \right\rangle = \frac{1}{T}\int_0^T \frac{1}{2}\mathrm{Re}\left[\vec{E} \times \vec{H}^* + \vec{E} \times \vec{H}e^{j2\omega t}\right]dt$$

$$= \frac{1}{2}\mathrm{Re}\left[\frac{1}{T}\int_0^T \vec{E} \times \vec{H}^* dt\right] + \frac{1}{2}\mathrm{Re}\left[\frac{1}{T}\int_0^T \vec{E} \times \vec{H}e^{j2\omega t} dt\right]$$

$$= \frac{1}{2}\mathrm{Re}\left[\vec{E} \times \vec{H}^*\right] + 0 \Rightarrow$$

$$\vec{\mathcal{S}}_{\text{ave}} = \left\langle \vec{\mathcal{S}} \right\rangle = \frac{1}{2}\mathrm{Re}\left[\vec{E} \times \vec{H}^*\right]$$

• Whereas, the **instantaneous Poynting vector** in terms of the time-harmonic fields is given by:

$$\vec{\mathcal{S}}(\vec{r},t) = \frac{1}{2}\mathrm{Re}\left[\vec{E} \times \vec{H}^*\right] + \frac{1}{2}\mathrm{Re}\left[\vec{E} \times \vec{H}e^{j2\omega t}\right] = \left\langle \vec{\mathcal{S}} \right\rangle + \frac{1}{2}\mathrm{Re}\left[\vec{E} \times \vec{H}e^{2j\omega t}\right]$$

A remark on time average of energy densities

• Recall we defined **magnetic energy as** $W_m(t) = \frac{1}{2}\iiint\limits_v \mu \left|\vec{\mathcal{H}}(\vec{r},t)\right|^2 dv$

• Now, let's calculate the **time average of this quantity i.e.,** $\langle W_m \rangle$

$W_m(t) = \frac{1}{2}\iiint\limits_v \mu\, \vec{\mathcal{H}}(\vec{r},t) \cdot \vec{\mathcal{H}}(\vec{r},t) dv$ but $\vec{\mathcal{H}}(\vec{r},t) = \mathrm{Re}\left[\vec{H}(\vec{r})e^{j\omega t}\right]$ then

$$W_m(t) = \frac{1}{2}\iiint\limits_v \mu\, \mathrm{Re}\left[\vec{H}e^{j\omega t}\right] \cdot \mathrm{Re}\left[\vec{H}e^{j\omega t}\right] dv$$

$$W_m = \frac{1}{2}\iiint\limits_v \mu \frac{\vec{H}e^{j\omega t} + \vec{H}^* e^{-j\omega t}}{2} \cdot \frac{\vec{H}e^{j\omega t} + \vec{H}^* e^{-j\omega t}}{2} dv$$

$$= \frac{1}{2} \cdot \frac{\mu}{4} \iiint\limits_v \{\vec{H} \cdot \vec{H}e^{2j\omega t} + \left(H \cdot He^{2j\omega t}\right)^* + \vec{H} \cdot \vec{H}^* + \left(\vec{H} \cdot \vec{H}^*\right)^*\}\, dv$$

$$= \frac{1}{4}\iiint\limits_v \left\{\mu\, \mathrm{Re}\left[\vec{H} \cdot \vec{H}e^{2j\omega t}\right] + \left|\vec{H}\right|^2 \mu\right\} dv$$

$$= \frac{1}{4}\iiint\limits_v \mu\, \mathrm{Re}\left[\vec{H} \cdot \vec{H}e^{2j\omega t}\right] dv + \frac{1}{4}\iiint\limits_v \mu \left|\vec{H}\right|^2 dv$$

• **The time average is given by**

$$\langle W_m \rangle = \frac{1}{4}\frac{1}{T}\iiint\limits_v \int_0^T \mu\, \mathrm{Re}\left[\vec{H} \cdot \vec{H}e^{2j\omega t}\right] dt dv + \frac{1}{4}\iiint\limits_v \int_0^T \mu \left|\vec{H}\right|^2 dv dt \Rightarrow$$

$$\langle W_m \rangle = \frac{1}{4}\iiint\limits_v \mu \left|\vec{H}\right|^2 dv .$$

• Similarly

$$\langle W_m \rangle = \frac{1}{4}\iiint \varepsilon \left|\vec{E}\right|^2 dv$$

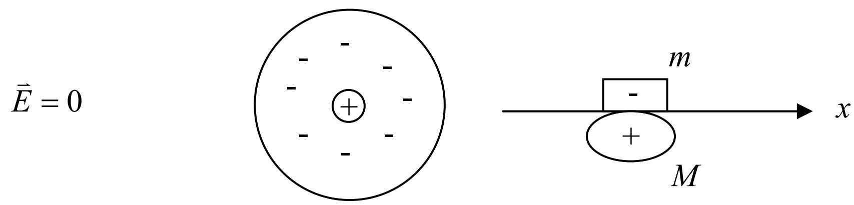
Lorentz-Lorenz Dispersion

• We **model the oscillating electron and nucleus as a mass and spring**

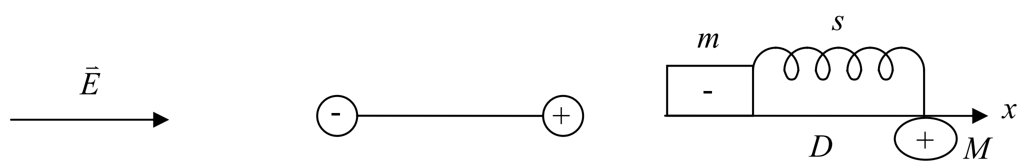
• This electron oscillator model is often called Lorentz model. It is **not really a model for atom as such**, but the way that an atom responds to a **perturbation**. At the time when Lorentz formulated the model, **it was not known that nuclei have massive mass** as compared to the electrons.

• The Lorentz assumption was that in **absence of applied electric** field the **centroids of positive and negative charges coincide**, but when a **field is applied**, the electrons will experience a **Lorentz force** and will be **displaced form their equilibrium position**.

• He then wrote "**the displacement immediately give raise to a new force by which the particle is pulled back toward its original position, and which we may therefore appropriately distinguish by the name of elastic force.**"



• Once **field is applied the electron moves**, but we assume **nucleus remains stationary**



• **Spring has a restoring force** $F_{\text{hook}} = -S\,x$

$S \equiv$ **Spring tension coefficient**

• There is also **friction within the system:** $F_{friction} = -D\dfrac{dx}{dt} = -D\,v$

$D \equiv$ **Friction coefficient**

• The friction (damping) is the result of **electron interacting** with other atoms, electrons, lattice potential, defects, vibrational mode of the material, etc.

• Equation of Motion:

$$m\frac{d^2x}{dt^2} = \sum_i F_i = F_{\text{ext}} + F_{\text{friction}} + F_{\text{hook}}$$

$F_{\text{ext}} =$ **External (applied) force** $= QE = QE_0 e^{j\omega t}$ (**assuming time harmonic fields**)

$F_{\text{hook}} = -Sx$ (**spring or hook force**)

$F_{\text{friction}} = -D\dfrac{dx}{dt}$ (**friction force**) then

• $m\dfrac{d^2x}{dt^2} + D\dfrac{dx}{dt} + Sx = QE_0 e^{j\omega t} \Rightarrow \dfrac{d^2x}{dt^2} + \dfrac{D}{m}\dfrac{dx}{dt} + \dfrac{S}{m}x = \dfrac{QE_0}{m}e^{j\omega t}$

• Let's define

$$\gamma = \frac{D}{m} \qquad \& \qquad \omega_0^2 = \frac{S}{m}$$

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + {\omega_0}^2 x = \frac{QE_0}{m}e^{j\omega t} \tag{1}$$

$$\frac{QE_0}{m} \text{ is } \frac{\text{force}}{\text{mass}} : \left[\frac{\text{N}}{\text{kg}} = \frac{\text{m}}{\text{s}^2} \equiv \text{acceleration}\right]$$

$$\gamma : \left[\frac{1}{s} = \text{Hertz}\right] \qquad \omega_0 : \left[\frac{1}{s} = \text{Hertz}\right]$$

• $\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + {\omega_0}^2 x = \frac{QE_0}{m}e^{j\omega t}$ is a **second order, linear, non-homogeneous** differential equation

• Solution to above consist of two parts: **complementary (** x_c **) and particular (** x_p **) solutions**

• **Complementary solution**, which is the **transient response**, is the solution of **homogeneous differential equation** (i.e. the forcing term $\frac{QE_0}{m}e^{j\omega t} = 0$)

• Complementary solution (**transient response)** $\rightarrow 0$ **as** $t \rightarrow \infty$

• Particular solution, which is the **steady state solution, is of interest to us**.

• Let us **assume time-harmonic solutions such as** $x_p = x_0 e^{j\omega t}$ and substitute this in our differential equation $\Rightarrow$

$$-x_0\omega^2 + jx_0\gamma\omega + x_0{\omega_0}^2 = \frac{QE_0}{m} \Rightarrow$$

$$x_0 = \frac{QE_0/m}{{\omega_0}^2 - \omega^2 + j\gamma\omega} \text{ with } \gamma = D/m \text{ and } {\omega_0}^2 = S/m$$

Calculating Permittivity & Susceptibility

• Recall $x = x_0 e^{j\omega t} = \frac{QE_0 e^{j\omega t}/m}{{\omega_0}^2 - \omega^2 + j\gamma\omega} = \frac{QE/m}{{\omega_0}^2 - \omega^2 + j\gamma\omega}$, where $E = E_0 e^{j\omega t}$

I) Assume that **dipoles are identical**
II) Assume **no coupling between dipoles**
III) There are **N dipoles per unit volume.** In other words, **N is the number of dipoles per unit volume**.

• **Polarization** $P(t)$ is given by $P(t) = NQx$ where Q is charge associated with dipole [C]. NQx has dimension of: $\left[\frac{1}{\mathrm{m}^3}\cdot \mathrm{C}\cdot m = \frac{C}{m^2}\right]$

• **Using** $P(t) = QNx$ **we have** $P(t) = \dfrac{Q^2 NE/m}{\omega_0{}^2 - \omega^2 + j\gamma\omega} \Rightarrow$

• We **calculate the ratio** $\dfrac{P}{E} = \dfrac{Q^2 N/m}{\omega_0{}^2 - \omega^2 + j\gamma\omega}$

• **Recall** $P = \varepsilon_0 \chi_e E \Rightarrow \chi_e = \dfrac{P}{\varepsilon_0 E}{}_e \Rightarrow$

$$\chi_e = \frac{Q^2 N/m\varepsilon_0}{\omega_0{}^2 - \omega^2 + j\gamma\omega}$$

• We **define** $\dfrac{Q^2 N}{m\varepsilon_0} = \omega_p{}^2$ where $\omega_p{}^2$ has the **dimension of:** $\left[\dfrac{1}{s^2}\right]$

Then

$$\chi_e = \frac{\omega_p{}^2}{\omega_0{}^2 - \omega^2 + j\gamma\omega} \qquad \Rightarrow \qquad \varepsilon_r = 1 + \chi_e = 1 + \frac{\omega_p{}^2}{\omega_0{}^2 - \omega^2 + j\gamma\omega}$$

• Compare ε_r above with **Jackson (3rd Edition)** Equation 107

$$\frac{\varepsilon}{\varepsilon_0} = 1 + \frac{\omega_p{}^2}{\omega_0{}^2 - \omega^2 - i\gamma\omega}$$

• Now, suppose there are N **molecules per unit volume** and **each molecule has** Z **electron**, and there are f_i **electrons per molecule** that have the **binding frequency (resonance frequency)** ω_i **and damping constant** γ_i then

$$\varepsilon_r = 1 + \frac{Q^2 N}{m\varepsilon_0} \sum \frac{f_i}{\omega_i{}^2 - \omega^2 + j\gamma_i\omega}, \text{ where}$$

$f_i \equiv$ **Oscillator strength** and $\sum f_i = Z$

• **Real and imaginary parts of** ε_r ($\varepsilon_r = \varepsilon_r' - j\varepsilon_r''$) are given by

$$\mathrm{Re}[\varepsilon_r] = \varepsilon_r' = \frac{\omega_p{}^2\left(\omega_0{}^2 - \omega^2\right)}{\left(\omega_0{}^2 - \omega^2\right)^2 + (\omega\gamma)^2} + 1$$

$$\mathrm{Im}[\varepsilon_r] = \varepsilon_r'' = \frac{\omega_p \omega\, \gamma}{\left(\omega_0{}^2 - \omega^2\right)^2 + (\omega\gamma)^2}$$

• Recall that the **displacement of electrons subject to the force** $QE_0 e^{j\omega t}$ was given by $x = x_0 e^{j\omega t} = \dfrac{QE_0 e^{j\omega t}/m}{\omega_0{}^2 - \omega^2 + j\gamma\omega}$. Note that the **displacement of electrons from equilibrium** is sinusoidal with time at the **frequency of the source**

• If there is **no damping** (no friction in our mechanical model), i.e., $D = 0 \Rightarrow \gamma = 0$ then

$$x = \frac{QE_0/m}{\omega_0{}^2 - \omega^2} e^{j\omega t}, \text{ and } \varepsilon_r = \frac{\varepsilon}{\varepsilon_0} = 1 + \frac{\omega_p{}^2}{\omega_0{}^2 - \omega^2} \qquad (1)$$

• Note as $\omega \to \omega_0, x \to \infty$. The **frequency** $\omega = \omega_0$ **is called the resonance of the system**. This model predicts a catastrophic response at $\omega = \omega_0$

• Note that if there is **no damping (** $\gamma = 0$ **),** $\varepsilon_r = \varepsilon'_r = 1 + \dfrac{\omega_p{}^2}{\omega_0{}^2 - \omega^2}$ **and** $\varepsilon_r'' = 0$.

• If **resonance frequency is also zer**o ($\omega_0 = 0$, the case of **free charges**), then $\varepsilon_r = \varepsilon'_r = 1 - \dfrac{\omega_p{}^2}{\omega^2}$, which is **negative** for $\omega_p > \omega$.

• While above considerations **do not predict losses in the case of free charges** ($\omega_0 = 0$), **there is in fact conduction losses associated with the free charges**. Recall the discussion of **static conductivity** and its origin.

• When **damping is present**, the resonance frequency is the **root of the characteristic equation of the homogeneous differential equation** $\dfrac{d^2x}{dt^2} + \gamma\dfrac{dx}{dt} + \omega_0{}^2 x = 0$, for real frequencies.

• **Resonance frequency** is then given by $\omega_r = \sqrt{\omega_0{}^2 - (\gamma/2)^2} = \sqrt{\omega_0{}^2 - \alpha^2}$ where $\alpha = \dfrac{\gamma}{2} = \dfrac{1}{2}\dfrac{D}{m}$ and $\omega_0{}^2 > \alpha^2$ (case of underdamped)

Note: if $\gamma = 0 \Rightarrow \alpha = 0 \Rightarrow \omega_r = \omega_0$

Wave Equation

• In the following the **field quantities are instantaneous**

$$\nabla\times\vec{\mathcal{E}} = -\frac{\partial\vec{\mathcal{B}}}{\partial t} - \vec{\mathcal{M}}_i = -\mu\frac{\partial}{\partial t}\vec{\mathcal{H}} - \vec{\mathcal{M}}_i \qquad (1)$$

$$\nabla\times\vec{\mathcal{H}} = \vec{\mathcal{J}}_i + \frac{\partial}{\partial t}\vec{\mathcal{D}} + \vec{\mathcal{J}}_c = \vec{\mathcal{J}}_i + \varepsilon\frac{\partial\vec{\mathcal{E}}}{\partial t} + \sigma_s\vec{\mathcal{E}} \qquad (2)$$

From (1) we have $\nabla\times\nabla\times\vec{\mathcal{E}} = -\nabla\times\left(\mu\frac{\partial}{\partial t}\vec{\mathcal{H}}\right) - \nabla\times\vec{\mathcal{M}}_i$ (3)

From (2) we have $\nabla\times\nabla\times\vec{\mathcal{H}} = \nabla\times\vec{\mathcal{J}}_i + \nabla\times\varepsilon\frac{\partial}{\partial t}\vec{\mathcal{E}} + \nabla\times\sigma_s\vec{\mathcal{E}}$
(4)

• Note that $\nabla\times\nabla\times\vec{\mathcal{A}} = \nabla\left(\nabla\cdot\vec{\mathcal{A}}\right) - \nabla^2\vec{\mathcal{A}}$ where $\nabla^2\vec{\mathcal{A}} = \nabla^2\mathcal{A}_x\hat{a}_x + \nabla^2\mathcal{A}_y\hat{a}_y + \nabla^2\mathcal{A}_z\hat{a}_z$ and $\nabla^2\mathcal{A}_x = \nabla\cdot\left(\nabla\mathcal{A}_x\right)$. **Laplacian is the divergence of gradient**

• Then **(3) can be written** as

$$\nabla\left(\nabla\cdot\vec{\mathcal{E}}\right) - \nabla^2\vec{\mathcal{E}} = -\nabla\times\mu\left(\frac{\partial}{\partial t}\vec{\mathcal{H}}\right) - \nabla\times\vec{\mathcal{M}}_i \qquad (5)$$

• Suppose that medium is **magnetically homogenous** (μ is independent of $\vec{r}$) then

$$\nabla\times\mu\left(\frac{\partial}{\partial t}\vec{\mathcal{H}}\right) = \mu\frac{\partial}{\partial t}\nabla\times\vec{\mathcal{H}}$$

• Use **Ampere Law** [Eq. (2)] for $\nabla\times\vec{\mathcal{H}}$ in Eq. (5)
We have

$$\nabla\left(\nabla\cdot\vec{\mathcal{E}}\right) - \nabla^2\vec{\mathcal{E}} = -\mu\frac{\partial}{\partial t}\left[\vec{\mathcal{J}}_i + \varepsilon\frac{\partial\vec{\mathcal{E}}}{\partial t} + \sigma_s\vec{\mathcal{E}}\right] - \nabla\times\vec{\mathcal{M}}_i$$

Or

$$\nabla^2\vec{\mathcal{E}} = \nabla\left(\nabla\cdot\vec{\mathcal{E}}\right) + \mu\frac{\partial}{\partial t}\vec{\mathcal{J}}_i + \mu\varepsilon\frac{\partial^2}{\partial t^2}\vec{\mathcal{E}} + \mu\sigma_s\frac{\partial}{\partial t}\vec{\mathcal{E}} + \nabla\times\vec{\mathcal{M}}_i$$

• From **Gauss Law** recall $\nabla\cdot\vec{\mathcal{E}} = \rho_{ev}/\varepsilon$ then

Wave equation for electric field:

$$\nabla^2\vec{\mathcal{E}} = \mu\frac{\partial}{\partial t}\vec{\mathcal{J}}_i + \nabla\times\vec{\mathcal{M}}_i + \varepsilon\mu\frac{\partial^2}{\partial t^2}\vec{\mathcal{E}} + \mu\sigma_s\frac{\partial}{\partial t}\vec{\mathcal{E}} + \nabla\left(\frac{\rho_{ev}}{\varepsilon}\right) \qquad (6)$$

• **Wave equation for magnetic field:**

$$\nabla^2 \vec{\mathcal{H}} = \varepsilon \frac{\partial}{\partial t}\vec{\mathcal{M}}_i + \sigma_s \vec{\mathcal{M}}_i - \nabla \times \vec{\mathcal{J}}_i + \varepsilon\mu \frac{\partial^2}{\partial t^2}\vec{\mathcal{H}} + \mu\sigma \frac{\partial \vec{\mathcal{H}}}{\partial t} + \nabla\left(\frac{\rho_{mv}}{\mu}\right) \tag{1}$$

• **Time harmonic wave equations:**

$$\nabla^2 \vec{E} = j\omega\mu\, \vec{J}_i + \nabla \times \vec{M}_i - \omega^2 \varepsilon\mu \vec{E} + j\omega\mu\sigma_s \vec{E} + \nabla\left(\frac{\rho_{\mathrm{ev}}}{\varepsilon}\right) \tag{2}$$

$$\nabla^2 \vec{H} = j\omega\varepsilon \vec{M}_i + \sigma_s \vec{M}_i - \nabla \times \vec{J}_i - \omega^2 \varepsilon\mu \vec{H} + j\omega\mu\sigma_s \vec{H} + \nabla\left(\frac{\rho_{\mathrm{mv}}}{\mu}\right) \tag{3}$$

• For **source free region** $\vec{\mathcal{M}}_i = \vec{\mathcal{J}}_i = \rho_{ev} = 0$ we have [see Eq. (5-last page)]

$$\nabla^2 \vec{\mathcal{E}} = \mu\varepsilon \frac{\partial^2}{\partial t^2}\vec{\mathcal{E}} + \mu\sigma_s \frac{\partial \vec{\mathcal{E}}}{\partial t} \tag{4}$$

• If **conductivity is also zero (** $\sigma = \sigma_s = 0$ **)** then

$$\nabla^2 \vec{\mathcal{E}} = \mu\varepsilon \frac{\partial^2}{\partial t^2}\vec{\mathcal{E}}$$

• In the case of **time harmonic fields for source free but lossy medium [Eq. (4)],** we have

$$\begin{aligned}\nabla^2 \vec{E} &= -\mu\varepsilon\omega^2 \vec{E} + j\omega\mu\sigma_s \vec{E} = -\mu(\varepsilon' - j\varepsilon'')\omega^2 \vec{E} + j\omega\mu\sigma_s \vec{E} = \\ &[-\mu\varepsilon'\omega^2 + j\omega\mu(\sigma_s + \omega\varepsilon'')]\,\vec{E} = [-\mu\varepsilon'\omega^2 + j\omega\mu\sigma_e]\,\vec{E}\end{aligned} \tag{6}$$

where $\sigma_s + \omega\varepsilon'' = \sigma_s + \sigma_a = \sigma_e$ is the effective conductivity.

• **Define:** $\gamma^2 = (\alpha + j\beta)^2 = -\mu\varepsilon'\omega^2 + j\omega\mu\sigma_e$ with α and β designating the real and imaginary parts of the γ, $\gamma = \alpha + j\beta$, where

$\alpha \equiv$ **Attenuation constant** [Np/m]

$\beta \equiv$ **Phase constant** [rad/m]

$\gamma \equiv$ **Propagation constant** [1/m]

then

$$\nabla^2 \vec{E} = [-\mu\varepsilon'\omega^2 + j\omega\mu\sigma_e]\,\vec{E} \quad \Rightarrow \nabla^2 \vec{E} = \gamma^2 \vec{E}$$

• **For lossless case** ($\sigma_e = 0$) from Eq. (6) we have

$$\nabla^2 \vec{E} = -\omega^2 \mu\varepsilon'\, \vec{E}$$

• **Note** $\gamma^2 = (\alpha + j\beta)^2 = \overbrace{j\omega\mu\sigma_e}^{0} - \mu\varepsilon'\omega^2 = -\mu\varepsilon'\omega^2$ **for lossless case**. Then $\gamma = \alpha + j\beta = \sqrt{-\mu\varepsilon'\omega^2} = j\omega\sqrt{\mu\varepsilon'} \rightarrow \alpha = 0$ *and* $\beta = \omega\sqrt{\mu\varepsilon'}$ in the case of lossless medium.

• Then $\nabla^2 \vec{E} = -\omega^2 \mu\varepsilon' \vec{E} = -\beta^2 \vec{E}$ where

$\beta^2 = \omega^2 \mu\varepsilon' = \omega^2 \mu_0 \varepsilon_0 \mu_r \varepsilon_r' = \frac{\omega^2}{c} \mu_r \varepsilon_r' = \frac{\omega^2}{c^2}\left(\sqrt{\mu_r \varepsilon_r'}\right)^2 = \frac{\omega^2}{c^2} n'^2$, or $\beta = \frac{\omega}{c} n'$

• Wave equation for **scalar components of** $\vec{E}$

$$\nabla^2 \vec{E} = -\beta^2 \vec{E} \Rightarrow \nabla^2 E_x \hat{a}_x + \nabla^2 E_y \hat{a}_y + \nabla^2 E_z \hat{a}_z = -\beta^2 \left[E_x \hat{a}_x + E_y \hat{a}_y + E_z \hat{a}_z\right] \Rightarrow$$

$$\left.\begin{array}{l} \nabla^2 E_x = -\beta^2 E_x \\ \nabla^2 E_y = -\beta^2 E_y \\ \nabla^2 E_z = -\beta^2 E_z \end{array}\right] \Rightarrow \text{with } \begin{array}{l} E_x = E_x(x,y,z) \\ E_y = E_y(x,y,z) \\ E_z = E_z(x,y,z) \end{array}$$

• As an **example the *x*-components of the electric filed** must satisfy the following:

$$\nabla^2 E_x(x,y,z) = -\beta^2 E_x(x,y,z) \Rightarrow$$

$$\frac{\partial^2}{\partial x^2} E_x(x,y,z) + \frac{\partial^2}{\partial y^2} E_x(x,y,z) + \frac{\partial^2}{\partial z^2} E_x(x,y,z) = -\beta^2 E_x(x,y,z)$$

The differential equations for other components of the field are similar

Solutions to Wave Equation

• To find the **solutions for** E_x **we assume** $E_x(x,y,z) = f(x)g(y)h(z)$ and use the **separation of variables** technique to get

$$\frac{1}{f}\frac{d^2 f(x)}{dx^2} + \frac{1}{g}\frac{d^2 g(y)}{dy^2} + \frac{1}{h}\frac{d^2 h(z)}{dz^2} + \beta^2 = 0 \Rightarrow$$

$$\frac{d^2 f(x)}{dx^2} = -\beta_x^2 \ f(x),$$

$$\frac{d^2 g(y)}{dy^2} = -\beta_y^2 \ g(y),$$

$$\frac{d^2 h(z)}{dz^2} = -\beta_z^2 \ h(z),$$

With $\beta_x{}^2 + \beta_y{}^2 + \beta_z{}^2 = \beta^2 = \omega^2 \mu\varepsilon' = \frac{\omega^2}{c^2} n'^2$, which sometime is called the **constraint equation.**

• **Solutions are**

$$\frac{d^2 f(x)}{dx^2} = -\beta_x{}^2 f(x) \Leftrightarrow \qquad f_1(x) = A_1 e^{-j\beta_x x} + B_1 e^{+j\beta_x x}$$

$$f_2(x) = C_1 \cos(\beta_x x) + D_1 \sin(\beta_x x)$$

$$\frac{d^2 g(y)}{dy^2} = -\beta_y{}^2 g(y) \Leftrightarrow \qquad g_1(y) = A_2 e^{-j\beta_y y} + B_2 e^{+j\beta_y y}$$

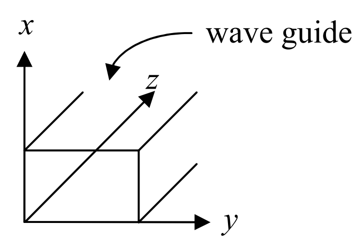
$$g_2(y) = C_2 \cos(\beta_y y) + D_2 \sin(\beta_y y)$$

$$\frac{d^2 h(z)}{dz^2} = -\beta_z{}^2 h \Leftrightarrow \qquad h_1(z) = A_3 e^{-j\beta_z z} + B_3 e^{+j\beta_z z}$$

$$h_2(z) = C_3 \cos(\beta_z z) + D_3 \sin(\beta_z z)$$

- $e^{\pm j\beta_x x}$ are called **traveling wave solutions**
- $\cos(\beta_x x)$ or $\sin(\beta_x x)$ are called **standing wave solutions**

• The type of **solution chosen** depends on the **problem and the boundary condition**.



• For example, for waves **confined in the *x*- and *y*-directions** and **traveling a long the *z*-direction** we have:

$$E_x(x,y,z) = f(x)g(y)h(z) =$$
$$[C_1 \cos(\beta_x x) + D_1 \sin(\beta_x x)] \cdot$$
$$[C_2 \cos(\beta_y y) + D_2 \sin(\beta_y y)] \cdot$$
$$A_3 e^{-j\beta_z z} + B_3 e^{+j\beta_z z}$$

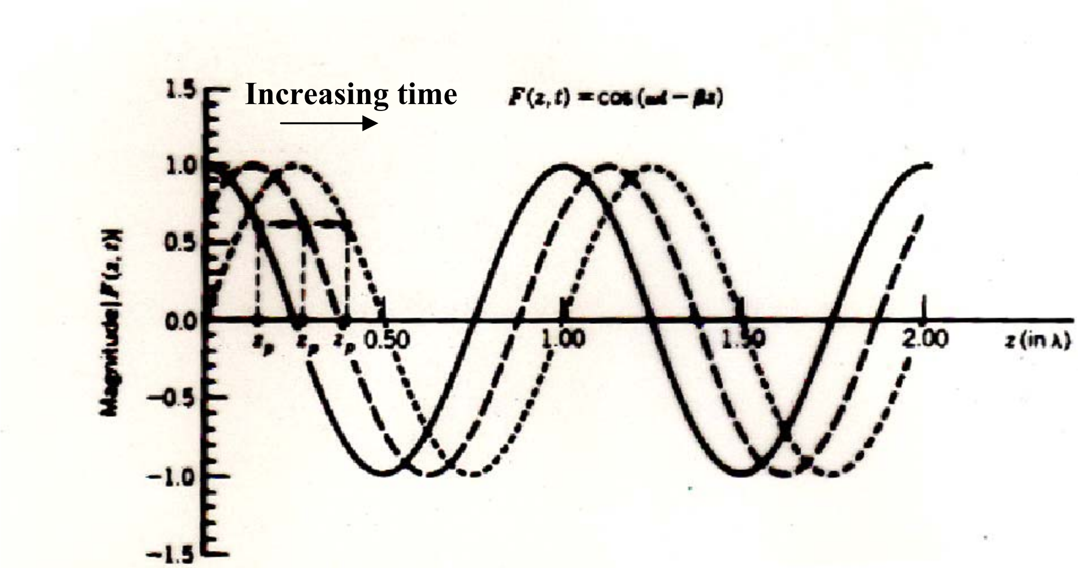
• $e^{-j\beta_z z}$ is the **positively traveling** wave and $e^{+j\beta_z z}$ is the **negatively traveling** wave (for time dependency of $e^{+j\omega t}$)

• To see this note the following

$$E = \mathrm{Re}\left[E_x(x,y,z) e^{j\omega t}\right] = [C_1 \cos(\beta_x x) + D_1 \sin(\beta_x x)] \cdot [C_2 \cos(\beta_y y) + D_2 \sin(\beta_y y)] A_3 \cos(\omega t - \beta_z z)$$

For our choice of $e^{-j\beta_z z} e^{j\omega t}$

• Let's **plot** $\cos(\omega t - \beta_z z)$ for **different times**



• To follow the point Z_p **at different times we must keep** $A_3 \cos(\omega t - \beta_z Z_p)$ **constant**
$\Rightarrow$ We must keep the phase $\omega t - \beta_z Z_p$ constant with time $\Rightarrow$

$$\frac{d}{dt}(\omega t - \beta_z Z_p) = 0 \Rightarrow \omega - \beta_z \frac{dZ_p}{dt} = 0 \Rightarrow \frac{dZ_p}{dt} = \frac{\omega}{\beta_z} = V_p$$

• $V_p = \dfrac{\omega}{\beta_z}$ is called **phase velocity**

Solution to Wave Equation in Source Free but Lossy Medium

• Recall **wave equation for lossy medium was given by**

$$\nabla^2 \vec{E} = \left[-\omega^2 \varepsilon' \mu + j\omega\mu\sigma_e\right]\vec{E} = \gamma^2 \vec{E} \qquad (1)$$

where $\gamma^2 = -\omega^2 \varepsilon' \mu + j\omega\mu\sigma_e = (\alpha + j\beta)^2$

• Once again Eq. (1) $\Rightarrow$
$\nabla^2 E_x(x,y,z)\hat{a}_x + \nabla^2 E_y(x,y,z)\hat{a}_y + \nabla^2 E_z(x,y,z) =$
$\gamma^2(E_x \hat{a}_x + E_y \hat{a}_y + E_z \hat{a}_z) \Rightarrow \nabla^2 E_x(x,y,z) = \gamma^2 E_x(x,y,z)$ **and so forth for** E_y **and** E_z

• Once again we propose a **solution of the form** $E_x(x,y,z)=f(x)g(y)h(z)$ and use **separation of variables to show**

$$\frac{d^2 f(x)}{dx^2}=+\gamma_x{}^2 f(x),$$
$$\frac{d^2 g(y)}{dy^2}=+\gamma_y{}^2 g(y),$$
$$\frac{d^2 h(z)}{dz^2}=+\gamma_z{}^2 h(z),$$

With $\gamma_x{}^2+\gamma_y{}^2+\gamma_z{}^2=\gamma^2$ **constrained equation**

• **Then** $E_x(x,y,z)=f(x)g(y)h(z)$ **is given by**

$$f_1(x)=A_1e^{-\gamma_x x}+B_1e^{\gamma_x x}$$
$$f_2(x)=C_1\cosh(\gamma_x x)+D_1\sinh(\gamma_x x)$$
$$g_1(y)=A_2e^{-\gamma_y y}+B_2e^{\gamma_y y}$$
$$g_2(y)=C_1\cosh(\gamma_y y)+D_2\sinh(\gamma_y y)$$
$$h_1(z)=A_3e^{-\gamma_z z}+B_3e^{\gamma_z z}$$
$$h_2(z)=C_3\cosh(\gamma_z z)+D_3\sinh(\gamma_z z)$$

• Exponential functions represent **attenuated traveling waves** and hyperbolic cosine and sine represent **attenuated standing waves**

• **Choices for the sign of** γ

• Recall we had $\gamma^2=(\alpha+j\beta)^2\Rightarrow\gamma=\pm(\alpha+j\beta)$. We could have equally defined $\gamma^2=(\alpha-j\beta)^2\Rightarrow\gamma=\pm(\alpha-j\beta)$ then we have **four choices**:

$$\left.\begin{array}{l}\gamma=\alpha+j\beta\\ \gamma=-\alpha-j\beta\\ \gamma=\alpha-j\beta\\ \gamma=-\alpha+j\beta\end{array}\right\}\textbf{which one should we choose}$$

$\gamma_z=\alpha_z+j\beta_z\Rightarrow e^{-\gamma_z z}=e^{-\alpha_z z}e^{-j\beta_z z}$ **travels along +*z*-axis, decays along +z-axis**

$\gamma_z=-\alpha_z-j\beta_z\Rightarrow e^{-\gamma_z z}=e^{+\alpha_z z}e^{j\beta_z z}$ **travels along -*z*-axis, decays along –z-axis**

$\gamma_z=-\alpha_z+j\beta_z\Rightarrow e^{-\gamma_z z}=e^{+\alpha_z z}e^{-j\beta_z z}$ travels along +*z*-axis, grows along +z-axis

$\gamma_z=\alpha_z-j\beta_z\Rightarrow e^{-\gamma_z z}=e^{-\alpha_z z}e^{+j\beta_z z}$ travels along -*z*-axis, grows along –z-axis

• For a **positively traveling wave** (+z-axis) in a **passive media** (media with no gain or external source of energy), we must have a **wave that decays as it moves further in the media**. Hence, the correct sign for a positively traveling wave in a passive media is

$$\gamma_z = \alpha_z + j\beta_z$$
$$e^{-\gamma_z z} = e^{-\alpha_z z} e^{-j\beta_z z}$$

with our choice of **time dependency of** $e^{+j\omega t}$

Summary

• **Traveling waves** $e^{-j\beta_z z}$ for positive z traveling
$e^{+j\beta_z z}$ for negative z traveling

• **Standing waves** $\cos(\beta_z z)$ for positive or negative z
$\sin(\beta_x z)$ for positive or negative z

• **Evanescent waves** $e^{-\alpha_z z}$ for positive z
$e^{\alpha_z z}$ for negative z

• **Attenuated traveling waves** $e^{-\gamma_z z} = e^{-\alpha_z z} e^{-j\beta_z z}$ for positive z traveling
$e^{\gamma_z z} = e^{\alpha_z z} e^{j\beta_z z}$ for negative z traveling

• **Attenuated standing waves**
$\cos(\gamma_z z) = \cos(\alpha_z z)\cosh(\beta_z z) - j\sin(\alpha_z z)\sinh(\beta_z z)$ for positive and negative z
$\sin(\gamma_z z) = \sin(\alpha_z z)\cosh(\beta_z z) + j\cos(\alpha_z z)\sinh(\beta_z z)$ for positive and negative z

• Note that:
$$\begin{aligned}\cos(\gamma_z z) &= \cos(\alpha_z z + j\beta_z z) = \cos(\alpha_z z)\cos(j\beta_z z) - \sin(\alpha_z z)\sin(j\beta_z z)\\ &= \cos(\alpha_z z)\cosh(\beta_z z) - j\sin(\alpha_z z)\sinh(\beta_z z)\end{aligned}$$

Wave Equation in Cylindrical Coordinates

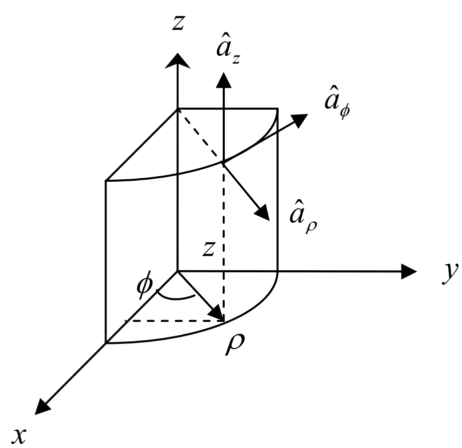
• Previously we **solved the wave equation** $\nabla^2 \vec{E} = -\beta^2 \vec{E}$ **in rectangular** coordinate system for lossless and source free region

• Suppose that **boundary condition (the geometrical consideration)** of the problem requires us to solve the wave equation in **cylindrical coordinates**. How do we go about this?

• In cylindrical coordinates
$\vec{E} = E_\rho(\rho,\phi,z)\hat{a}_\rho + E_\phi(\rho,\phi,z)\hat{a}_\phi + E_z(\rho,\phi,z)\hat{a}_z$

• Then $\nabla^2\vec{E} = -\beta^2\vec{E} \;\Rightarrow \nabla^2\left[E_\rho\hat{a}_\rho + E_\phi\hat{a}_\phi + E_z\hat{a}_z\right] = -\beta^2\left(E_\rho\hat{a}_\rho + E_\phi\hat{a}_\phi + E_z\hat{a}_z\right)$

• But
$\nabla^2\left(E_\rho\hat{a}_\rho\right) \neq \hat{a}_\rho\nabla^2 E_\rho$ and
$\nabla^2\left(E_\phi\hat{a}_\phi\right) \neq \hat{a}_\phi\nabla^2 E_\phi$
while,
$\nabla^2\left(E_z\hat{a}_z\right) = \hat{a}_z\nabla^2 E_z$



• Then **how do we solve** $\nabla^2\vec{E} = -\beta^2\vec{E}$ for $\vec{E} = E_\rho\hat{a}_\rho + E_\phi\hat{a}_\phi + E_z\hat{a}_z$. In other words, **what is** $\nabla^2\vec{E}$?

• Note that $\nabla^2\vec{E} = -\beta^2\vec{E}$ **was obtained by using**
$\nabla^2\vec{E} = \nabla\left(\nabla\cdot\vec{E}\right) - \nabla\times\nabla\times\vec{E}$

• **Using above in** $\nabla^2\vec{E} = -\beta^2\vec{E}$ **we have**
$\nabla\left(\nabla\cdot\vec{E}\right) - \nabla\times\nabla\times\vec{E} = -\beta^2\vec{E}$ (Wave equation in **lossless source free region**)
Where $\beta = \omega\sqrt{\mu\varepsilon'} = \dfrac{\omega}{c}n'$ is a constant

• In cylindrical coordinates

$$\nabla\cdot\vec{E} = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho E_p\right) + \frac{1}{\rho}\frac{\partial}{\partial\phi}E_\phi + \frac{\partial}{\partial z}E_z$$

and

$$\nabla\psi(\rho,\phi,z) = \hat{a}_\rho\frac{\partial\psi}{\partial\rho} + \hat{a}_\phi\frac{1}{\rho}\frac{\partial\psi}{\partial\phi} + \hat{a}_z\frac{\partial\psi}{\partial z}$$

and

$$\nabla\times\vec{E} = \hat{a}_\rho\left[\frac{1}{\rho}\frac{\partial E_z}{\partial\phi} - \frac{\partial E_\phi}{\partial z}\right] + \hat{a}_\phi\left[\frac{\partial}{\partial z}E_\rho - \frac{\partial}{\partial\rho}E_z\right] + \hat{a}_z\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho E_\phi\right) - \frac{1}{\rho}\frac{\partial E_\rho}{\partial\phi}\right]$$

• **The use of** $\nabla\cdot$, ∇ **and** $\nabla\times$ **in cylindrical coordinate** in $\nabla\left(\nabla\cdot\vec{E}\right) - \nabla\times\nabla\times\vec{E} = -\beta^2\vec{E}$ will result in **three partial differential equations**:

$$\nabla^2 E_\rho + \left(-\frac{E_\rho}{\rho^2} - \frac{2}{\rho^2}\frac{\partial E_\phi}{\partial\phi}\right) = -\beta^2 E_\rho$$

$$\nabla^2 E_\phi + \left(-\frac{E_\phi}{\rho^2} + \frac{2}{\rho^2}\frac{\partial E_\rho}{\partial \phi} \right) = -\beta^2 E_\phi$$

$$\nabla^2 E_z = -\beta^2 E_z$$

where,

$$\nabla^2 \psi = \frac{\partial^2 \psi}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial \psi}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 \psi}{\partial \phi^2} + \frac{\partial^2 \psi}{\partial z^2}$$

with $\psi(\rho, \phi, z) \equiv E_\rho,\ E_\phi$, or E_z

• Note that **differential equations for** E_ρ **and** E_ϕ **are coupled partial differential equations** while the differential equation for E_z **is not coupled**

• The solutions of $\nabla^2 E_z = -\beta^2 E_z$ are most useful in **constructing TE^z and TM^z modes** (TE and TM with respect to WRT *z*-direction) boundary value problems and will be considered here.

• **From** $\nabla^2 E_z = -\beta^2 E_z$ **and the expression for** $\nabla^2 \psi$ $(\psi \equiv E_z)$ we have

$$\frac{\partial^2 \psi}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial \psi}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 \psi}{\partial \phi^2} + \frac{\partial^2 \psi}{\partial z^2} = -\beta^2 \psi \quad \text{where} \qquad (1)$$

$$\psi = \psi(\rho, \phi, z) \qquad (2)$$

• Let $\psi(\rho, \phi, z) = f(\rho) g(\phi) h(z)$. **Substitute (2) in (1)** and we have:

$$g(\phi)h(z)\frac{d^2}{d\rho^2} f(\rho) + \frac{g(\phi)h(z)}{\rho}\frac{d}{d\rho} f(\rho) + \frac{f(\rho)h(z)}{\rho^2}\frac{d^2 g(\phi)}{d\phi^2} + f(\rho)g(\phi)\frac{d^2 h(z)}{dz^2} = -\beta\ f(\rho)g(\phi)h(z)$$

• **Divide both sides by** *fgh* and we get:

$$\frac{1}{f(\rho)}\frac{d^2}{d\rho^2} f(\rho) + \frac{1}{\rho\, f(\rho)}\frac{df(\rho)}{d\rho} + \frac{1}{\rho^2 g(\phi)}\frac{d^2}{d\phi^2} g(\phi) + \frac{1}{h(z)}\frac{d^2}{dz^2} h(z) = -\beta^2 \qquad (4)$$

Where β^2 is a constant

• Since $\frac{1}{h(z)}\frac{d^2}{dz^2} h(z)$, which is only a function of z, **added to other terms** (which are functions of ρ and ϕ) **must equal to a constant** ($-\beta^2$) **for all values of** z, we must have

$$\frac{1}{h(z)}\frac{d^2 h(z)}{dz^2} = -\beta_z^{\,2}$$, **where** $\beta_z^{\,2}$ **is another constant**

• Then, **Eq. (4) can be written as**

$$\frac{\rho^2}{f(\rho)}\frac{d^2}{d\rho^2}f(\rho)+\frac{\rho}{f(\rho)}\frac{d}{d\rho}f(\rho)+\frac{1}{g(\phi)}\frac{d^2}{d\phi^2}g(\phi)+\left(\beta^2-\beta_z^{\ 2}\right)\rho^2=0$$

• Note that in the above, $\frac{1}{g(\phi)}\frac{d^2g(\phi)}{d\phi^2}$, **which is only a function of ϕ, added to other terms must equal to a constant (0 here),** then similar to the previous case we sety

$\frac{1}{g(\phi)}\frac{d^2g(\phi)}{d\phi^2}=-m^2$, **where m^2 is a constant**

• Let us also **define** $\beta^2-\beta_z^{\ 2}=\beta_\rho^{\ 2}\Leftrightarrow\beta_z^{\ 2}+\beta_\rho^{\ 2}=\beta^2$ (**constraint equation** for wave equation in cylindrical coordinates)

• **using the constraint equation we see**

$$\frac{\rho^2}{f(\rho)}\frac{d^2}{d\rho^2}f(\rho)+\frac{\rho}{f(\rho)}\frac{d}{d\rho}f(\rho)-m^2+\left(\beta^2-\beta_z^{\ 2}\right)\rho^2=0\quad\Rightarrow$$

$$\rho^2\frac{d^2f(\rho)}{d\rho^2}+\rho\frac{df(\rho)}{d\rho}-m^2f(\rho)+\beta_\rho^{\ 2}\rho^2f(\rho)=0$$

Where $\beta_\rho^{\ 2}$ **and** m^2 **are constant**. Above is the **classical Bessel Differential Equation**.

$$\rho^2\frac{d^2f(\rho)}{d\rho^2}+\rho\frac{df(\rho)}{d\rho}+\left(\beta_\rho^{\ 2}\rho^2-m^2\right)f(\rho)=0$$

Summary

• **The solution to** $\nabla^2\psi=-\beta^2\psi$ **where** $\psi(\rho,\phi,z)\equiv E_z(\rho,\phi,z)$ is given by $\psi=f(\rho)g(\phi)h(z)$ **where** $f(\rho)$**,** $g(\phi)$**, and** $h(z)$ **are themselves solutions to**

$$\frac{1}{h(z)}\frac{d^2}{dz^2}h(z)=-\beta_z^{\ 2}\Leftrightarrow\frac{d^2h(z)}{dz^2}=-\beta_z^{\ 2}h(z) \tag{1}$$

$$\frac{1}{g(\phi)}\frac{d^2}{d\phi^2}g(\phi)=-m^2\Leftrightarrow\frac{d^2g(\phi)}{d\phi^2}=-m^2g(\phi) \tag{2}$$

$$\rho^2\frac{df(\rho)}{d\rho^2}+\rho\frac{df(\rho)}{d\rho}+\left(\beta_\rho^{\ 2}\rho^2-m^2\right)f(\rho)=0 \tag{3}$$

With constraint equation $\beta_z^{\ 2}+\beta_\rho^{\ 2}=\beta^2=\omega^2\mu\varepsilon$

• **Solutions to** $\frac{1}{h(z)}\frac{d^2}{dz^2}h(z) = -\beta_z^{\,2} \Leftrightarrow \frac{d^2 h(z)}{dz^2} = -\beta_z^{\,2} h(z)$ are given by

Standing wave $\leftarrow$ $h_1(z) = A_1 \cos(\beta_z z) + B_1 \sin(\beta_z z)$

or

Traveling wave $\leftarrow$ $h_2(z) = C_1 e^{-j\beta_z z} + D_1 e^{+j\beta_z z}$

• **Solution to** $\frac{1}{g(\phi)}\frac{d^2}{dz^2}g(\phi) = -m^2 \Leftrightarrow \frac{d^2 g(\phi)}{d\phi^2} = -m^2 g(\phi)$ are given by

Standing wave $\leftarrow$ $g_1(\phi) = A_2 \cos(m\phi) + B_2 \sin(m\phi)$

or

Traveling wave $\leftarrow$ $g_2(\phi) = C_2 e^{-jm\phi} + D_2 e^{+jm\phi}$

• **Solution to** $\rho^2 \frac{df(\rho)}{d\rho^2} + \rho\frac{df(\rho)}{d\rho} + \left(\beta_\rho^{\,2}\rho^2 - m^2\right)f(\rho) = 0$ **(Bessel Diff. Eq.)** is given by

Traveling wave $\leftarrow$ $f_1(\rho) = A_3 H_m^{(1)}(\beta_\rho \rho) + B_3 H_m^{(2)}(\beta_\rho \rho)$

or

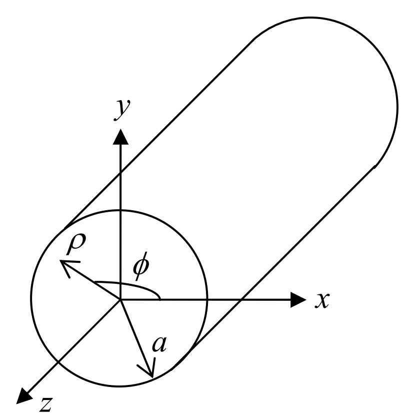
Standing wave $\leftarrow$ $f_2(\rho) = C_3 J_m(\beta_\rho \rho) + D_3 Y_m(\beta_\rho \rho)$

$H_m^{(1)}(\beta_\rho \rho) \equiv$ **Hankel function of the first kind**

$H_m^{(2)}(\beta_\rho \rho) \equiv$ **Hankel function of the second kind**

$J_m(\beta_\rho \rho) \equiv$ **Bessel function of the first kind**

$Y_m(\beta_\rho \rho) \equiv$ **Bessel function of the second kind**



• The functions $e^{\pm j\cdots}, \cos(\cdots), \sin(\cdots), J_m, Y_m, H_m^{(1)}, H_m^{(2)}$ **are all valid solutions**. Which one is used in a given problem, **depends on the problems at hand (particularly the boundary conditions).**

• As an example **consider a metallic cylindrical waveguide.** The solution i**nside of the guide**, $0 \le \rho < a$ is given by:

$$\psi_{in}(\rho,\phi,z) = f(\rho)g(\phi)h(z)$$
$$= \left[C_3 J_m(\beta_\rho \rho) + D_3 Y_m(\beta_\rho \rho)\right]\cdot\left[A_2 \cos(m\phi) + B_2 \sin(m\phi)\right]\cdot\left[C_1 e^{-j\beta_z z} + D_1 e^{+j\beta_z z}\right]$$

• Note that **inside the guide** the **solution in** ρ **must be standing waves**, the solution in ϕ **must be periodic,** and solution in z **must be traveling waves**.

• Furthermore, since $Y_m(\beta_\rho \rho)$ **is singular at** $\rho = 0$, then $D_3 = 0 \Rightarrow$

$$\psi_{in} = C_3 J_m(\beta_\rho \rho)[A_2 \cos(m\phi) + B_2 \sin(m\phi)][C_1 e^{-j\beta_z z} + D_1 e^{+j\beta_z z}]$$

• The **field outside of the guide** ($\rho > a$) must be **traveling in both** z **and** ρ **and be periodic in** ϕ, then

$$\psi_{\text{out}}(\rho,\phi,z) = B_3 H_m^{(2)}(\beta_\rho \rho)[A_2 \cos(m\phi) + B_2 \sin(m\phi)][C_1 e^{-j\beta_z z} + D_1 e^{j\beta_z z}]$$

Where $H_m^{(2)}(\beta_\rho \rho)$ is **positively traveling wave**

• Note the following relations for Hankel functions of the first and second kind.

$$H_m^{(1)}(\beta_\rho \rho) = \sqrt{\frac{2}{\pi \beta_\rho \rho}}\, e^{j\left[\beta_\rho \rho - m\left(\frac{\pi}{2}\right) - \frac{\pi}{4}\right]}_{\beta_\rho \rho \to \infty}$$

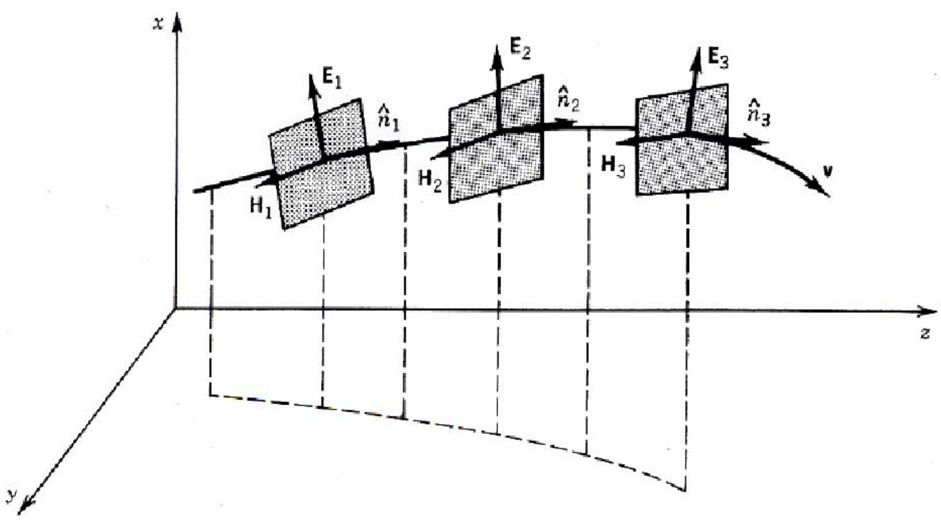
$$H_m^{(2)}(\beta_\rho \rho) = \sqrt{\frac{2}{\pi \beta_\rho \rho}}\, e^{-j\left[\beta_\rho \rho - m\left(\frac{\pi}{2}\right) - \frac{\pi}{4}\right]}_{\beta_\rho \rho \to \infty}$$

Fields, Modes, TEM, Plane wave and Uniform plane waves

• Field is a **modification of space-time coordinates**

• **Mode is a particular field configuration** for a given **boundary value problem**. **Many field** configurations (modes) may **satisfy the Maxwell equations** (wave equation). These usually are **referred to as the modes**.

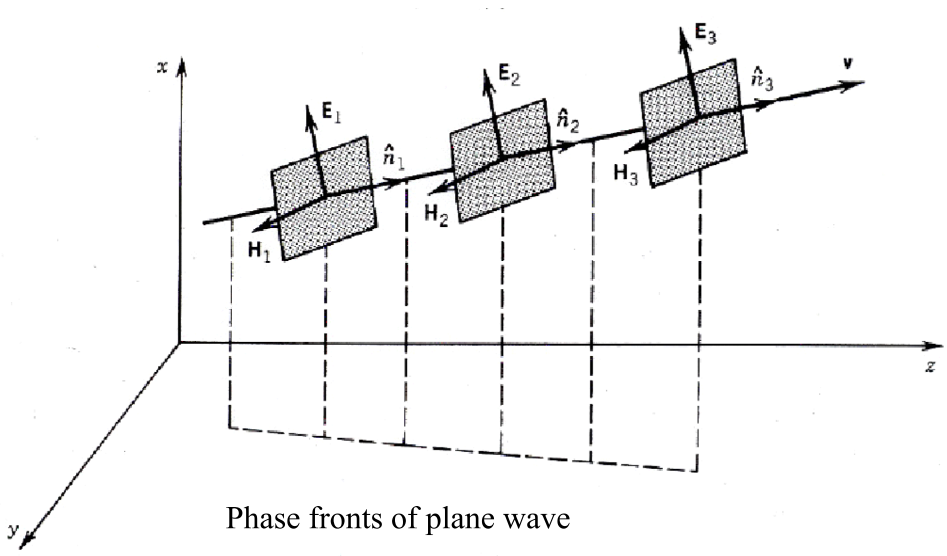
• In **TEM mode,** $\bar{E}$ **and** $\bar{H}$ at every point in space **are constrained in a local plane**, independent of time. This plane is called **equiphase Plane**. **In general equiphase planes are not parallel at two different points along the trajectory of the wave**



Phase Front of TEM wave

• If **equiphase planes are parallel** (i.e. the space orientation of the planes for TEM mode are the same), **then we say we have a plane wave. In other words, the equiphase surfaces are parallel planar surfaces.**

• **If in addition to parallel planar equiphase surfaces, the field has Equiamplitude planar surfaces (the amplitude is the same over each plane), we say we have a uniform plane wave. In this case field is not a function of the coordinates that make up equiamplitude and equiphase plane**



Phase fronts of plane wave

• We mentioned wave trajectory, **what do we mean by wave trajectory**

• Consider the following **plane wave**:
$\vec{E} = \vec{E}_0 e^{-j\vec{k}\cdot\vec{r}+j\omega t}$ **when** $\vec{E}_0$ **is a constant** and $\vec{k} = \vec{\beta}$

• Since $\nabla \cdot \vec{D} = 0$ **for source free region** $\Rightarrow \nabla \cdot \vec{E} = 0$ then $\nabla \cdot \vec{E} = \nabla \cdot \left(\vec{E}_0 e^{-j\vec{k}\cdot\vec{r}+j\omega t}\right) = 0$
Recall $\nabla \cdot \left(f\vec{F}\right) = f\nabla \cdot \vec{F} + \vec{F} \cdot \nabla f$
Then $\nabla \cdot \vec{E} = e^{-j\vec{k}\cdot\vec{r}+j\omega t}\nabla \cdot \vec{E}_0 + \vec{E}_0 \cdot \nabla\left[e^{-j\vec{k}\cdot\vec{r}+j\omega t}\right] = 0$, but $\nabla \cdot \vec{E}_0 = 0$
$-j\vec{k} \cdot \vec{E}_0 e^{-j\vec{k}\cdot\vec{r}+j\omega t} = 0 \Rightarrow \vec{k} \cdot \vec{E} = 0$

• Using $\nabla \cdot \vec{H} = 0$ **we can also show** $\vec{k} \cdot \vec{H} = 0$

• **It can also be shown** (HW) $\vec{k} \times \vec{E} = \omega\mu\vec{H}$ and $\vec{k} \times \vec{H} = -\varepsilon\omega\vec{E}$

Summary:

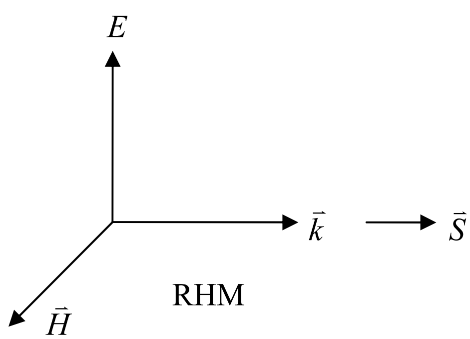
$\vec{k} \cdot \vec{E} = 0$

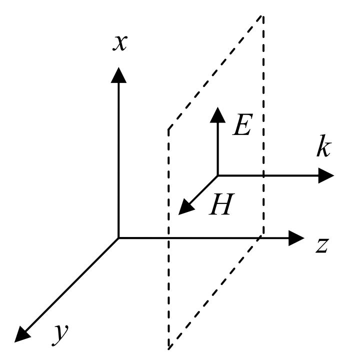
$\vec{k} \cdot \vec{H} = 0$

$\vec{k} \times \vec{E} = \omega\mu\vec{H}$

$\vec{k} \times \vec{H} = -\varepsilon\omega\vec{E}$

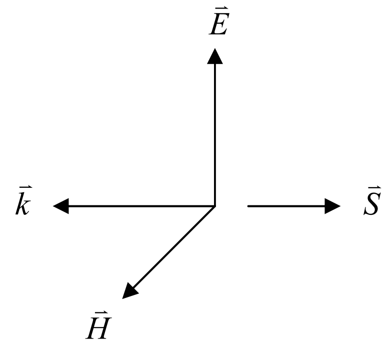
$\langle \vec{s} \rangle \propto \vec{E} \times \vec{H}$





• Let’s assume there are situations **for which** ε **and** μ **are both negative** $\varepsilon \to -|\varepsilon|$ **and** $\mu \to -|\mu|$ **then**

$$\vec{k}\cdot\vec{E}=0$$
$$\vec{k}\cdot\vec{H}=0$$
$$\vec{k}\times\vec{E}=-\omega|\mu|\vec{H}$$
$$\vec{k}\times\vec{H}=+|\varepsilon|\omega\vec{E}$$
$$\langle\vec{s}\rangle\propto\vec{E}\times\vec{H}$$



Meta-materials or LHM (left-handed media)

Relation between $\vec{E}$ and $\vec{H}$ for plane waves

• **From** $\vec{k}\times\vec{E}=\omega\mu\vec{H} \Rightarrow k\,\hat{a}_k\times\vec{E}=\omega\mu\vec{H} \Rightarrow \vec{H}=\dfrac{k}{\omega\mu}\hat{a}_k\times\vec{E}$

where $\hat{a}_k$ is the **unit vector along** $\vec{k}$.

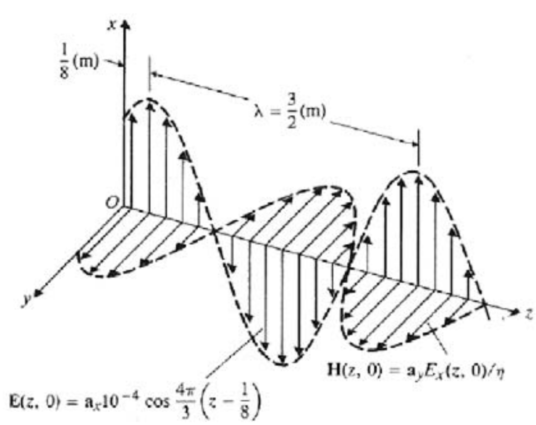
• **With** $k=\dfrac{\omega}{c}n=\omega\sqrt{\mu_0\varepsilon_0}\sqrt{\mu_r}\sqrt{\varepsilon_r}$ **Expression for** $\vec{H}$ **can be written as**

$$\vec{H}=\frac{\omega\sqrt{\mu_0\varepsilon_0}\sqrt{\mu_r\varepsilon_r}}{\omega\mu_r\mu_0}\hat{a}_k\times\vec{E}\Rightarrow \vec{H}=\frac{\sqrt{\varepsilon_0\varepsilon_r}}{\sqrt{\mu_0\mu_r}}\hat{a}_k\times\vec{E}=\frac{\hat{a}_k\times\vec{E}}{\sqrt{\mu/\varepsilon}}=\frac{\hat{a}_k\times\vec{E}}{\eta}$$ where

$\eta=\sqrt{\mu/\varepsilon}$ is the **medium intrinsic impedance** and we can define $\eta_0=\sqrt{\mu_0/\varepsilon_0}=120\pi=377\,[\Omega]$ as the **free space intrinsic impedance**.

• **Similar expression for** $\vec{E}$ **in terms of** $\vec{H}$ can be found to be

$$\vec{E}=-\eta\,\hat{a}_k\times\vec{H}$$



Fresnel Reflection & Transmission Coefficients

• The case of $\vec{E}$ **Perpendicular Polarization:**

• **The interface is in** xy **plane**

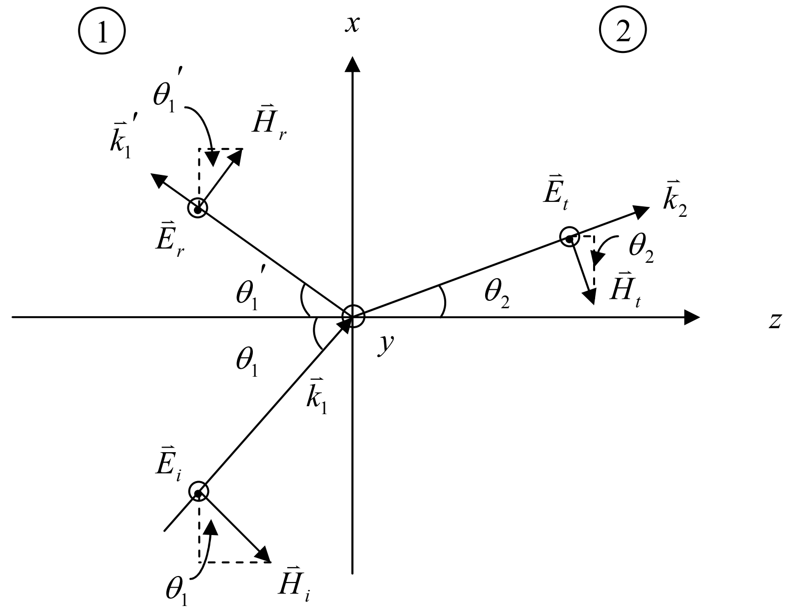
• **Plane of incidence is** xz **plane**

• **Incident waves are** $\vec{E}_i, \vec{H}_i, \vec{K}_1$

• **Reflected waves are** $\vec{E}_r, \vec{H}_r, \vec{K}_1'$

• **Transmitted waves are** $\vec{E}_t, \vec{H}_t, \vec{K}_2$

• $\theta_1 \equiv$ **Angle of incidence,**
$\theta_1' \equiv$ **Reflection angle**
$\theta_2 \equiv$ **Transmitted angle**



• As stated earlier this geometry is for $E\perp$($\vec{E}$ **perpendicular to the plane of incidence) or TE (electric field is transverse to the propagation direction) or** σ **polarization**

• $\vec{E}_i = E_0 e^{-j\vec{k}_1\cdot\vec{r}} e^{+j\omega t}\hat{a}_y$ **where** $\vec{k}_1 = k_{1x}\hat{a}_x + k_{1z}\hat{a}_z$ with $k_{1x} = k_1\sin\theta_1 = \frac{\omega}{c}n_1\sin\theta_1$ **and** $k_{1z} = k_1\cos\theta_1 = \frac{\omega}{c}n_1\cos\theta_1$ **and** $k_1 = \frac{\omega}{c}n_1 = \frac{\omega}{c}\sqrt{\mu_1\varepsilon_1}$.
Then $\vec{k}_1 = k_1(\sin\theta_1\hat{a}_x + \cos\theta_1\hat{a}_z)$ and **we have** $\vec{E}_i = E_0 e^{-jk_1(\sin\theta_1 x + \cos\theta_1 z)} e^{j\omega t}\hat{a}_y$

• **Note also that**

$$k_{1x}^{\ 2} + k_{1z}^{\ 2} = k_1^{\ 2} \Rightarrow k_{1z} = \sqrt{k_1^{\ 2} - k_{1x}^{\ 2}} = \sqrt{\frac{\omega^2}{c^2}n_1^{\ 2} - \frac{\omega^2}{c^2}n_1^{\ 2}\sin\theta_1^{\ 2}} = \frac{\omega}{c}n_1\sqrt{1-\sin^2\theta_1} = \frac{\omega}{c}n_1\cos\theta_1$$

• From $\vec{H}_i = \frac{\hat{a}_{ki}\times\vec{E}_i}{\eta_1}$ **we have** $\vec{H}_i = \frac{E_0}{\eta_1}(-\hat{a}_x\cos\theta_1 + \sin\theta_1\hat{a}_z)e^{-jk_1(\sin\theta_1 x + \cos\theta_1 z)}e^{j\omega t}$

• For **Reflected wave** we have

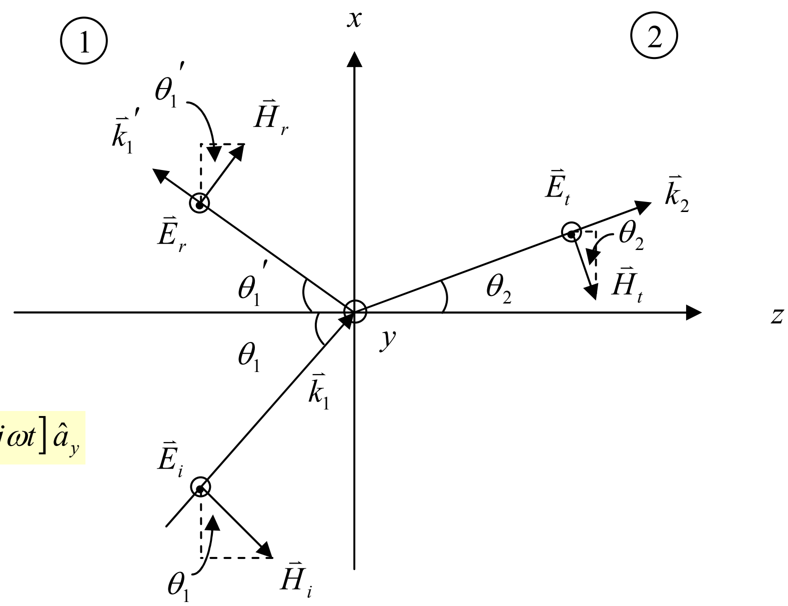
$\vec{E}_r = rE_0 e^{-j\vec{k}_1'\cdot\vec{r}} e^{j\omega t}\hat{a}_y$ **with** $\left|\vec{k}_1'\right| = k_1' = \frac{\omega}{c}\sqrt{\mu_1\varepsilon_1} = \frac{\omega}{c}n = \left|\vec{k}_1\right| = k_1 \Rightarrow k_1' = k_1$

$k_1' = -k_1'\cos\theta_1'\hat{a}_z + k_1'\sin\theta_1'\hat{a}_x$ since $k_1' = k_1$ then $k_1' = -k_1\cos\theta_1'\hat{a}_z + k_1\sin\theta_1'\hat{a}_x$

The reflected $\vec{E}$ and $\vec{H}$ are then

$$\vec{E}_r = rE_0\exp[-jk_1(\sin\theta_1'x - \cos\theta_1'z)]\exp[j\omega t]\,\hat{a}_y$$

$$\vec{H}_r = \frac{rE_0}{\eta_1}[\cos\theta_1'\,\hat{a}_x + \sin\theta_1'\,\hat{a}_z]\exp[-jk_1(\sin\theta_1'x - \cos\theta_1'z)]\exp[j\omega t]$$



• Transmitted $\vec{E}$ and $\vec{H}$ are

$$E_t = tE_0 e^{-j\vec{k}_2\cdot\vec{r}} e^{j\omega t}\hat{a}_y$$

$$\vec{k}_2 = k_{2x}\hat{a}_x + k_{2z}\hat{a}_z$$

$$k_{2x} = k_2\sin\theta_2 = \frac{\omega}{c}n_2\sin\theta_2$$

$$k_{2z} = k_2\cos\theta_2 = \frac{\omega}{c}n_2\cos\theta_2$$

$$k_2 = \frac{\omega}{c}n_2 = \frac{\omega}{c}\sqrt{\mu_2\varepsilon_2}$$

$$\vec{E}_t = tE_0\exp[-jk_2(\sin\theta_2 x + \cos\theta_2 z)]\exp[j\omega t]\,\hat{a}_y$$

• Note that $k_{2x}{}^2 + k_{2z}{}^2 = k_2{}^2 \Rightarrow$

$$k_{2z} = \sqrt{k_2{}^2 - k_{2x}{}^2} \Rightarrow$$

$$k_{2z} = \sqrt{\frac{\omega^2}{c^2}n_2{}^2 - \frac{\omega^2}{c^2}n_2{}^2\sin^2\theta_2} = \frac{\omega}{c}n_2\sqrt{1-\sin^2\theta_2} = \frac{\omega}{c}n_2\cos\theta_2 \text{ i.e. } k_{2z} = \frac{\omega}{c}n_2\cos\theta_2$$

$$\vec{H}_t = \frac{tE_0}{\eta_2}(-\cos\theta_2\hat{a}_x + \sin\theta_2\hat{a}_z)\exp[-jk_2(\sin\theta_2 x + \cos\theta_2 z)]\exp[j\omega t]$$

• We now **apply the B.C.** at xy plane and $z = 0$, **requiring tangential $\vec{E}$ and $\vec{H}$ to be continuous (two good dielectric)**

$$(E_i + E_r)_{\text{tangential}} = (E_t)_{\text{tangential}}$$

$$(H_i + H_r)_{\text{tangential}} = (H_t)_{\text{tangential}}$$

• Note that **tangential components are along** x **and** y

$$E_0 e^{-jk_1\sin\theta_1 x} + rE_0 e^{-jk_1\sin\theta_1' x} = tE_0 e^{-jk_2\sin\theta_2 x}$$

$$-\frac{E_0}{\eta_1}\cos\theta_1 e^{-jk_1\sin\theta_1 x} + \frac{rE_0}{\eta_1}\cos\theta_1'\,e^{-jk_1\sin\theta_1' x} = -\frac{tE_0}{\eta_2}\cos\theta_2 e^{-jk_2\sin\theta_2 x}$$

• The above is a **set of 4 equations and 4 unknowns** ($\theta_1', \theta_2, r, t$). It can be **reduced to 2 equations and 2 unknown**. Once this is done we have

$$\theta_1 = \theta_1'$$

$$k_1 \sin\theta_1 = k_2 \sin\theta_2$$

• $\theta_1 = \theta_2$ **is the first Snell's Law of Refraction** (i.e. the incident & reflected angles are equal)

• **Second Snell's Law of Refraction**

$$k_1 \sin\theta_1 = k_2 \sin\theta_2 \Leftrightarrow k_{1x} = k_{2x} \Leftrightarrow \frac{\omega}{c} n_1 \sin\theta_1 = \frac{\omega}{c} n_2 \sin\theta_2 \Rightarrow n_1 \sin\theta_1 = n_2 \sin\theta_2$$

This says that **tangential component of the propagation vector across the interface is continuous.**

• The **use of the two Snell's law (above)** will reduce our set of 4 equations and 4 unknowns to **2 equations and 2 unknowns**:

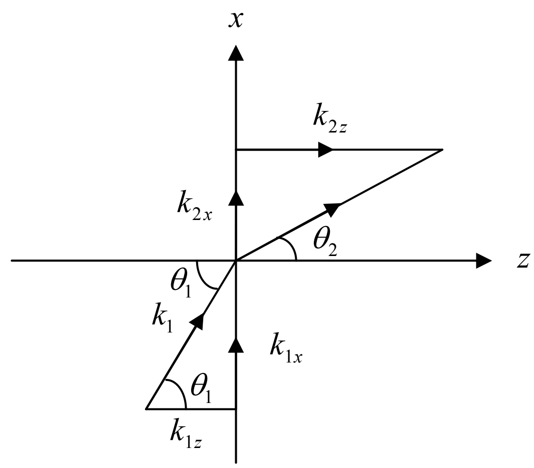
$$r + 1 = t \qquad (1)$$

$$\frac{\cos\theta_1}{\eta_1}(r-1) = \frac{-t}{\eta_2}\cos\theta_2 \qquad (2)$$

• **(1) and (2) can be solved to give**

$$r = \frac{|\vec{E}_r|}{|\vec{E}_i|} = \frac{\eta_2 \cos\theta_1 - \eta_1 \cos\theta_2}{\eta_2 \cos\theta_1 + \eta_1 \cos\theta_2}$$

$$t = \frac{|\vec{E}_t|}{|\vec{E}_i|} = \frac{2\eta_2 \cos\theta_1}{\eta_2 \cos\theta_1 + \eta_1 \cos\theta_2}$$



• Using $\eta = \sqrt{\mu/\varepsilon}$ and **multiplying top and bottom** by $\sqrt{\varepsilon_1\varepsilon_2\mu_1\mu_2}$ we have

$$r = \frac{\sqrt{\mu_2/\varepsilon_2}\cos\theta_1 - \sqrt{\mu_1/\varepsilon_1}\cos\theta_2}{\sqrt{\mu_2/\varepsilon_2}\cos\theta_1 + \sqrt{\mu_1/\varepsilon_1}\cos\theta_2} = \frac{\sqrt{\varepsilon_1\mu_1}\mu_2\cos\theta_1 - \sqrt{\mu_2\varepsilon_2}\mu_1\cos\theta_2}{\sqrt{\varepsilon_1\mu_1}\mu_2\cos\theta_1 + \sqrt{\mu_2\varepsilon_2}\mu_1\cos\theta_2}$$

$$= \frac{\frac{\omega}{c}\sqrt{\varepsilon_1\mu_1}\mu_2\cos\theta_1 - \frac{\omega}{c}\sqrt{\varepsilon_2\mu_2}\mu_1\cos\theta_2}{\frac{\omega}{c}\sqrt{\varepsilon_1\mu_1}\mu_2\cos\theta_1 + \frac{\omega}{c}\sqrt{\varepsilon_2\mu_2}\mu_1\cos\theta_2}$$

• **Recall**

$$k_{1z} = k_1\cos\theta_1 = \frac{\omega}{c} n_1 \cos\theta_1 = \frac{\omega}{c}\sqrt{\mu_1\varepsilon_1}\cos\theta_1$$

$$k_{2z} = k_2\cos\theta_2 = \frac{\omega}{c} n_2 \cos\theta_2 = \frac{\omega}{c}\sqrt{\mu_2\varepsilon_2}\cos\theta_2$$

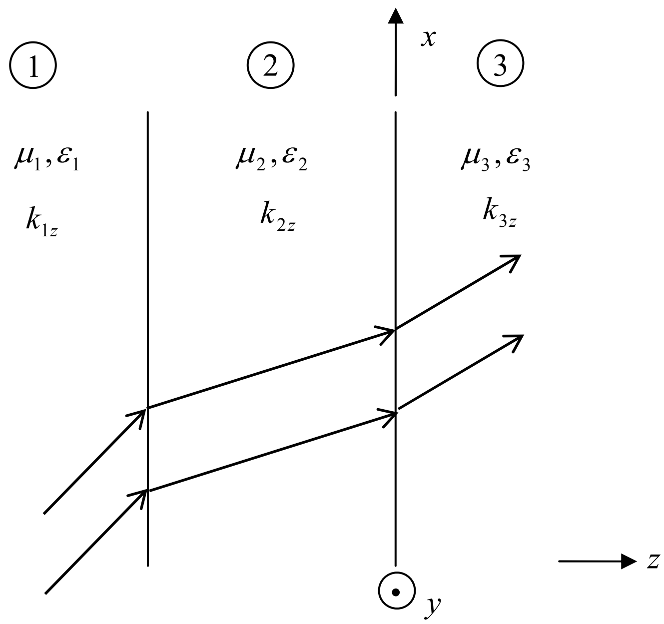
• **Hence**

$$r = \frac{\mu_2 k_{1z} - \mu_1 k_{2z}}{\mu_2 k_{1z} + \mu_1 k_{2z}} \qquad (1)$$

and similarly

$$t = \frac{2\mu_2 k_{1z}}{\mu_2 k_{1z} + \mu_1 k_{2z}} \qquad (2)$$

• Note that (1) and (2) are **reflection and transmission coefficient (Fresnel field coefficients) for TE or $\vec{E}_\perp$ polarization.**



Two Interface Problem

• We consider **TE or $\vec{E}$ perpendicular polarization. The Fresnel reflection coefficients at each interface can be written as:**

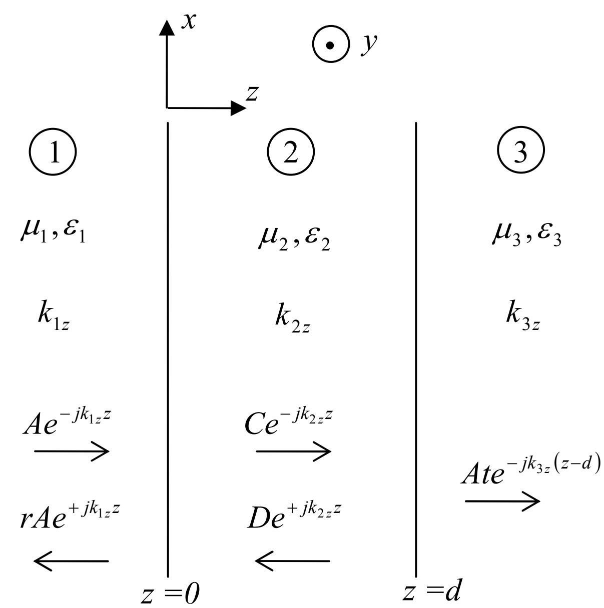
$$r_{12} = \frac{\mu_2 k_{1z} - \mu_1 k_{2z}}{\mu_2 k_{1z} + \mu_1 k_{2z}}$$

$$t_{12} = \frac{2\mu_2 k_{1z}}{\mu_2 k_{1z} + \mu_1 k_{2z}}$$

$$r_{23} = \frac{\mu_3 k_{2z} - \mu_2 k_{3z}}{\mu_3 k_{2z} + \mu_2 k_{3z}} \qquad \begin{matrix} 1 \to 2 \\ 2 \to 3 \end{matrix}$$

$$r_{21} = \frac{\mu_1 k_{2z} - \mu_2 k_{1z}}{\mu_1 k_{2z} + \mu_2 k_{1z}} = -r_{12} \qquad \begin{matrix} 1 \to 2 \\ 2 \to 1 \end{matrix}$$

$$t_{23} = \frac{2\mu_3 k_{2z}}{\mu_3 k_{2z} + \mu_2 k_{3z}} \qquad \begin{matrix} 1 \to 2 \\ 2 \to 3 \end{matrix}$$



• **At** $z = 0$

(1) $rA = r_{12}A + t_{21}D$

(2) $C = t_{12}A + r_{21}D$

• **At** $z = d$ **(slab thickness is** d **)**

(3) $At = t_{23}Ce^{-jk_{2z}d}$

(4) $De^{jk_{2z}d} = r_{23}Ce^{-jk_{2z}d} \Rightarrow$

(5) $D = r_{23}Ce^{-2jk_{2z}d} = r_{23}Ce^{+2j\phi}$ **where**

(6) $\phi = -k_{2z}d = -\dfrac{\omega}{c} n_2 \cos\theta_2 d$

• Use (5) in (2) then

(1) $C = t_{12}A + r_{21}r_{23}Ce^{+2j\phi} \Rightarrow$

(2) $C = \dfrac{t_{12}}{1 - r_{21}r_{23}e^{+2j\phi}}A$

• **Using (2) in** $At = t_{23}Ce^{-jk_{2z}d} = t_{23}Ce^{j\phi}$ **(Eq. 3-page 55), we have**

$$At = t_{23}e^{+j\phi}\frac{t_{12}}{1 - r_{21}r_{23}e^{+2j\phi}}A \quad \Rightarrow \quad t^{\mathrm{TE}} = t = \frac{t_{12}t_{23}e^{+j\phi}}{1 - r_{21}r_{23}e^{+2j\phi}} \qquad (1)$$

• **In a similar manner** (HW) we can show

$$r^{\mathrm{TE}} = r = r_{12} + \frac{t_{12}t_{21}r_{23}e^{+2j\phi}}{1 - r_{21}r_{23}e^{+2j\phi}} \qquad (2)$$

• **Note if medium 1 and 3 are the same**

then $r_{21} = r_{23} = \dfrac{\mu_1 k_{2z} - \mu_2 k_{1z}}{\mu_1 k_{2z} + \mu_2 k_{1z}} = -r_{12}$ and

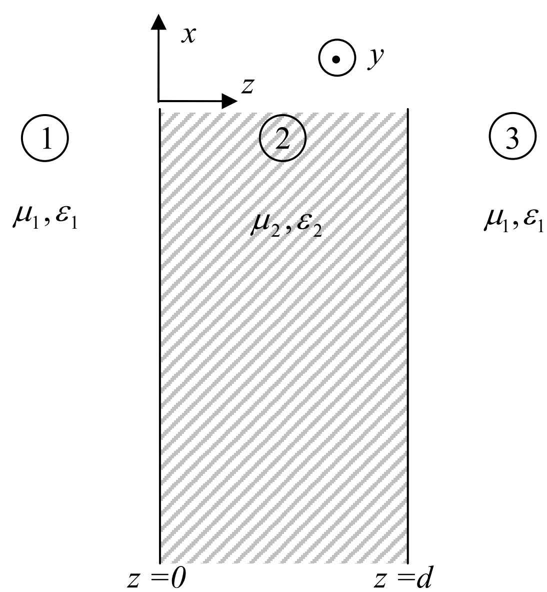
$t_{23} = t_{21} = \dfrac{2\mu_1 k_{2z}}{\mu_1 k_{2z} + \mu_2 k_{1z}}$.

Then (1) and (2) can be written as

Then $t^{\mathrm{TE}} = \dfrac{t_{12}t_{21}e^{+j\phi}}{1 - (r_{21})^2 e^{+2j\phi}}$

$$r^{\mathrm{TE}} = r_{12} + \frac{t_{12}t_{21}r_{21}e^{+2j\phi}}{1 - (r_{21})^2 e^{+2j\phi}}$$

$$\phi = -k_{2z}d = -\frac{\omega}{c}n_2 \cos\theta_2 d$$



• From the expression for r^{TE} we see that if $r_{12} = r_{21} = 0$, then $r^{\mathrm{TE}} = 0$**; i.e. there is no reflection from the slab. This is called the matched condition.**

• Recall $r_{21} = \dfrac{\mu_1 k_{2z} - \mu_2 k_{1z}}{\mu_1 k_{2z} + \mu_2 k_{1z}} = -r_{12}$, then $r_{12} = r_{21} = 0$ if

$$\mu_1 k_{2z} = \mu_2 k_{1z} \Rightarrow \mu_1 \frac{\omega}{c} n_2 \cos\theta_2 = \mu_2 \frac{\omega}{c} n_1 \cos\theta_1 \Rightarrow$$

$$\mu_1\sqrt{\mu_2\varepsilon_2}\cos\theta_2 = \mu_2\sqrt{\mu_1\varepsilon_1}\cos\theta_1 \Rightarrow \eta_1\cos\theta_2 = \eta_2\cos\theta_1$$

• What happens to t^{TE} **(transmission coefficient) under matched condition**.

• Note that **with** $r_{12} = r_{21} = 0 \Rightarrow t^{\mathrm{TE}} = \dfrac{t_{12}t_{21}e^{+j\phi}}{1-(r_{21})^2 e^{+2j\phi}} = t_{12}t_{21}e^{+j\phi}$. Recall that

$t_{12} = \dfrac{2\mu_2 k_{1z}}{\mu_2 k_{1z} + \mu_1 k_{2z}}$ and $t_{21} = \dfrac{2\mu_1 k_{2z}}{\mu_1 k_{2z} + \mu_2 k_{1z}}$. **Hence under matched condition** ($\mu_1 k_{2z} = \mu_2 k_{1z}$) **we have** $t_{12} = t_{21} = 1$**, which then implies** $t^{\mathrm{TE}} = t_{12}t_{21}e^{+j\phi} = e^{+j\phi} = e^{-jk_{2z}d}$ and $r^{\mathrm{TE}} = 0$. This says that under matched condition the **slab only inserts a phase on the traveling wave.**

• At **normal incidence** $\theta_1 = \theta_2 = 0$, **the matching condition** (no reflection from the slab) **given by** $\eta_1 \cos\theta_2 = \eta_2 \cos\theta_1$ **will simplify to** $\eta_1 = \eta_2$.

• Note that **under matched condition with** $t^{\mathrm{TE}} = e^{+j\phi} = e^{-jk_{2z}d}$ **we can write** $-\dfrac{\partial\phi}{\partial\omega} = \dfrac{\partial}{\partial\omega}[k_{2z}d] = d\dfrac{\partial k_{2z}}{\partial\omega} = \dfrac{d}{\partial\omega/\partial k_{2z}} = \dfrac{d}{v_g} \Rightarrow v_g = \dfrac{d}{-\partial\phi/\partial\omega}$, where we will later see $-\partial\phi/\partial\omega$ **is called the group delay**.

• Final Remarks: you should study (self study) topics such as critical and Brewster angles.