HW #6

Q: In deriving the non-homogeneous wave equations for scalar Potential (V) & vector potential ($\vec{A}$) we used the Lorentz-gauge given by $\nabla\cdot\vec{A}+\mu\varepsilon\frac{\partial V}{\partial t}=0$. Obtain the Differential equations relating V & $\vec{A}$ to the sources (ρ & $\vec{J}$) under different gauge for which $\nabla\cdot\vec{A}=0$. This Condition is called Coulomb gauge. Comment on the use of Coulomb gauge vs Lorentz gauge.
(Assume a simple medium)

sol: Recall that we obtained the non-homogeneous wave equations for $\vec{A}$ & V by considering the following

(1) $\nabla\cdot\vec{B}=0 \Rightarrow$ (2) $\boxed{\vec{B}=\nabla\times\vec{A}}$ (5) $\nabla\times\vec{E}+\frac{\partial}{\partial t}\nabla\times\vec{A}=0 \Rightarrow$

(3) $\nabla\times\vec{E}=-\frac{\partial}{\partial t}\vec{B} \Rightarrow$ (4) $\nabla\times E+\frac{\partial}{\partial t}\vec{B}=0 \Rightarrow$ $\vec{E}+\frac{\partial}{\partial t}\vec{A}=-\nabla V \Rightarrow$ (8) $\boxed{\vec{E}=-\nabla V-\frac{\partial}{\partial t}\vec{A}}$

(6) $\nabla\times(E+\frac{\partial}{\partial t}\vec{A})=0 \Rightarrow$ (7)

* Recall in Ampere's law we have (2) $\nabla\times\vec{H}=\vec{J}+\frac{\partial}{\partial t}\vec{D} \Rightarrow$

(11) (10) $\mu(\cancel{x})$ & $\varepsilon(\cancel{x})$

(9) $\nabla\times\frac{\vec{B}}{\mu}=\vec{J}+\frac{\partial}{\partial t}\varepsilon\vec{E}$ for homogeneous medium

use (2) & (8) in (9) $\Rightarrow$

(12) $\nabla\times(\nabla\times\vec{A})=\mu\vec{J}+\varepsilon\mu\frac{\partial}{\partial t}\left[-\nabla V-\frac{\partial}{\partial t}\vec{A}\right] \Rightarrow$

(13) $\nabla(\nabla\cdot\vec{A})-\nabla^2\vec{A}=\mu\vec{J}-\nabla(\varepsilon\mu\frac{\partial}{\partial t}V)-\varepsilon\mu\frac{\partial^2}{\partial t^2}\vec{A}$

(14) $-\nabla^2\vec{A}+\varepsilon\mu\frac{\partial^2}{\partial t^2}\vec{A}=\mu\vec{J}-\nabla\left[\nabla\cdot\vec{A}+\varepsilon\mu\frac{\partial}{\partial t}V\right]$

* Now if we had use Lorentz gauge i.e (15) $\nabla\cdot\vec{A}+\varepsilon\mu\frac{\partial}{\partial t}V=0$

we would have obtained (16) $-\nabla^2 A+\varepsilon\mu\frac{\partial^2}{\partial t^2}\vec{A}=\mu\vec{J}$, but here we are

to use Coulomb gauge (17) $\nabla\cdot\vec{A}=0$ then (14) $\Rightarrow$

$$\boxed{-\nabla^2\vec{A}+\varepsilon\mu\frac{\partial^2}{\partial t^2}\vec{A}=\mu\vec{J}-\varepsilon\mu\nabla\left(\frac{\partial}{\partial t}V\right)} \quad (18)$$

* How about the diff. equation for V. To find that Eq. we use

the Gauss law (19) $\nabla\cdot\varepsilon\vec{E}=\rho \Rightarrow$ (20) $\nabla\cdot\vec{E}=\rho/\varepsilon$, since $\rho(\cancel{x})$

use (8) in (20) $\Rightarrow$ (21) $\nabla\cdot\left[-\nabla V-\frac{\partial}{\partial t}\vec{A}\right]=\rho/\varepsilon \Rightarrow$

(22) $-\nabla^2 V-\frac{\partial}{\partial t}\nabla\cdot\vec{A}=\rho/\varepsilon$ but from coulomb's gauge $\nabla\cdot\vec{A}=0 \Rightarrow$

$$\boxed{\nabla^2 V=-\rho/\varepsilon} \quad (23)$$

* We see that by using Coulomb gauge the equation governing V is simplified (as compared to Lorentz gauge) but equation for $\vec{A}$ is more complicated & furthermore is no longer decoupled from V as it can be seen from Eq (18).

Q: The inhomogeneous wave equation for V & $\vec{A}$ were obtained for simple medium (Linear, isotropic & homogeneous), subject to Lorentz gauge. Here try to obtain the differential equations governing the dynamical behavior of V & $\vec{A}$, when ε & μ depend on position, using the Lorentz gauge. Comment on your results.

Sol:

we start with $\nabla \cdot \vec{B} = 0 \Rightarrow \boxed{\vec{B} = \nabla \times \vec{A}}$ ① &

$\nabla \times \vec{E} = -\frac{\partial}{\partial t}\vec{B} \Rightarrow \nabla \times \vec{E} = -\frac{\partial}{\partial t}\nabla \times \vec{A} \Rightarrow \nabla \times (\vec{E} + \frac{\partial}{\partial t}\vec{A}) = 0 \Rightarrow$

② $\boxed{E = -\frac{\partial}{\partial t}\vec{A} - \nabla V}$

* Eqs (1) & (2) are not changed from previous case.

using Ampere's law

③ $\nabla \times \vec{H} = \vec{J} + \frac{\partial}{\partial t}\varepsilon\vec{E} \Rightarrow$ ④ $\nabla \times \left[\frac{\vec{B}}{\mu(r)}\right] = \vec{J} + \varepsilon\frac{\partial}{\partial t}\vec{E}$

Recall $\nabla \times (f\vec{F}) = f\nabla \times \vec{F} + \nabla f \times \vec{F}$ then (4) $\Rightarrow$

⑤ $\frac{1}{\mu}\nabla \times \vec{B} + \nabla(\frac{1}{\mu}) \times \vec{B} = \vec{J} + \varepsilon\frac{\partial}{\partial t}\vec{E}$ Use (1) & (2) in (5) $\Rightarrow$

$\frac{1}{\mu}\nabla \times [\nabla \times \vec{A}] + \nabla(\frac{1}{\mu}) \times (\nabla \times \vec{A}) = \vec{J} + \varepsilon\frac{\partial}{\partial t}\left[-\frac{\partial}{\partial t}\vec{A} - \nabla V\right] \Rightarrow$

$\frac{1}{\mu}\left[\nabla(\nabla \cdot \vec{A}) - \nabla^2\vec{A}\right] + \nabla(\frac{1}{\mu}) \times (\nabla \times \vec{A}) = \vec{J} - \varepsilon\frac{\partial^2}{\partial t^2}\vec{A} - \varepsilon\frac{\partial}{\partial t}\nabla V \Rightarrow$

(6)
$$\nabla(\nabla\cdot\vec{A}) - \nabla^2\vec{A} + \mu\nabla\left(\frac{1}{\mu}\right)\times(\nabla\times\vec{A}) = \mu\vec{J} - \varepsilon\mu\frac{\partial^2\vec{A}}{\partial t^2} - \frac{\partial}{\partial t}\varepsilon\mu\nabla V$$

since $\varepsilon(\cancel{t})$ & $\mu(\cancel{t})$

(7)

we can write the following $\nabla(\psi f) = f\nabla\psi + \psi\nabla f$ then

(8) $\nabla(\varepsilon\mu V) = V\nabla(\varepsilon\mu) + \varepsilon\mu\nabla V \Rightarrow \nabla(\varepsilon\mu V) - V\nabla(\varepsilon\mu) = \varepsilon\mu\nabla V$

* so we rewrite (6) as

(9)
$$-\nabla^2\vec{A} + \mu\nabla\left(\frac{1}{\mu}\right)\times(\nabla\times\vec{A}) = \mu\vec{J} - \varepsilon\mu\frac{\partial^2\vec{A}}{\partial t^2} - \nabla(\nabla\cdot\vec{A}) - \frac{\partial}{\partial t}\left[\nabla(\varepsilon\mu V) - V\nabla(\varepsilon\mu)\right] \Rightarrow$$

(10)
$$-\nabla^2\vec{A} + \mu\nabla\left(\frac{1}{\mu}\right)\times(\nabla\times\vec{A}) = \mu\vec{J} - \varepsilon\mu\frac{\partial^2\vec{A}}{\partial t^2} - \nabla\left[\nabla\cdot\vec{A} + \varepsilon\mu\frac{\partial}{\partial t}V\right] + \frac{\partial}{\partial t}\left[V\nabla(\varepsilon\mu)\right]$$

(11)

From Lorentz gauge $\nabla\cdot\vec{A} + \varepsilon\mu\frac{\partial}{\partial t}V = 0 \Rightarrow$

$$-\nabla^2\vec{A} + \mu\nabla\left(\frac{1}{\mu}\right)\times(\nabla\times\vec{A}) = \mu\vec{J} - \varepsilon\mu\frac{\partial^2\vec{A}}{\partial t^2} + \nabla(\varepsilon\mu)\frac{\partial}{\partial t}V \Rightarrow$$

(12)
$$\boxed{-\left(\frac{\partial}{\partial t}V\right)\nabla(\varepsilon\mu) + \mu\nabla\left(\frac{1}{\mu}\right)\times(\nabla\times\vec{A}) - \mu\vec{J} = \nabla^2\vec{A} - \varepsilon\mu\frac{\partial^2\vec{A}}{\partial t^2}}$$

* As equation for V, we use the Gauss law

(13) $\nabla\cdot\vec{D}=\rho \Rightarrow$ (14) $\nabla\cdot\varepsilon\vec{E}=\rho$ but (15) $\nabla\cdot f\vec{F}=\nabla f\cdot\vec{F}+f\nabla\cdot\vec{F}$

then (14) $\Rightarrow$

(16) $\nabla\varepsilon\cdot\vec{E}+\varepsilon\,\nabla\cdot\vec{E}=\rho$ use (17) $\vec{E}=-\dfrac{\partial\vec{A}}{\partial t}-\nabla V$ in (16) $\Rightarrow$

$$\nabla\varepsilon\cdot\left[-\frac{\partial\vec{A}}{\partial t}-\nabla V\right]+\varepsilon\,\nabla\cdot\left[-\frac{\partial\vec{A}}{\partial t}-\nabla V\right]=\rho \Rightarrow$$

$$\nabla\varepsilon\cdot\frac{\partial\vec{A}}{\partial t}+\nabla\varepsilon\cdot\nabla V+\varepsilon\frac{\partial}{\partial t}\nabla\cdot\vec{A}+\varepsilon\nabla^2V=-\rho$$

Divide by $\varepsilon \Rightarrow$ (18) $\dfrac{\nabla\varepsilon}{\varepsilon}\cdot\dfrac{\partial\vec{A}}{\partial t}+\dfrac{\nabla\varepsilon\cdot\nabla V}{\varepsilon}+\dfrac{\partial}{\partial t}\nabla\cdot\vec{A}+\nabla^2V=-\rho/\varepsilon$

use Lorentz gauge (19) $\nabla\cdot\vec{A}=-\mu\varepsilon\dfrac{\partial V}{\partial t}$ in (18) $\Rightarrow$

$$\frac{\nabla\varepsilon}{\varepsilon}\cdot\frac{\partial\vec{A}}{\partial t}+\frac{\nabla\varepsilon\cdot\nabla V}{\varepsilon}+\frac{\partial}{\partial t}\left[-\mu\varepsilon\frac{\partial V}{\partial t}\right]+\nabla^2V=-\rho/\varepsilon \Rightarrow$$

$$\frac{\nabla\varepsilon}{\varepsilon}\cdot\frac{\partial\vec{A}}{\partial t}+\frac{\nabla\varepsilon\cdot\nabla V}{\varepsilon}-\mu\varepsilon\frac{\partial^2}{\partial t^2}V+\nabla^2V=-\rho/\varepsilon \Rightarrow$$

(20) $$\boxed{\nabla^2V-\mu\varepsilon\frac{\partial^2}{\partial t^2}V=-\frac{\rho}{\varepsilon}-\frac{\nabla\varepsilon}{\varepsilon}\cdot\frac{\partial\vec{A}}{\partial t}-\frac{\nabla\varepsilon\cdot\nabla V}{\varepsilon}}$$

* The complexity of Eq (12) & (20) for the case of $\varepsilon(r)$ & $\mu(r)$ as compared to homogeneous medium [$\varepsilon(\cancel{r})$, $\mu(\cancel{r})$] speaks for itself.

Q: Derive the two divergence equations $\nabla\cdot\vec{B}=0$ & $\nabla\cdot\vec{D}=\rho$ from the two curl equations (Faraday's & Ampere's law) & the continuity equation

sol:

Faradays law ① $\nabla\times\vec{E}=-\frac{\partial\vec{B}}{\partial t} \Rightarrow$ ② $\nabla\cdot(\nabla\times\vec{E})=-\frac{\partial}{\partial t}\nabla\cdot\vec{B}$

but div (curl any vector function) $=0 \Rightarrow$

③ $0=-\frac{\partial}{\partial t}\nabla\cdot\vec{B}$. since we are concern with electrodynamics for which we suppose $\vec{B}$ is a function of time, then the only way (3) is satisfied is for $\nabla\cdot\vec{B}=0$ ④

From Ampere's law ⑤ $\nabla\times\vec{H}=\vec{J}+\frac{\partial}{\partial t}\vec{D} \Rightarrow$ ⑥ $\nabla\cdot(\nabla\times\vec{H})=\nabla\cdot\vec{J}+\nabla\cdot\frac{\partial}{\partial t}\vec{D}$

but $\nabla\cdot(\nabla\times$ any vector function$)=0 \Rightarrow$ continuity equation ⑧ $\nabla\cdot\vec{J}=-\frac{\partial\rho}{\partial t}$ then

⑦ $0=\nabla\cdot\vec{J}+\frac{\partial}{\partial t}\nabla\cdot\vec{D}$ from

$(7)\Rightarrow$ ⑧ $0=-\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial t}\nabla\cdot\vec{D} \Rightarrow$ ⑨ $\frac{\partial}{\partial t}\nabla\cdot\vec{D}=\frac{\partial\rho}{\partial t}$

for (9) to hold for all times we must have

$$\boxed{\nabla\cdot\vec{D}=\rho}$$

Q: obtain the time-dependant & time harmonic wave equations for $\vec{E}$ & $\vec{H}$ in the case of linear, isotropic, homogeneous medium for which there are free charges & conduction current present.

sol: From Maxwell equation (time-dependent)

① $\nabla \times \vec{E} = -\frac{\partial B}{\partial t} \Rightarrow \nabla \times \vec{E} = -\mu \frac{\partial}{\partial t} \vec{H}$

② $\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \Rightarrow \nabla \times \vec{H} = \sigma \vec{E} + \varepsilon \frac{\partial \vec{E}}{\partial t}$

③ $\nabla \cdot \vec{D} = \rho \Rightarrow \nabla \cdot \vec{E} = \rho / \varepsilon$

④ $\nabla \cdot \vec{B} = \nabla \cdot \vec{H} = 0$

(1) ⇒ ⑤ $\nabla \times \nabla \times \vec{E} = -\mu \frac{\partial}{\partial t} \nabla \times \vec{H} \Rightarrow$ ⑥ $\nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E} = -\mu \frac{\partial}{\partial t}\left[\sigma \vec{E} + \varepsilon \frac{\partial \vec{E}}{\partial t}\right]$

use (3) ⇒ ⑦ $\nabla(\rho/\varepsilon) - \nabla^2 \vec{E} = -\mu\sigma \frac{\partial}{\partial t} \vec{E} - \mu\varepsilon \frac{\partial^2}{\partial t^2} \vec{E} \Rightarrow$

⑧ $$\boxed{\frac{1}{\varepsilon} \nabla \rho + \mu\sigma \frac{\partial}{\partial t} \vec{E} + \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} = \nabla^2 \vec{E}}$$

* For $\vec{H}$ take curl of (2) ⇒

⑨ $\nabla \times \nabla \times \vec{H} = \nabla \times (\sigma \vec{E}) + \nabla \times (\varepsilon \frac{\partial \vec{E}}{\partial t}) \Rightarrow$

$\nabla(\nabla \cdot \vec{H}) - \nabla^2 \vec{H} = \sigma \nabla \times \vec{E} + \varepsilon \frac{\partial}{\partial t} \nabla \times \vec{E}$

$0 - \nabla^2 \vec{H} = \sigma\left[-\mu \frac{\partial}{\partial t} \vec{H}\right] + \varepsilon \frac{\partial}{\partial t}\left[-\mu \frac{\partial}{\partial t} \vec{H}\right] \Rightarrow$

⑩ $-\nabla^2 \vec{H} = -\mu\sigma \frac{\partial}{\partial t} \vec{H} - \varepsilon\mu \frac{\partial^2}{\partial t^2} \vec{H} \Rightarrow$

(11) $$\boxed{\nabla^2 \vec{H} = \mu\sigma \frac{\partial}{\partial t}\vec{H} + \varepsilon\mu \frac{\partial^2}{\partial t^2}\vec{H}}$$

* For time Harmonic fields $\frac{\partial}{\partial t} \rightarrow j\omega$ & $\frac{\partial^2}{\partial t^2} \rightarrow -\omega^2$ then

(8) $\Rightarrow$ $$\boxed{\frac{1}{\varepsilon}\nabla\rho + j\omega\mu\sigma\vec{E}(r) - \omega^2\mu\varepsilon\vec{E}(r) = \nabla^2\vec{E}(r)}$$

(11) $\Rightarrow$ $$\boxed{\nabla^2\vec{H}(r) = j\omega\mu\sigma\vec{H}(r) - \omega^2\varepsilon\mu\vec{H}(r)}$$

Q: prove that $U(R,t) = f(t - R/c)$ is a solution of diff. equation (1)

$$\frac{\partial^2 U}{\partial R^2} - \frac{1}{c^2}\frac{\partial^2 U}{\partial t^2} = 0$$

Remark: Recall that Eq (1) was obtained in our attempt to find a solution for wave equation for scaler Potential $V(R,t) = \frac{U(R,t)}{R}$

Sol: let $t - \frac{R}{c} = y$ (2) then $\frac{\partial y}{\partial R} = -1/c$ (3) & $\frac{\partial y}{\partial t} = 1$ (4) &

$U(R,t) = f(t - R/c) = f(y)$

* Using the chain rule we can write

(5) $$\frac{\partial U}{\partial R} = \frac{\partial f(y)}{\partial R} = \frac{\partial y}{\partial R} \cdot \frac{\partial f(y)}{\partial y} = \frac{-1}{c}\frac{\partial f(y)}{\partial y}.$$

Applying the chain rule one more time we have

(6) $$\boxed{\frac{\partial^2 U}{\partial R^2} = +\frac{1}{c^2}\frac{\partial^2 f(y)}{\partial y^2}}.$$

Also (7) $$\frac{\partial u}{\partial t} = \frac{\partial f(y)}{\partial t} = \frac{\partial y}{\partial t}\frac{\partial f(y)}{\partial y} = \frac{\partial f(y)}{\partial y}$$ & also

(8) $$\boxed{\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 f(y)}{\partial t^2} = \frac{\partial^2 f(y)}{\partial y^2}}$$

use (6) & (8) in (1) we have

$$+\frac{1}{c^2}\frac{\partial f(y)}{\partial y^2} - \frac{1}{c^2}\frac{\partial^2 f(y)}{\partial y^2} = 0,$$ hence $U(R,t) = f(t - R/c) = f(y)$ is a solution to Eq (1)

An electric field confined between two metallic plate (with air between them) is propagating along z-direction with phase constant β. The instantaneous electric field is given by

$$① \ \vec{E} = \hat{a_y}\, 0.1 \sin(10\pi x) \cos(6\pi 10^9 t - \beta z) \ [V/m]$$

Find the $\vec{H}$ between the two plate & the value of β. Express $\vec{H}$ as instantaneous field.

Sol:

We use time Harmonic field representation for which

$$② \ \vec{E}(r) = \hat{a_y}\, 0.1 \sin(10\pi x)\, e^{-j\beta z} = \hat{a_y} E_y$$

[Check that multiplying (2) with $e^{j\omega t}$ & taking the real part will produce the equation (1) for $\vec{E}(r,t)$, where $\omega = 6\pi 10^9$]

* From Faraday's law $\ ③ \ \nabla \times \vec{E} = -j\omega\mu \vec{H} \Rightarrow ④ \ \vec{H} = \frac{j}{\omega\mu_0} \nabla \times \vec{E}$

μ_0 since we are in air.

$$⑤ \ \nabla \times \vec{E} = \begin{vmatrix} \hat{a_x} & \hat{a_y} & \hat{a_z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & E_y & 0 \end{vmatrix} = -\hat{a_x}\left[\frac{\partial}{\partial z} E_y\right] + \hat{a_z}\left[\frac{\partial}{\partial x} E_y\right]$$

$$⑥ \ = \hat{a_x}\left[(j\beta)\, 0.1 \sin(10\pi x)\, e^{-j\beta z}\right] + \hat{a_z}\left[\pi \cos(10\pi x)\, e^{-j\beta z}\right]$$

then

$$⑦ \ \vec{H} = \frac{j}{\omega\mu_0} \nabla \times \vec{E} = \frac{1}{\omega\mu_0} \left\{ \begin{array}{l} -\hat{a_x}\, 0.1\, \beta \sin(10\pi x)\, e^{-j\beta z} + \\ \hat{a_z}\, j\pi \cos(10\pi x)\, e^{-j\beta z} \end{array} \right\}$$

* To find β we try to find $\vec{E}$ from Ampere's law & compare it to Eq (2). Note that $\beta \neq K_0$

From Ampere's law (8) $\nabla \times \vec{H} = j\omega\varepsilon\vec{E} \Rightarrow$ (9) $\vec{E} = \frac{1}{j\omega\varepsilon_0}\nabla\times\vec{H}$

(9) $\varepsilon = \varepsilon_0$ since the medium between to plate is air

From (7) $\vec{H} = H_x\hat{a}_x + H_z\hat{a}_z$ then

(10) $\nabla\times\vec{H} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ H_x & 0 & H_z \end{vmatrix} = \hat{a}_x\left[\frac{\partial}{\partial y}H_z\right] - \hat{a}_y\left[\frac{\partial}{\partial x}H_z - \frac{\partial}{\partial z}H_x\right] + -\hat{a}_z\left[\frac{\partial}{\partial y}H_x\right]$ using (7) we have

(11) $\nabla\times\vec{H} = \frac{1}{\omega\mu_0}\left\{\hat{a}_x(0) - \hat{a}_y\left[-j10\pi^2\sin(10\pi x)e^{-j\beta z} - j\beta^2 0.1\sin(10\pi x)e^{-j\beta z}\right] - \hat{a}_z(0)\right\}$

$\Rightarrow$

(12) $\nabla\times\vec{H} = \frac{1}{\omega\mu_0}\left\{\hat{a}_y\left(j10\pi^2 + j0.1\beta^2\right)\sin(10\pi x)e^{-j\beta z}\right\}$

Using (12) in (9) we have

(13) $$\boxed{\vec{E} = \frac{1}{\cancel{j}\omega\varepsilon_0}\,\frac{\cancel{j}}{\omega\mu_0}\left\{\hat{a}_y\left(10\pi^2 + 0.1\beta^2\right)\sin(10\pi x)e^{-j\beta z}\right\}}$$

* Now compare (13) obtained from Ampere's law to what was given in the problem statement, i.e Eq (2). For (2) & (13) to be the same we must have

(14) $\frac{1}{\omega^2\mu_0\varepsilon_0}\left(10\pi^2 + 0.1\beta^2\right) = 0.1 \Rightarrow$ OR

(15) $(10\pi)^2 + \beta^2 = \omega^2 \mu_0 \varepsilon_0 \Rightarrow$

(16) $\beta = \sqrt{\omega^2 \mu_0 \varepsilon_0 - (10\pi)^2} = \sqrt{(6\pi \times 10^9)^2 \times \frac{1}{(3\times 10^8)^2} - (10\pi)^2} = 54.4$ [rad/s]

(17)

*Note that $\beta = 54.4 \neq k_0$ when $k_0 = \frac{\omega}{c} = \frac{6\pi \times 10^9}{3\times 10^8} = 62.83$ [rad/s]

*Also note that from the problem statement $\tilde{E} = E_y \hat{e_y} = \hat{a_y}\, 0.1 \sin(10\pi x) \cos(6\pi 10^9 t - \beta z)$

we could have defined a β along x, i.e $\beta_x = 10\pi$ & a β along z, i.e $\beta_z = \beta$ then condition (15) states—

$$\beta_x^2 + \beta_z^2 = \omega^2 \mu_0 \varepsilon_0 \quad \text{where } \omega^2 \mu_0 \varepsilon_0 = \frac{\omega^2}{c^2} = k_0^2$$

* Now that we have $\vec{H}(r)$ [Eq (7)] & value of β [Eq (16)]

Using (7) we can write the instantaneous $\vec{H}$ as

$$\vec{H}(r,t) = \frac{1}{\omega \mu_0} \left\{ \begin{array}{l} -\hat{a_x}\, 0.1 \times 54.4 \sin(10\pi x) \cos(\omega t - \beta z) + \\ \hat{a_z}\, \pi \cos(10\pi x) \cos(\omega t - \beta z + \pi/2) \end{array} \right\}$$

$$= \frac{1}{6\pi \times 10^9 \times 4\pi \times 10^{-7}} \left\{ \begin{array}{l} -\hat{a_x}\, 5.44 \sin(10\pi x) \cos(6\pi 10^9 t - 54.4 z) \\ -\hat{a_z}\, \pi \cos(10\pi x) \sin(6\pi \times 10^9 t - 54.4 z) \end{array} \right\}$$

$$= -\hat{a_x}\, 2.3 \times 10^{-4} \sin(10\pi x) \cos(6\pi \times 10^9 t - 54.4 z)$$
$$- \hat{a_z}\, 1.33 \times 10^{-4} \cos(10\pi x) \sin(6\pi \times 10^9 t - 54.4 z)$$