

Question 1: For the transmission line shown below

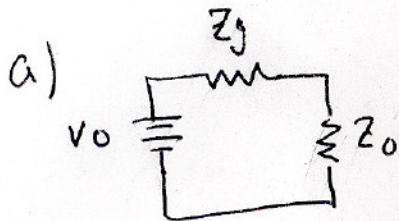
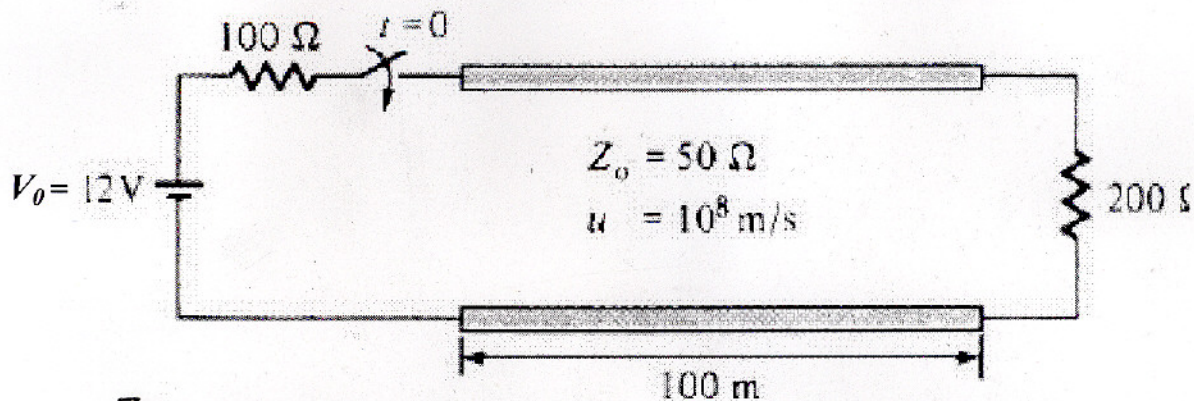
a) Draw the voltage reflection diagram. (5 pts)

b) Plot the voltage at the source for $0 < t < 6 \mu s$. What is the final voltage value? (5 pts)

c) Plot the voltage at the load for $0 < t < 6 \mu s$. (5 pts)

d) Plot the current at the load for $0 < t < 6 \mu s$. (5 pts)

Note: In all your plots clearly mark the values on the figures.

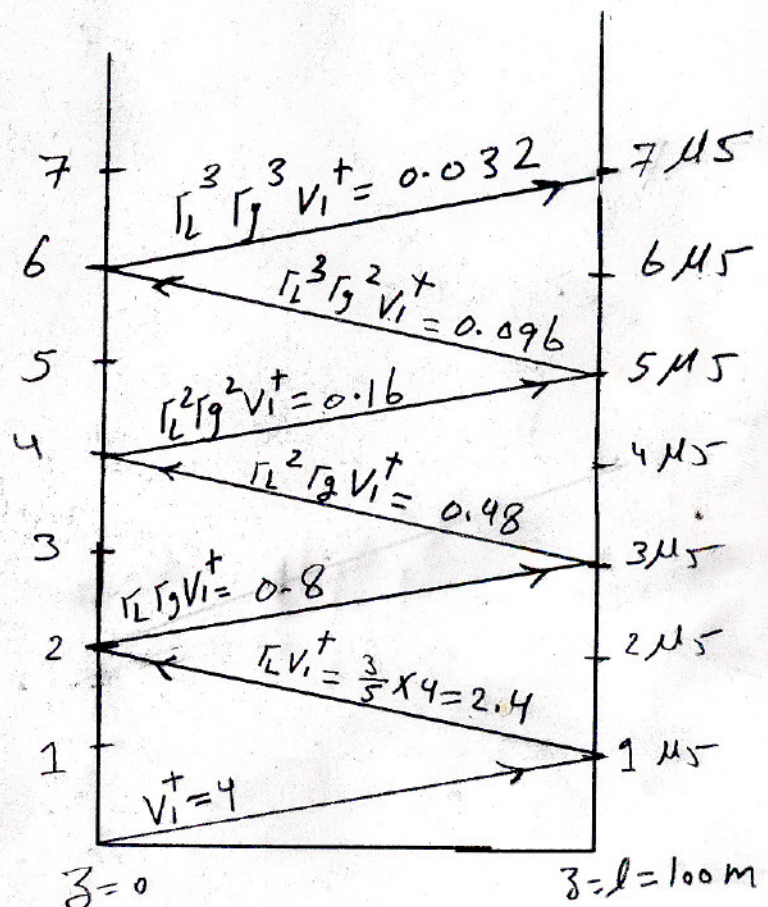


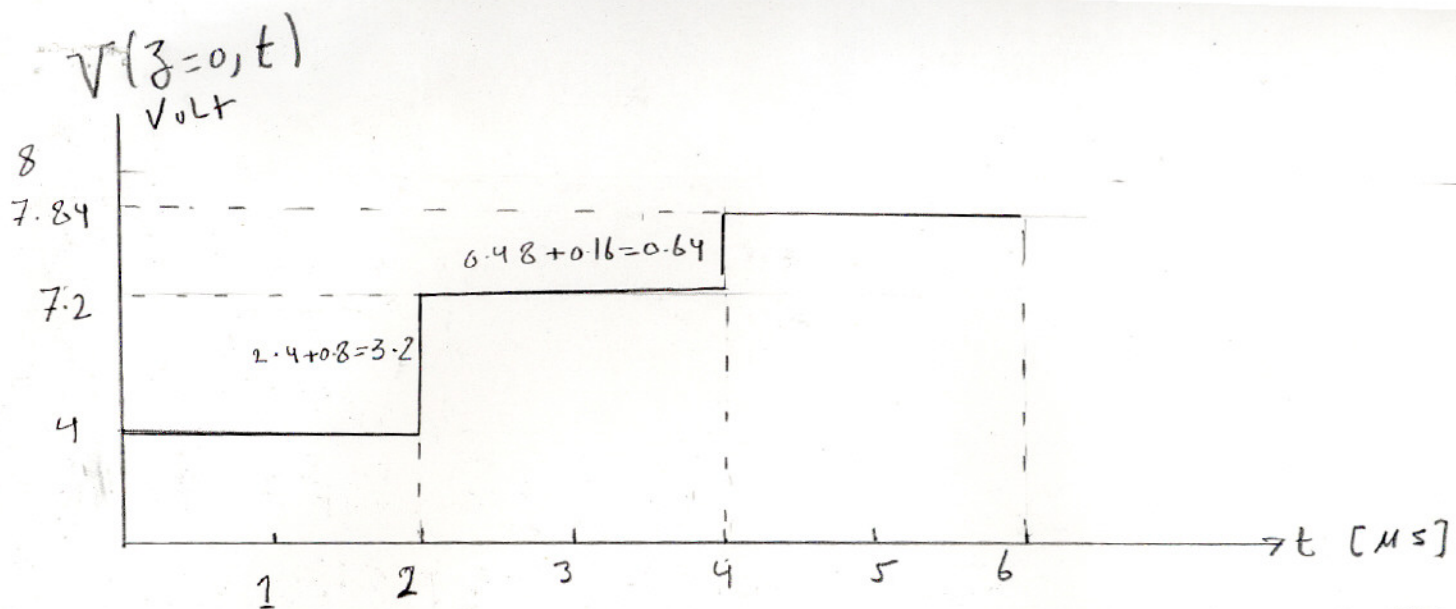
$$V_1^+ = V_0 \frac{Z_0}{Z_g + Z_0} = 12 \frac{50}{100 + 50} = 12 \times \frac{5}{150} = \boxed{4 \text{ Volt}}$$

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{200 - 50}{200 + 50} = \boxed{3/5}$$

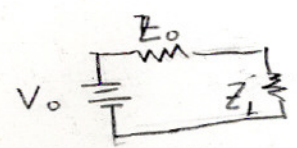
$$\Gamma_g = \frac{Z_g - Z_0}{Z_g + Z_0} = \frac{100 - 50}{100 + 50} = \boxed{1/3}$$

$$T = \frac{l}{u} = \frac{100}{10^8} = 1 \mu s$$

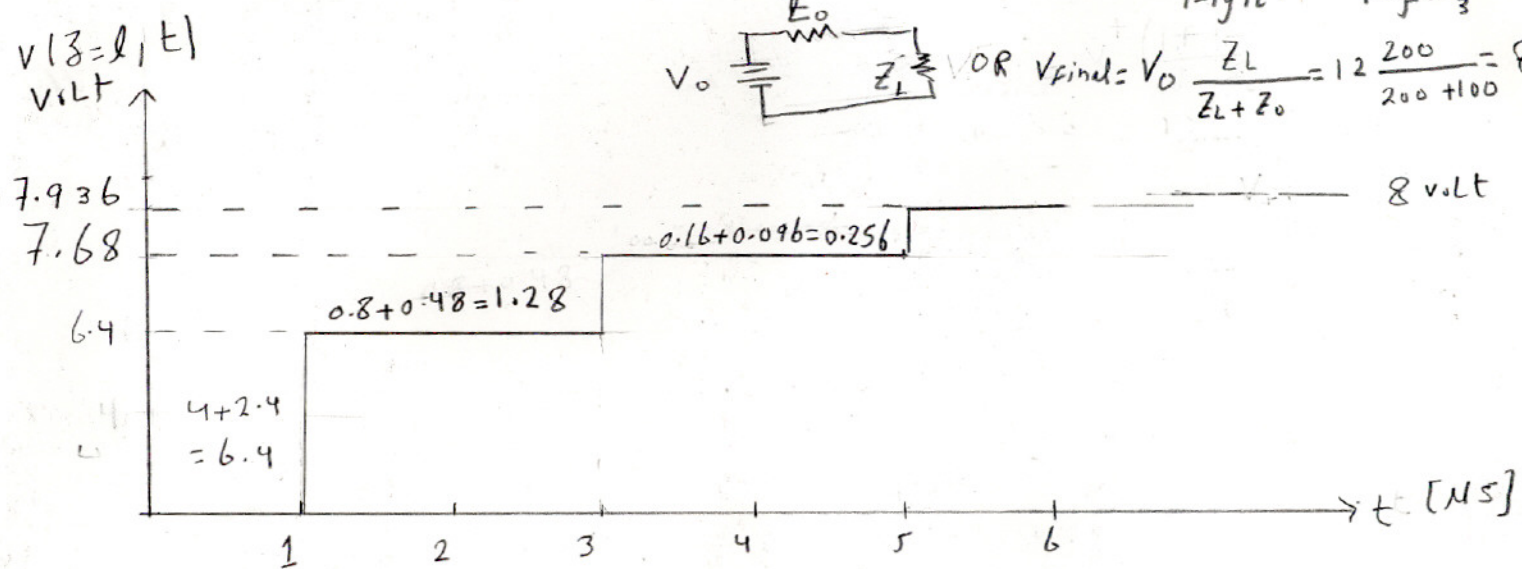




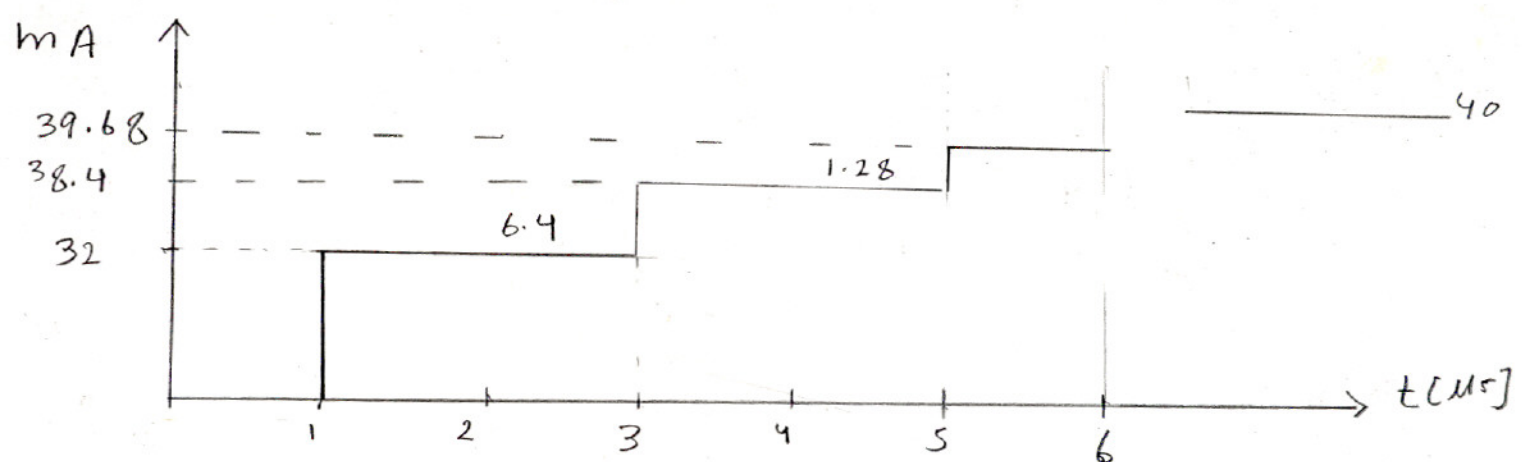
$$V_{find} = V_i + \left(\frac{1 + \frac{R_L}{1 - \Gamma_g R_L}}{1 - \frac{3}{4} \times \frac{1}{3}} \right) = 4 \left(\frac{1 + \frac{3}{15}}{1 - \frac{3}{4} \times \frac{1}{3}} \right) = 8$$



OR $V_{find} = V_0 \frac{Z_L}{Z_L + Z_0} = 12 \frac{200}{200 + 100} = 8$



$I(z=1, t) = V(z=1, t) / R_L$ where $R_L = 200$



Question 2: A segment of lossy transmission line of length l , characteristic impedance Z_0 , and propagation constant γ is shown in Fig. 1, where $(1, 1')$ and $(2, 2')$ are the input and output terminals respectively. The transmission line can also be modeled as a symmetric 2-port T-network as shown in Fig. 2.

- Find a relation between Z_1 , Y_2 , and $\cosh(\gamma l)$.
- Find a relation between Z_1 , Y_2 , Z_0 , and $\sinh(\gamma l)$.
- Find a relation between Y_2 , Z_0 , and $\sinh(\gamma l)$.

Hint: You may begin by formulating the problem in terms of a 2-port Network, relating voltages and currents according to

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix}$$

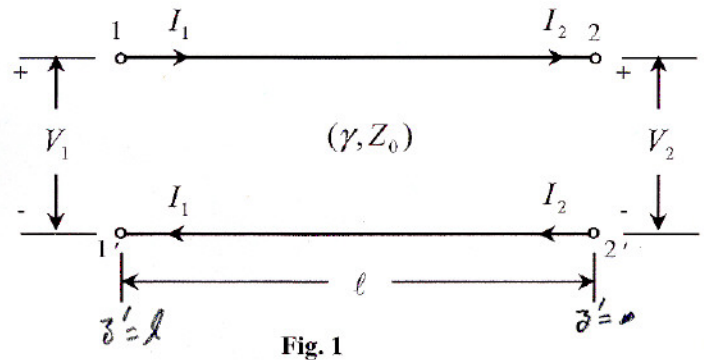


Fig. 1

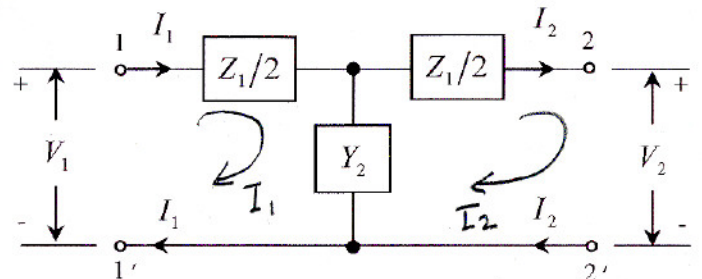


Fig. 2

From TL Eq 5

$$\begin{aligned} \textcircled{1} V(z') &= I_L Z_0 \cosh \gamma z' + I_2 Z_0 \sinh \gamma z' \\ \textcircled{2} I(z') &= \frac{I_L Z_0}{Z_0} \sinh \gamma z' + I_2 \cosh \gamma z' \end{aligned}$$

For $z' = l \Rightarrow$

$$\textcircled{3} V(z'=l) = V_L \cosh \gamma l + I_2 Z_0 \sinh \gamma l$$

$$\textcircled{4} I(z'=l) = \frac{V_L}{Z_0} \sinh \gamma l + I_2 \cosh \gamma l$$

Using $(1, 1')$ for input & $(2, 2')$ for output

$$V_1 = V_2 \cosh(\gamma l) + I_2 Z_0 \sinh(\gamma l)$$

$$I_1 = \frac{V_2}{Z_0} \sinh(\gamma l) + I_2 \cosh(\gamma l)$$

$$\begin{pmatrix} V_1 \\ I_1 \end{pmatrix} = \begin{pmatrix} \cosh(\gamma l) & Z_0 \sinh(\gamma l) \\ \frac{\sinh(\gamma l)}{Z_0} & \cosh(\gamma l) \end{pmatrix} \begin{pmatrix} V_2 \\ I_2 \end{pmatrix} \text{ where } \begin{pmatrix} A & B \\ C & D \end{pmatrix} \text{ are evident.}$$

①

For 2-Port network

$$\text{loop 1: } \textcircled{1} V_1 = I_1 \frac{Z_1}{2} + (I_1 - I_2) \frac{1}{Y_2} \Rightarrow V_1 = I_1 \left(\frac{Z_1}{2} + \frac{1}{Y_2} \right) - \frac{I_2}{Y_2} \quad \textcircled{2}$$

$$\text{loop 2: } \textcircled{3} I_2 \frac{Z_1}{2} + (I_2 - I_1) \frac{1}{Y_2} + V_2 = 0 \Rightarrow$$

$$\text{From (3), } \textcircled{4} I_2 \left(\frac{Z_1}{2} + \frac{1}{Y_2} \right) + V_2 = I_1 \frac{1}{Y_2} \Rightarrow$$

$$\textcircled{5} \boxed{I_1 = V_2 Y_2 + \left(\frac{Z_1 Y_2}{2} + 1 \right) I_2}$$

$$\text{use (5) in (2)} \Rightarrow \textcircled{6} V_1 = \left[V_2 Y_2 + \left(\frac{Z_1 Y_2}{2} + 1 \right) I_2 \right] \left(\frac{Z_1}{2} + \frac{1}{Y_2} \right) - \frac{I_2}{Y_2} \Rightarrow$$

$$\textcircled{7} V_1 = V_2 \left(Y_2 \right) \left(\frac{Z_1}{2} + \frac{1}{Y_2} \right) + I_2 \left\{ \left(\frac{Z_1 Y_2}{2} + 1 \right) \left(\frac{Z_1}{2} + \frac{1}{Y_2} \right) - \frac{1}{Y_2} \right\}$$

$$\textcircled{8} V_1 = V_2 \left(\frac{Z_1 Y_2}{2} + 1 \right) + I_2 \left(\frac{Z_1^2 Y_2}{4} + \frac{Z_1}{2} + \frac{Z_1}{2} + \frac{1}{Y_2} - \frac{1}{Y_2} \right) \Rightarrow$$

$$\textcircled{9} \boxed{V_1 = V_2 \left(1 + \frac{Z_1 Y_2}{2} \right) + I_2 \left(Z_1 \left(1 + \frac{Z_1 Y_2}{4} \right) \right)} = \textcircled{10}$$

$$\text{From (5) \& (9)} \Rightarrow \begin{pmatrix} V_1 \\ I_1 \end{pmatrix} = \begin{pmatrix} 1 + \frac{Z_1 Y_2}{2} & Z_1 \left(1 + \frac{Z_1 Y_2}{4} \right) \\ Y_2 & 1 + \frac{Z_1 Y_2}{2} \end{pmatrix} \begin{pmatrix} V_2 \\ I_2 \end{pmatrix}$$

Compare (10) with (1) (previous page) $\Rightarrow$

$$\boxed{1 + \frac{Z_1 Y_2}{2} = \cosh(\gamma l)}$$

$$\boxed{Z_1 \left(1 + \frac{Z_1 Y_2}{4} \right) = Z_0 \sinh(\gamma l)}$$

$$\boxed{Y_2 = \frac{\sinh(\gamma l)}{Z_0}}$$

Question 3: A uniform plane wave is traveling downward in +z-direction in seawater, with the x-y plane denoting the sea surface and z=0 denoting a point just below the surface. The constitutive parameters of seawater are $\epsilon_r = 80$, $\mu_r = 1$, and $\sigma = 4$ [S/m].

If the magnetic field at z=0 is given by $\vec{H}(0,t) = \hat{a}_y 100 \cos(2\pi \times 10^3 t + 15^\circ)$ [mA/m].

a) Obtain expression for $\vec{E}(z,t)$ and $\vec{H}(z,t)$.

b) Determine the depth at which the amplitude of $\vec{E}$ is 1% of its value at z=0.

Let us calculate $\frac{\sigma}{\omega \epsilon}$ to see if sea water is a good conductor or good dielectric. From $\vec{H}(0,t) = \hat{a}_y 100 \cos(2\pi \times 10^3 t + 15^\circ) \Rightarrow$

① $\omega = 2\pi \nu = 2\pi \times 10^3$ then

② $\frac{\sigma}{\omega \epsilon} = \frac{4}{2\pi \times 10^3 \times 8.85 \times 10^{-12} \times 80} = 8.992 \times 10^5 \gg 1 \Rightarrow$

Good conductor

In General $\vec{E}(z) = E_0 e^{-\alpha z} e^{-j\beta z} \hat{a}_x = |E_0| e^{j\phi_0 - \alpha z - j\beta z} \hat{a}_x$ (3)

$\vec{H}(z) = \frac{E_0}{\eta} e^{-\alpha z} e^{-j\beta z} \hat{a}_y = \frac{|E_0|}{\eta} e^{j\phi_0 - \alpha z - j\beta z} \hat{a}_y$ (4)

[or if you like $\vec{H} = \frac{E_0}{\eta} e^{-\alpha z} e^{-j\beta z} \hat{a}_y$ & $\vec{E} = -\eta \hat{a}_B \times \vec{H} = -\eta \frac{E_0}{\eta} e^{-\alpha z} e^{-j\beta z} \hat{a}_x \hat{a}_y$
 $\Rightarrow \vec{E} = E_0 e^{-\alpha z} e^{-j\beta z} \hat{a}_x$ as given in (3)]

⑤ $\alpha = \sqrt{\frac{\omega \mu \sigma}{2}} = \sqrt{\frac{2\pi \nu \mu \sigma}{2}} = \sqrt{\frac{2\pi \times 10^3 \times 4\pi \times 10^{-7} \times 4}{2}} = 0.1257$ [NP/m]

⑥ $\beta = \sqrt{\frac{\omega \mu \sigma}{2}} = \sqrt{\frac{2\pi \nu \mu \sigma}{2}} = 0.1257$ [rad/m]

⑦ $\eta = \sqrt{\frac{\omega \mu}{2\sigma}} (1+j) = \sqrt{\frac{2\pi \nu \mu}{2\sigma}} (1+j) = \sqrt{\frac{2\pi \times 10^3 \times 4\pi \times 10^{-7}}{2 \times 4}} (1+j) = 3.1416 (1+j) = 0.044 \angle 45^\circ$

From (4) ⑦ $\vec{H}(z) = \frac{|E_0|}{0.044} e^{j\phi_0 - 0.1257z} e^{-j0.1257z} \hat{a}_y \Rightarrow$

⑨ $\vec{H}(z, t) = \frac{|E_0|}{0.044} e^{-0.1257z} \cos(2\pi \times 10^3 t + \phi_0 - 45^\circ - 0.1257z) \hat{a}_y$

$\Rightarrow H(z=0, t) = \frac{|E_0|}{0.044} \cos(2\pi \times 10^3 t + \phi_0 - 45^\circ) \hat{a}_y$

⑩

Compare (10) with given $\vec{H}(0, t)$ in problem statement, i.e

⑪ $\frac{|E_0|}{0.044} \cos(2\pi \times 10^3 t + \phi_0 - 45^\circ) = 100 \cos(2\pi \times 10^3 t + 15^\circ) \Rightarrow$

From (11) it is clear that

⑫ $\frac{|E_0|}{0.044} = 100 \Rightarrow |E_0| = 4.4 \text{ [mV/m]}$

⑬ $\phi_0 = 60^\circ$ then

⑭ $\vec{E}(z) = 4.4 e^{j60^\circ - 0.1257z} e^{-j0.1257z} \hat{a}_x \text{ [mV/m]}$

⑮ $\vec{E}(z, t) = 4.4 e^{-0.1257z} \cos(2\pi \times 10^3 t - 0.1257z + 60^\circ) \hat{a}_x \text{ [mV/m]}$

⑯ $\vec{H}(z) = 4.4 e^{j60^\circ} \frac{1}{0.044 e^{j45^\circ}} e^{-0.1257z} e^{-j0.1257z} \hat{a}_y \text{ or}$

$\vec{H}(z, t) = 100 e^{-0.1257z} \cos(2\pi \times 10^3 t - 0.1257z + 15^\circ) \hat{a}_y$

b) At $z=0$ $|E|=4.4$, i.e. $4.4 = \frac{4.4}{100}$ then we want z' for which

$\frac{4.4}{100} = 4.4 e^{-0.1257z'} \Rightarrow \frac{1}{100} = e^{-0.1257z'} \Rightarrow$

$z' = \frac{\ln(0.01)}{-0.1257} = 36.636 \text{ [m]}$

Question 4: In class we saw that for ionized gas $v_p v_g = c^2$; where v_g is the group velocity, v_p is the phase velocity and c is the speed of light in vacuum. Prove that in general $v_p v_g = c^2$ implies a hyperbolic dispersion relation (k vs. ω). Recall that the equation for hyperbola is given by $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. For simplicity only consider the one-dimensional propagation case.

① $v_p = \frac{\omega}{k}$, ② $v_g = \frac{d\omega}{dk}$ then

③ $v_p v_g = c^2 \Rightarrow$ ④ $\frac{\omega}{k} \frac{d\omega}{dk} = c^2 \Rightarrow$ ⑤ $\omega d\omega = c^2 k dk \Rightarrow$

⑥ $\int \omega d\omega = c^2 \int k dk \Rightarrow$ ⑦ $\frac{\omega^2}{2} + A_0 = c^2 \left[\frac{k^2}{2} + A_1 \right]$ where

A_0 & A_1 are constants

(7) $\Rightarrow$ $A_0 - c^2 A_1 = c^2 \frac{k^2}{2} - \frac{\omega^2}{2} \Rightarrow$

Let $A_2 = A_0 - c^2 A_1$ where A_2 is a constant then

⑧ $\frac{c^2 k^2}{2} - \frac{\omega^2}{2} = A_2 \Rightarrow$ ⑨ $\frac{k^2}{2A_2/c^2} - \frac{\omega^2}{2A_2} = 1$

Let $a^2 = 2A_2/c^2$ & $b^2 = 2A_2$ then

(9) $\Rightarrow$ $\boxed{\frac{k^2}{a^2} - \frac{\omega^2}{b^2} = 1}$ a hyperbola

OR

$$V_p V_g = c^2$$

$$V_p = \frac{c}{n} \quad V_g = \frac{1}{dk/d\omega} \quad \& \quad k = \frac{\omega}{c} n \Rightarrow$$

$$V_g = \frac{c}{\frac{d}{d\omega} \omega n} \quad \text{then}$$

$$V_g V_p = c^2 \Rightarrow \frac{c}{n} \frac{c}{d\omega n / d\omega} = c^2 \Rightarrow$$

$$n \frac{d\omega n}{d\omega} = 1 \quad \text{let } z = \omega n \Rightarrow n = \frac{z}{\omega}$$

$$\text{then } \frac{z}{\omega} \frac{dz}{d\omega} = 1 \Rightarrow z dz = \omega d\omega \Rightarrow \int z dz = \int \omega d\omega$$

$$\frac{z^2}{2} = \frac{\omega^2}{2} + A \quad A \text{ is a constant}$$

$$\text{Recall } z^2 = \omega^2 n^2 = c^2 k^2 \quad \text{then}$$

$$\frac{c^2 k^2}{2} = \frac{\omega^2}{2} + A \Rightarrow \frac{k^2}{(2/c^2)} - \frac{\omega^2}{2} = A \Rightarrow$$

$$\boxed{\frac{k^2}{(2A/c^2)} - \frac{\omega^2}{2A} = 1} \Rightarrow \boxed{\frac{k^2}{(\sqrt{2A}/c)^2} - \frac{\omega^2}{(\sqrt{2A})^2} = 1} \equiv$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Question 5: Figure shows a uniform plane wave obliquely incident on the interface between two perfect dielectrics.

For this configuration, the so called parallel polarization, where electric field is on the plane of incidence, answer the following questions:

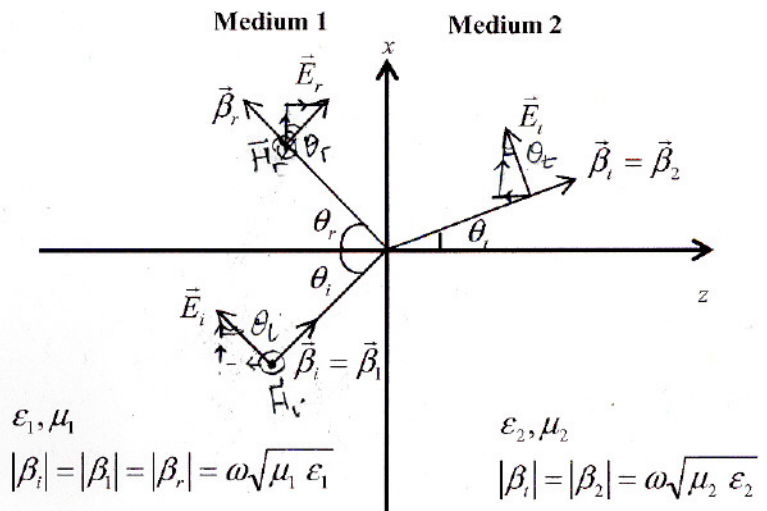
a) What are the time harmonic field expressions for the incident electric and magnetic fields? (4 Pts)

b) What are the time harmonic field expressions for the reflected electric and magnetic fields? (4 Pts)

c) What are the time harmonic field expressions for the transmitted electric and magnetic fields? (4 Pts)

d) Describe the procedure by which the Fresnel transmission and reflection coefficients can be obtained. (3 Pts)

e) Give the expression for the Fresnel reflection coefficient in terms of the propagation constant β_{1z} and β_{2z} , (i.e., the projection of $\vec{\beta}_1$ and $\vec{\beta}_2$ along the z-direction.) (5 pts)



a)

$$\vec{E}_i = E_{i0} (\cos \theta_i \hat{a}_x - \sin \theta_i \hat{a}_z) e^{-j\vec{\beta}_i \cdot \vec{r}} \Rightarrow$$

$$\vec{E}_i = E_{i0} (\cos \theta_i \hat{a}_x - \sin \theta_i \hat{a}_z) e^{-j(\sin \theta_i x + \cos \theta_i z) \beta_1}$$

$$\vec{H}_i = \frac{E_{i0}}{\eta_1} e^{-j\vec{\beta}_i \cdot \vec{r}} \hat{a}_y \Rightarrow$$

$$\vec{H}_i = \frac{E_{i0}}{\eta_1} e^{-j(\sin \theta_i x + \cos \theta_i z) \beta_1} \hat{a}_y$$

$$\vec{E}_r = E_{r0} (\cos \theta_r \hat{a}_x + \sin \theta_r \hat{a}_z) e^{-j\vec{\beta}_r \cdot \vec{r}} \Rightarrow$$

$$\vec{E}_r = E_{r0} (\cos \theta_r \hat{a}_x + \sin \theta_r \hat{a}_z) e^{-j(\sin \theta_r x - \cos \theta_r z) \beta_1}$$

$$\vec{H}_r = -\frac{E_{r0}}{\eta_1} e^{-j\vec{\beta}_r \cdot \vec{r}} \hat{a}_y \Rightarrow$$

$$\vec{H}_r = -\frac{E_{r0}}{\eta_1} e^{-j(\sin \theta_r x - \cos \theta_r z) \beta_1} \hat{a}_y$$

$$\vec{\beta}_i = \beta_1 (\sin \theta_i \hat{a}_x + \cos \theta_i \hat{a}_z) \quad (5)$$

$$\vec{\beta}_r = \beta_1 (\sin \theta_r \hat{a}_x - \cos \theta_r \hat{a}_z) \quad (6)$$

$$\vec{\beta}_t = \beta_2 (\sin \theta_t \hat{a}_x + \cos \theta_t \hat{a}_z) \quad (7)$$

$$E_{r0} = \Gamma E_{i0}$$

$$c) \quad \textcircled{12} \quad \vec{E}_t = E_{t0} (\cos \theta_t \hat{a}_x - \sin \theta_t \hat{a}_z) e^{-j \vec{B}_t \cdot \vec{r}} \Rightarrow$$

$$\textcircled{13} \quad \boxed{\vec{E}_t = E_{t0} (\cos \theta_t \hat{a}_x - \sin \theta_t \hat{a}_z) e^{-j (\sin \theta_t x + \cos \theta_t z) \beta_2}}$$

$$\textcircled{14} \quad \boxed{\vec{H}_t = \frac{E_{t0}}{\eta_2} e^{-j (\sin \theta_t x + \cos \theta_t z) \beta_2} \hat{a}_y} \quad E_{t0} = \tau E_{i0}$$

d) We apply B.C at $z=0$ plane which requires tangential $\vec{E}$ & $\vec{H}$ to be continuous across the boundary

$$\textcircled{15} \quad E_{i0} \cos \theta_i e^{-j \sin \theta_i x \beta_1} + E_{r0} \cos \theta_r e^{-j \sin \theta_r x \beta_1} = E_{t0} \cos \theta_t e^{-j \sin \theta_t x \beta_2}$$

$$\textcircled{16} \quad \frac{E_{i0}}{\eta_1} e^{-j \sin \theta_i x \beta_1} - \frac{E_{r0}}{\eta_1} e^{-j \sin \theta_r x \beta_1} = \frac{E_{t0}}{\eta_2} e^{-j \sin \theta_t x \beta_2}$$

$$(15) \& (16) \Rightarrow \textcircled{17} \quad \boxed{\theta_i = \theta_r} \quad \& \quad \textcircled{18} \quad \boxed{\beta_1 \sin \theta_i = \beta_2 \sin \theta_t}$$

↑ Snell's laws ↑

$$\begin{aligned} e) \quad \Gamma_{11} &= \frac{\eta_2 \cos \theta_t - \eta_1 \cos \theta_i}{\eta_2 \cos \theta_t + \eta_1 \cos \theta_i} = \frac{\sqrt{\frac{\mu_2}{\epsilon_2}} \cos \theta_t - \sqrt{\frac{\mu_1}{\epsilon_1}} \cos \theta_i}{\sqrt{\frac{\mu_2}{\epsilon_2}} \cos \theta_t + \sqrt{\frac{\mu_1}{\epsilon_1}} \cos \theta_i} \quad \text{Multiply by } \epsilon_1 \epsilon_2 \\ &= \frac{\epsilon_1 \epsilon_2}{\epsilon_1 \epsilon_2} \frac{\sqrt{\frac{\mu_2}{\epsilon_2}} \cos \theta_t - \sqrt{\frac{\mu_1}{\epsilon_1}} \cos \theta_i}{\sqrt{\frac{\mu_2}{\epsilon_2}} \cos \theta_t + \sqrt{\frac{\mu_1}{\epsilon_1}} \cos \theta_i} = \frac{\epsilon_1 \sqrt{\epsilon_2 \mu_2} \cos \theta_t - \epsilon_2 \sqrt{\epsilon_1 \mu_1} \cos \theta_i}{\epsilon_1 \sqrt{\epsilon_2 \mu_2} \cos \theta_t + \epsilon_2 \sqrt{\epsilon_1 \mu_1} \cos \theta_i} \\ &= \frac{\epsilon_1 \omega \sqrt{\epsilon_2 \mu_2} \cos \theta_t - \epsilon_2 \omega \sqrt{\epsilon_1 \mu_1} \cos \theta_i}{\epsilon_1 \omega \sqrt{\epsilon_2 \mu_2} \cos \theta_t + \epsilon_2 \omega \sqrt{\epsilon_1 \mu_1} \cos \theta_i} = \frac{\epsilon_1 \beta_2 z - \epsilon_2 \beta_1 z}{\epsilon_1 \beta_2 z + \epsilon_2 \beta_1 z} \quad \text{where} \\ &\quad \boxed{\beta_1 z = \omega \sqrt{\epsilon_1 \mu_1} \cos \theta_i = \beta_1 \cos \theta_i} \quad \& \quad \boxed{\beta_2 z = \omega \sqrt{\epsilon_2 \mu_2} \cos \theta_t = \beta_2 \cos \theta_t} \end{aligned}$$