UNIVERSITY OF TORONTO

Department of Electrical and Computer Engineering

Fields and Waves Laboratory

Courses ECE 320F and ECE 357S

III Year

RESONANT CAVITY

Supplementary Instructions

RESONANCE

1. Introduction, general resonance

A linear, single input resonant system is an assembly of objects exhibiting all of the following properties:

(i) Power input by external source at a frequency produces a steady state response at the same frequency.

(ii) Some of the energy supplied by the source is stored in the system.

(iii) There exists at least one frequency, such that no portion of power absorbed by the system at this frequency is returned to the source.

The definition given above will be illustrated for three systems considered below. In all three cases the source of energy is a generator of EMF=$V_o \cos \omega t$ and internal resistance R_S.

Case (i): The generator driving a load resistance R_L as shown in Fig. 1.

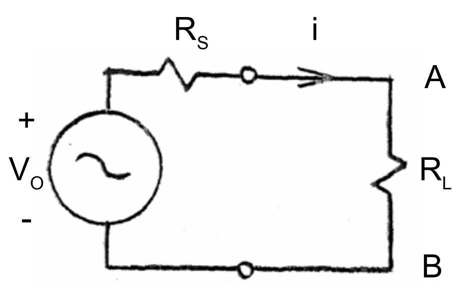


Figure 1. Generator driving a load resistance R_L.

(a) System response: current i,

$$i = \frac{V_0}{R_S + R_L} \cos \omega t. \quad (1)$$

(b) Power supplied by the source p_S,

$$p_s = v_{AB} i = \frac{V_0^2 R_L}{(R_L + R_S)^2} \cos^2 \omega t. \quad (2)$$

(c) Power absorbed by the load p_L,

$$p_L = v_L i = \frac{V_0{}^2 R_L}{(R_L + R_S)^2}\cos^2 \omega t. \tag{3}$$

(d) Energy stored by the load e_L,

$$e_L = \int_o^T (p_s - p_L)dt = \int_o^T 0 \cdot dt = 0. \tag{4}$$

The system is linear but not resonant.

Case (ii) The generator driving an R_L, C circuit shown in Fig. 2

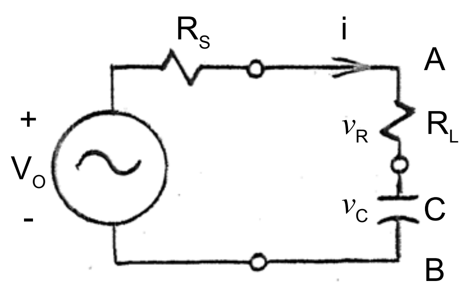


Figure 2. Generator driving R_L, C circuit.

(a) System responses: current i, capacitor voltage v_C, resistor voltage v_R and load voltage v_{AB}.

$$i = R_e\left[\frac{V_0 e^{j\omega t}}{(R_L + R_S - j/\omega C)}\right] = R_e\left[\frac{V_0 e^{j(\omega t+\varphi)}\omega C}{\left\{1 + [(R_L + R_S)\omega C]^2\right\}^{\frac{1}{2}}}\right]. \tag{5a}$$

with $\tan\varphi = 1/(R_L + R_S)\omega C$.

$$i = \frac{V_0 \omega C}{\left\{1 + [(R_L + R_S)\omega C]^2\right\}^{\frac{1}{2}}}\cos(\omega t + \varphi). \tag{5b}$$

$$v_C = \mathrm{Re}\left[V_0 e^{j\omega t}\left(\frac{-j}{\omega C}\right)\frac{1}{[R_S + R_L - j/\omega C]}\right] = \frac{V_0 \cos(\omega t + \varphi - 90^\circ)}{\left\{1 + [(R_L + R_S)\omega C]^2\right\}^{\frac{1}{2}}} \tag{5c}$$

(b) Power supplied by the source p_S,

$$p_s = v_{AB} i = \mathrm{Re}\left[V_0 e^{i\omega t}\frac{R_L - j/\omega C}{R_S + R_L - j/\omega C}\right]\cdot i =$$

$$p_s = V_0 \cos(\omega t + \varphi - \theta)\left\{\frac{1 + (R_L C\omega)^2}{1 + [(R_L + R_S)\omega C]^2}\right\}^{\frac{1}{2}} \frac{V_0 \omega C \cos(\omega t + \varphi)}{\left\{1 + [(R_L + R_S)\omega C]^2\right\}^{\frac{1}{2}}}, \tag{6a}$$

with $\tan\theta = 1/R_L \omega C$.

$$ps = V_o{}^2 \omega C \frac{\left[1 + (R_L \omega C)^2\right]^{\frac{1}{2}}}{1 + [(R_L + R_S)\omega C]^2} \cos(\omega t + \varphi - \theta)\cos(\omega t + \varphi) =$$

$$= V_o{}^2 \omega C \frac{\left[1 + (R_L \omega C)^2\right]^{\frac{1}{2}}}{1 + [(R_L + R_S)\omega C]^2} \frac{1}{2}\{\cos\theta + \cos[2(\omega t + \varphi) - \theta]\}. \tag{6b}$$

The term $\cos\theta + \cos[2(\omega t + \varphi) - \theta]$ varies in the course of a cycle between positive values, indicating energy flow into the system, and negative values indicating flow of energy into the source.

(d) Energy stored in the load e_L,

$$e_L = \tfrac{1}{2} C v_c^2 = \tfrac{1}{2} C\left\{\mathrm{Re}\left[V_0 \frac{e^{j\omega t}(-j/\omega C)}{R_S + R_L - j/\omega C}\right]\right\}^2 =$$

$$= \tfrac{1}{2} C V_0^2 \frac{1}{1 + [(R_S + R_L)\omega C]^2} \cos^2(\omega t + \varphi - 90^\circ) \tag{7}$$

It is apparent from the above that energy varying with time is stored in the system due to the presence of the capacitor, a circuit element capable of storing electric energy.

The system thus is linear, capable of storing energy but is not a resonant system because it exchanges energy with the source.

Case (iii) The generator driving an R_L, L, C circuit shown in Fig. 3.

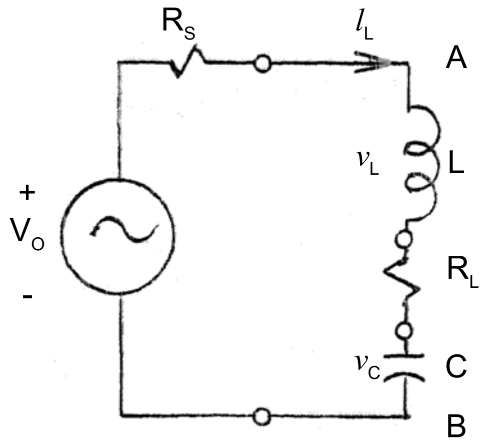


Figure 3. Generator driving R_L, L, C circuit.

(a) System responses: current i, capacitor voltage v_c, inductor voltage v_L and load voltage v_{AB}.

The current i,

$$i = R_e\left[\frac{V_0 e^{j\omega t}}{R_L + R_S + j\left(\omega L - \frac{1}{\omega C}\right)}\right] = R_e \frac{V_0 e^{j(\omega t - \varphi)}}{\left[(R_L + R_S)^2 + \left(\omega L - \frac{1}{\omega C}\right)^2\right]^{\frac{1}{2}}} \qquad (8a)$$

with $\tan\varphi = (\omega L - 1/\omega C)/(R_L + R_S)$.

If one introduces the symbol $\omega_0^2 = 1/LC$ the expression for i becomes

$$i = R_e\left[\frac{V_0 e^{j(\omega t + \varphi)}\omega_0 C}{\left\{[\omega_0 C(R_L + R_S)]^2 + (\omega/\omega_0 - \omega_0/\omega)^2\right\}^{\frac{1}{2}}}\right] = \qquad (8b)$$

$$= \frac{V_0 \omega_0 C}{\left\{[\omega_0 C(R_L + R_S)]^2 + (\omega/\omega_0 - \omega_0/\omega)^2\right\}^{\frac{1}{2}}}\cos(\omega t + \varphi).$$

The capacitance voltage v_C is

$$v_C = R_e\left[\frac{V_0 e^{j(\omega t+\varphi)}(\omega_0/\omega)(-j)}{\left\{[\omega_0 C(R_L+R_S)]^2+(\omega/\omega_0-\omega_0/\omega)^2\right\}^{\frac{1}{2}}}\right]$$
$$= \frac{V_0(\omega_0/\omega)\cos(\omega t+\varphi-90^\circ)}{\left\{[\omega_0 C(R_L+R_S)]^2+(\omega/\omega_0-\omega_0/\omega)^2\right\}^{\frac{1}{2}}}. \quad (8c)$$

The inductance voltage V_L is, by analogous procedure,

$$v_L = \frac{V_0(\omega/\omega_0)\cos(\omega t+\varphi+90^\circ)}{\left\{[\omega_0 C(R_L+R_S)]^2+(\omega/\omega_0-\omega_0/\omega)^2\right\}^{\frac{1}{2}}}. \quad (8d)$$

The load voltage v_{AB} is,

$$v_{AB} = V_0\left\{\frac{(R_L\omega_0 C)^2+(\omega/\omega_0-\omega_0/\omega)^2}{(R_L+R_S)\omega_0 C+(\omega/\omega_0-\omega_0/\omega)^2}\right\}^{\frac{1}{2}}\cos(\omega t+\varphi+\theta)$$
$$\text{with } \tan\theta = \left(\omega L-\frac{1}{\omega C}\Big/R_L\right). \quad (8e)$$

(b) Power supplied by the source, p_s is

$$p_s = v_{AB}i = V_0^2\frac{\left[(R_L\omega_0 C)^2+(\omega/\omega_0-\omega_0/\omega)^2\right]^{\frac{1}{2}}}{\left\{[(R_L+R_S)\omega_0 C]^2+(\omega/\omega_0-\omega_0/\omega)^2\right\}^{\frac{1}{2}}}\cos(\omega t+\varphi+\theta)\cos(\omega t+\varphi) \quad (9a)$$

One observes that for the case of $\omega L - 1/\omega C = 0$ both phase angles φ and θ are zero so that the expression for power p_s delivered by the source is proportional to $\cos^2\omega t$ and is always positive, implying that no power is returned to the source. The requirement that $\omega L - 1/\omega C = 0$ can be reformulated in the form $\omega^2 = 1/LC = \omega_0^2$.

Because of the presence of capacitance C and inductance L the system is capable of storing energy. It is thus apparent that it satisfies all three requirements (i), (ii), and (iii) characterizing a resonant system, and is therefore an example of such, resonating at frequency f_0,

$$f_0 = \omega_0/2\pi = \left(\frac{1}{LC}\right)^{\frac{1}{2}}\Big/2\pi\,. \quad (10)$$

Some additional features of the R.L.C. circuit considered will be presented in paragraphs (c) and (d) below.

(c) Power p_{L0} dissipated by the load at resonance, with $\omega = \omega_0$ is

$$p_{L_0} = R_L i^2 = V_0^2 (\omega_0 C)^2 \frac{R_L}{[\omega_0 C (R_L + R_S)]^2} \cos^2 \omega t \, , \qquad (11)$$

which is equal to the power supplied by the source at resonance as evident from Eq. 9a.

(d) Energy e_{L_0} stored by the load at resonance, i.e. at $\omega = \omega_0$, is

$$e_{L0} = \frac{1}{2} L i_0^2 + \frac{1}{2} C v_{c0}^2 \, ,$$

with i_0 and v_{c0} designating current and capacitance voltage amplitudes respectively at

$$\omega = \omega_0. \qquad (12a)$$

Substitution from Eq. 8b and 8c reduces the above relation to

$$e_{L0} = \frac{1}{2} \frac{L V_0^2}{(R_L + R_S)^2} \cos^2 \omega t + \frac{1}{2} \frac{C V_0^2}{(C\omega_0)(R_L + R_S)^2} \sin^2 \omega t \qquad (12b)$$

But $\dfrac{1}{C\omega_0^2} = L$, so that

$$e_{L0} = \frac{1}{2} \frac{L V_0^2}{(R_L + R_S)^2} \cos^2 \omega t + \frac{1}{2} \frac{L V_0^2}{(R_L + R_S)^2} \sin^2 \omega t = \frac{1}{2} \frac{L V_0^2}{(R_L + R_S)^2} \qquad (12c)$$

It is apparent from the above result that in the R_L, L, C resonant system considered the total energy stored at resonance does not vary with time. This feature as observed in the special system discussed is an illustration of general property of all linear resonant systems, stated here without proof, of storing time independent total energy at resonance.

2. Series resonant circuit

The case (iii) system described above and shown in Fig. 3 is a series resonant circuit, a system commonly employed as such, or as a convenient approximation of other resonant systems. For these reasons some additional properties thereof will be described below.

A feature of interest is the behaviour of the circuit at frequencies close to resonance. In the discussion to follow we shall employ standard phasor quantities I and V instead of instantaneous circuit quantities i and v used earlier.

The common loop current I is, from the diagram of Fig. 3, given by the relation

$$I = \frac{V_0}{[(R_S + R_L)] + j(\omega L - 1/\omega C)} \tag{13a}$$

When the expression for the resonant (angular) frequency $\omega_0 = 1/\sqrt{LC}$ is introduced into the above equation the expression for the current I becomes

$$\begin{aligned} I &= \frac{V_0 \omega_0 C}{(R_S + R_L)\omega_0 C + j(\omega/\omega_0 - \omega_0/\omega)} = \\ &= \frac{V_0 \omega_0 C}{(R_S + R_L)\omega_0 C + j\left(\omega^2 - \omega_0^2\right)/\omega\omega_0} = \\ I &= \frac{V_0 \omega_0 C}{(R_S + R_L)\omega_0 C + j\left(\dfrac{\omega - \omega_0}{\omega_0}\right)\left(\dfrac{\omega + \omega_0}{\omega}\right)}. \end{aligned} \tag{13b}$$

The final form for I obtained above will now be formulated in convenient dimensionless parameters. We shall also limit the frequencies considered to values close to resonant frequency ω_0. It is useful to observe that the term $1/\omega_0 C$ is the magnitude of the reactance of the capacitance C at frequency ω_0. Furthermore the term $(R_S + R_L)\omega_0 C = (R_S + R_L)/X_{C_0}$ and is the ratio of lossy portion of the circuit impedance and a representative of reactive portions. It should be borne in mind that for $\omega_0 = 1/\sqrt{LC}$ the magnitude of the capacitive reactance X_{C_0} is equal to the magnitude of the inductive reactance X_{L_0}. The reciprocal of the term $(R_S + R_L)\omega_0 C = (R_S + R_L)/X_{C_0} = (R_S + R_L)/X_{L_0}$, $X_{C_0}/(R_S + R_L)$ is called the quality factor Q of the circuit and will be shown below to be a measure of the relationship between power dissipated in the circuit and energy stored therein, and will also govern the frequency behaviour of the circuit in the vicinity of the resonance.

It should also be noted that for frequencies close to resonance it is convenient to approximate the value ω by ω_0, except in the term ω-ω_0.

When the term Q and the approximation of ω to ω_0 is introduced into Eq. 13b, the expression for the current I reduces to

$$I = \frac{V_0}{X_{C0}\left[1/Q + j2(\omega - \omega_0)/\omega_0\right]}. \tag{14a}$$

The above relation may also be reformulated by stating that the loop impedance Z in the vicinity of resonance is

$$Z = X_{C_0}\left[1/Q + j2(\omega - \omega_0)/\omega_0\right]. \tag{14b}$$

The equation for current I derived above exhibits several features characteristic of all resonant phenomena:

(i) At resonance the driving voltage V_0 and the response I are in phase due to the disappearance of the imaginary term

$j(\omega/\omega_0 - \omega_0/\omega)$ and,

(ii) the driven current amplitude is maximum, I_{max}=$V_0 Q/X_{C_0}$.

The two observations can be considered to be the consequence of the fact that at resonance the loop impedance is pure resistance. It should be mentioned that although in the circuit considered the response current amplitude is maximum at resonance, there are circuits where the current amplitude is minimum. It is, however, in all cases in phase

with the driving voltage V_0.

We shall next consider the off resonance behaviour of the circuit. As the frequency moves away from the resonance the imaginary term $j(\omega/\omega_0 - \omega_0/\omega) \doteq j2(\omega - \omega_0)/\omega_0$ increases the magnitude of the denominator in Eq. 14a for the current I, reducing the amplitude thereof and introducing a phase shift between the phasors of the driving voltage V_0 and the current I. Interesting conditions are obtained when the magnitude of the imaginary term in the loop impedance reaches the value of the real value therein, i.e. when

$$\pm 2(\omega - \omega_0)/\omega_0 = 1/Q \,. \tag{15a}$$

At this frequency the magnitude of the impedance increases from its resonant value of QX_{C0} to $\sqrt{2}$ QX_{C0}, reducing the value of loop current amplitude to $1/\sqrt{2}$ of its resonant value. The power dissipated in the circuit is $(R_S + R_L)|I|^2$ and the reduction of the current amplitude by the factor $1/\sqrt{2}$ is accompanied by the reduction of the power dissipated to one-half of its value at resonance. Expressed in decibel units the level of power reduction is 3dB. The frequency shift ω-ω_0 from resonance to 3dB power reduction is, from Eq. 15a, $\pm\frac{1}{2}\omega_0/Q$. The range of frequencies $\Delta\omega$ for which power absorbed (and dissipated) lies between maximum at resonance and 3dB power reduction is thus

$$\Delta\omega = \omega_0/Q \,, \tag{15b}$$

and is called the 3dB bandwidth of the circuit, and is often employed as a measure of frequency range of the resonant circuit effective loop impedance. It follows from Eq. 14b that the loop impedance Z, approximated in the neighbourhood of resonance by expression $X_{C_0}\left[1/Q + 2j(\omega - \omega_0)/\omega_0\right]$ increases rapidly as the frequency moves away from the resonance with the reactive part dominating the magnitude. The exact value of the reactive part, $jX_{C_0}(\omega/\omega_0 - \omega_0/\omega)$ is seen to approach large capacitive reactances for low frequencies, and large inductive for high frequencies. As a consequence the ability of the circuit to absorb power at frequencies significantly removed from the resonance is strongly reduced.

An important aspect of the circuit performance is the phase between the driving voltage V_0 and the driven current I(angle φ of Eq. 8a). As was mentioned earlier, at resonance the two quantities are in phase. As the frequency moves away from the resonance the loop impedance acquires reactive components affecting the phase relationships between input voltage and current. At 3dB points, i.e. at $\omega - \omega_0 = \pm\frac{1}{2}\omega_0/Q$ the phase is ±45°, the current leading the voltage at the lower edge of the band, where the reactive component of the input is capacitive, and the current lagging the voltage where the reactive component is inductive.

The features of the resonant circuit mentioned above are diagrammatically represented in Fig. 4.

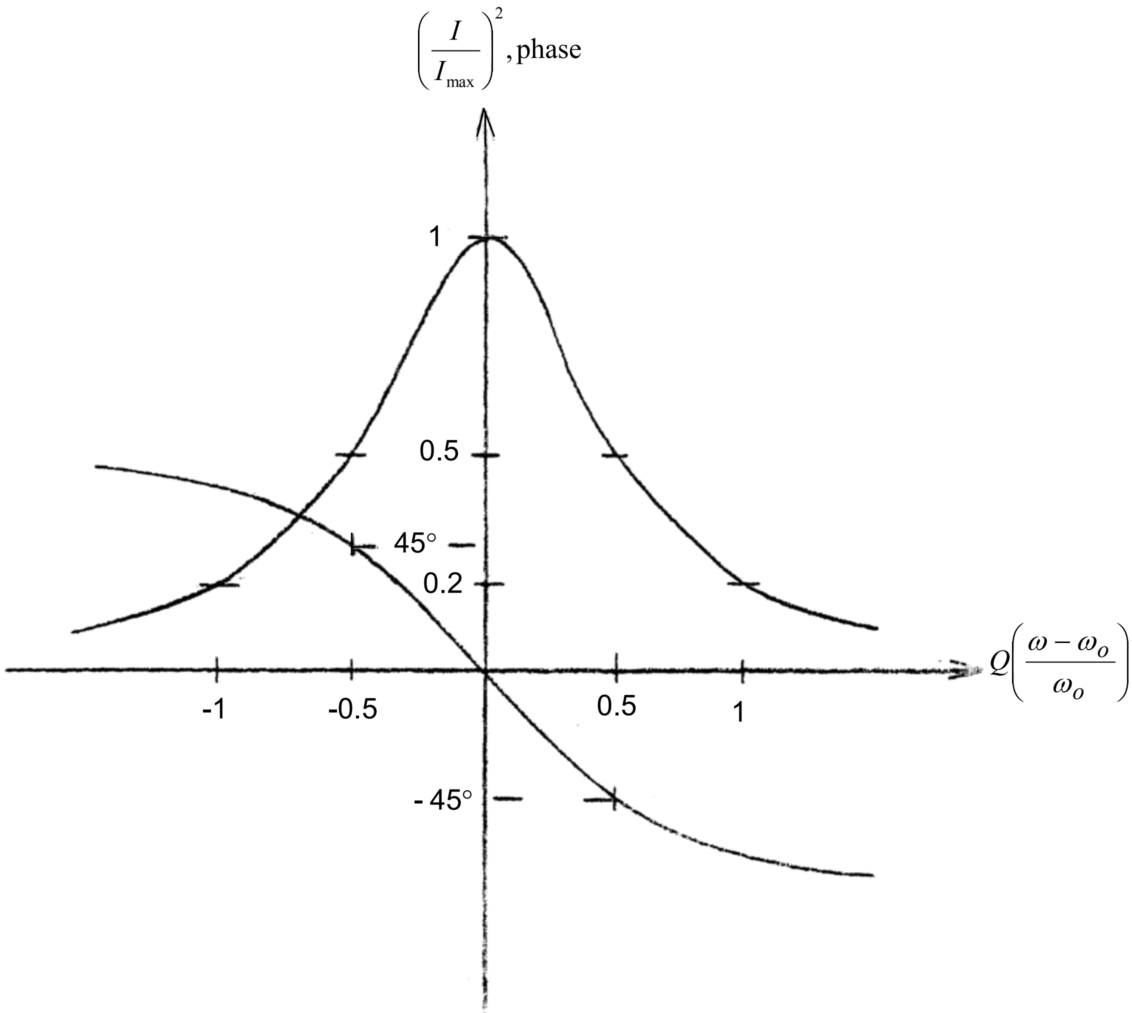


Figure 4. Amplitude and phase relationships near resonance.

Because the ability of the resonant circuit to absorb power is frequency-sensitive, it is commonly used as a device to select a desirable frequency from a manifold thereof. In communications applications it is often employed as a frequency selective element of a tuner.

It may be useful to consider in some detail the significance of the quality factor Q. It governs two important properties of a resonant circuit: the frequency selectivity as expressed in 3dB bandwidth $\Delta\omega = \omega_0/Q$ and the value of impedance at resonance, $R_S + R_L = X_{C_0}/Q$. It is important to note that the factor depends not only on the resistive element of the load, but also on the resistive element of the source. Thus frequency selective properties of a resonant circuit do not depend solely on the loss mechanism of the load, but also on the loss mechanism of the source.

Another important property of the Q factor is its effect on the capacitance and inductance voltages at resonance. At resonance the impedance of the capacitance and inductance are $-jX_{C_0} = -j/\omega_0 C$ and $jX_{L_0} = j\omega_0 L = j/\omega_0 C$. The circuit current I_0 at resonance is $V_0/(R_L + R_S)$. Thus the reactance voltages are

$$V_{C0} = \frac{V_0}{R_L + R_S}(-j/\omega_0 C) = -jV_0 Q,$$

and,

$$V_{L0} = \frac{V_0}{R_L + R_S}(j\omega_0 L) = jV_0 Q. \qquad (16)$$

The magnitudes of reactance voltages are thus Q times larger than the voltages that would have appeared across them, had they been driven by the source independently. Because the reactive voltages are of opposite polarity the total voltage across the series combination of the inductance and capacitance is zero and the voltage appearing across the load is $V_0 R_L/(R_L + R_S)$ and depends only on the loss elements of the loop.

The quality factor also bears on the energy balance in the circuit at resonance. The instantaneous energy stored in the inductance and capacitance are $Li^2/2$ and $C\upsilon_c{}^2/2$ respectively. At resonance the phasor of the current i is $V_0/(R_L + R_S)$ while the phasor of the capacitance voltage v_c is $-jV_0/[\omega_0 C(R_S + R_L)]$. The current i and voltage v_c are thus 90° out of phase so that

$$i = V_0 \cos\omega_0 t/(R_S + R_L)$$

and,

$$v_c = V_0 \sin\omega_0 t/[\omega_0 C(R_S + R_L)]. \qquad (17)$$

The total energy stored W_0 is

$$W_0 = \frac{1}{2}LV_0^2 \cos^2 \omega t/(R_S + R_L)^2 + \frac{1}{2}LV_0^2 \sin^2 \omega t/[\omega_0 C(R_S + R_L)]^2. \qquad (18a)$$

But, because $L\omega_0 = 1/\omega_0 C$ and

$$L\omega_0/(R_S + R_L) = \left(\frac{1}{\omega_0 C}\right)\Big/(R_S + R_L) = Q,$$

the equation for W_0 reduces

$$W_0\omega_0 = \frac{1}{2} Q V_0^2 \big/ \left(R_S + R_L\right) \tag{18b}$$

The following conclusions can be drawn from the last equation:

(i) At resonance the total energy stored in a series resonant circuit is independent of time.

(ii) $V_0 \big/ \sqrt{2}$ is the RMS value of the amplitude of the driving EMF. The term $\frac{1}{2} V_0^2 \big/ \left(R_S + R_L\right) = V_0^2\, RMS / \left(R_S + R_L\right)$ is the power P_0 dissipated in the total resistive portion of the circuit. Eq. 18b can thus be expressed in the form

$$Q = \frac{\omega_0 W_0}{P_0}. \tag{19}$$

The result obtained above can be formulated in terms of the statement that the quality factor Q of a series resonant circuit is the ratio at resonance of the total energy stored multiplied by angular resonant frequency, and the total power dissipated in the circuit.

The results (i) and (ii) listed above have been derived analytically for a series resonant circuit. They apply, however, to all resonant circuits. Also, expression for the 3dB bandwidth $\Delta\omega = \omega_0 / Q$ is exactly valid for series and parallel resonant circuits. For other resonant configuration the relation is of the form $\Delta\omega = \gamma\omega_0 / Q$ where γ is a numerical factor commonly lying between one and two.

The relationships involving Q were derived for steady state conditions. They also occur in analysis of transient behaviour. Equation of motion for a harmonically excited series resonant circuit is

$$L\frac{d^2 i}{dt^2} + R\frac{di}{dt} + \frac{i}{c} = j\omega V_0 e^{j\omega t}. \tag{20}$$

The general solution incorporating effects of initial condition is

$$i = I_1 e^{\beta t} + I_2 e^{j\omega t}, \tag{21}$$

where $I_1 e^{\beta t}$ is the solution of the homogeneous equation

$$L\frac{d^2i}{dt^2}+R\frac{di}{dt}+\frac{i}{c}=0\,, \tag{22a}$$

which, on substitution for $i = I_1 e^{\beta t}$ becomes

$$L\beta^2+R\beta+\frac{1}{C}=0 \tag{22b}$$

or,

$$\beta^2+\beta\frac{R}{L}+\omega_0^2=0\,. \tag{22c}$$

The solutions of this equation are

$$\beta=-\frac{R}{2L}\pm\sqrt{\left(\frac{R}{2L}\right)^2-\omega_0^2}\,. \tag{23a}$$

For the common case of low loss systems the term $R/2L$ is smaller than ω_0 and it is convenient to express β in the form

$$\beta=\pm j\omega_0\sqrt{1-\left(\frac{R}{2L\omega_0}\right)^2}-\frac{R}{2L\omega_0}\cdot\omega_0\,. \tag{23b}$$

Substitution of Q for the term $L\omega_0/R$ reduces the equation to the form

$$\beta=\pm j\omega_0\sqrt{1-\left(\frac{1}{2Q}\right)^2}-\frac{\omega_0}{2Q}\,. \tag{23c}$$

The transient solution i_1 is thus

$$i1=\left[I_1^+e^{j\omega_0' t}+I_1^-e^{-j\omega_0' t}\right]e^{-\frac{t}{2\tau}} \tag{24}$$

where $\omega_0'=\omega_0\sqrt{1-\left(\frac{1}{2Q}\right)^2}$ and $\tau=Q/\omega_0$.

It is apparent from Eq. 24 that the time constant 2τ for the linear transient is $2Q/\omega_0 = 2/\Delta\omega_0$, where $\Delta\omega_0$ is the 3dB bandwidth of the steady state response. In as much as the quadratic quantities such as energy and power are proportional to the squares of linear quantities, the decays thereof are determine by the term

$$\left(e^{-\frac{\omega_0 t}{2Q}}\right)^2 = e^{-\frac{\omega_0 t}{Q}} = e^{-\frac{t}{\Delta\omega_0}} \qquad (25)$$

The reciprocal of the 3dB bandwidth $\Delta\omega_0$ is thus the time constant of the transient behaviour of stored energy and power in a series resonant circuit. Similar relations also obtain for more complicated resonant systems.

3. Parallel resonant circuit

A circuit dual to a series resonant circuit is a parallel resonant circuit shown in Fig. 5.

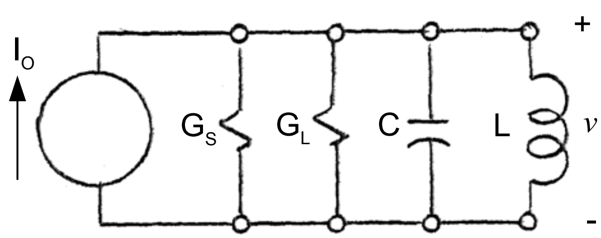


Figure 5. Parallel resonant circuit

Results obtained for series resonant circuit can be adapted to describe properties of a parallel resonant circuit by carrying out the duality transformation switching the words series ↔ parallel, impedance ↔ admittance and voltage ↔ current. Thus whereas in series resonant circuit the driving quantity was voltage and the responding quantity was current as common to all components of the circuit, in parallel resonant circuit the driver is current and responding quantity is the common voltage.

Presented in Table 1 are dual relationships pertaining to the two circuits discussed.

Table 1

Quantity	**Series resonant circuit**	**Parallel resonant circuit**
impedance - admittance	$Z = R_S + R_L + j\left(\omega L - \frac{1}{\omega_0}\right)$	$Y = G_S + G_L + j\left(\omega C - \frac{1}{\omega L}\right)$
impedance - admittance near resonance	$Z = \frac{1}{\omega_0 C}\left[\frac{1}{Q} + 2j\frac{(\omega - \omega_0)}{\omega_0}\right]$	$Y = \omega_0 C\left[\frac{1}{Q} + 2j\frac{(\omega - \omega_0)}{\omega_0}\right]$
impedance - admittance at resonance	$Z_0 = L\omega_0/Q = R_S + R_L = Z_{\min}$ $Y_0 = Q/L\omega_0 = Y_{\max}$	$Y_0 = C\omega_0/Q = G_S + G_L = Y_{\min}$ $Z_0 = Q/C\omega_0 = Z_{\max}$
quality factor Q	$\frac{L\omega_0}{R_S + R_L} = Q$	$\frac{C\omega_0}{G_S + G_L} = Q$
resonant frequency f_0	$f_0 = \frac{1}{2\pi\sqrt{LC}}$	$f_0 = \frac{1}{2\pi\sqrt{LC}}$
3dB bandwidth Δf_0	$\Delta f_0 = f_0/Q$	$\Delta f_0 = f_0/Q$

ELECTROMAGNETIC CAVITY

1. Electromagnetic cavity

Electromagnetic cavity is a volume of space enclosed by electromagnetically impenetrable, usually metallic walls. If the cavity is to interact with outside space, the cavity walls are breached by a small opening commonly called an iris through which energy can pass into or out of the cavity. Examples of cavities are sections of transmission systems such as coaxial lines or waveguides terminated at both ends by elements impenetrable, or nearly so for electromagnetic modes of the corresponding transmission systems.

A prototype of a transmission system cavity is a section of transmission line terminated at both ends by devices inhibiting totally or partially passage of electromagnetic energy through them. The prototype may often serve as an equivalent circuit of wide range of cavities and will be analysed below.

2. Transmission line cavity with single iris

The system considered is a section of transmission line of characteristic impedance Z_0 and phase velocity u, of length l. One end of the section is terminated in a short circuit while the other end is connected to a source of frequency ω through identical transmission line. An iris allowing passage of some electromagnetic energy into the cavity is inserted between the feed line and the cavity section. Electromagnetic properties of the iris are equivalent to a susceptance B. The circuit representation of the system is shown in Fig. 1.

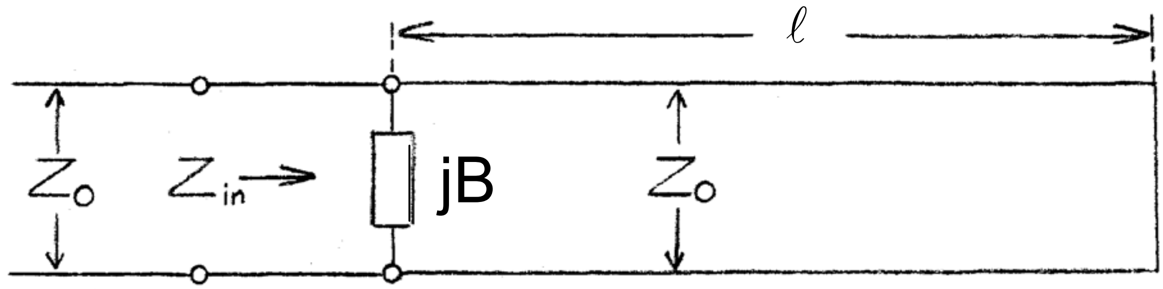


Figure 1. Transmission line cavity.

The parameter describing most of the circuit properties of the cavity is the input impedance Z_{in} which, however, must be associate with a specific pair of terminals.

A useful relationship in the discussion of the problem is the relation given below in Fig. 2 and proved in the Appendix.

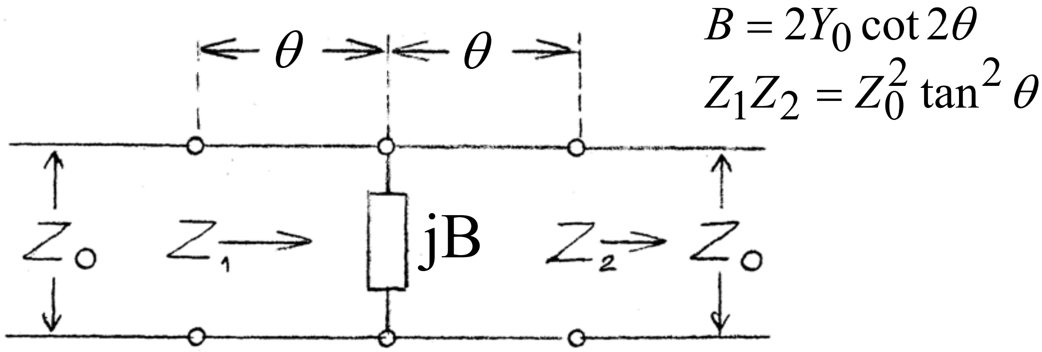


Figure 2. Inverter circuit.

The equivalent circuit of the cavity as shown in Fig. 1 can be modified to the form shown in Fig. 3.

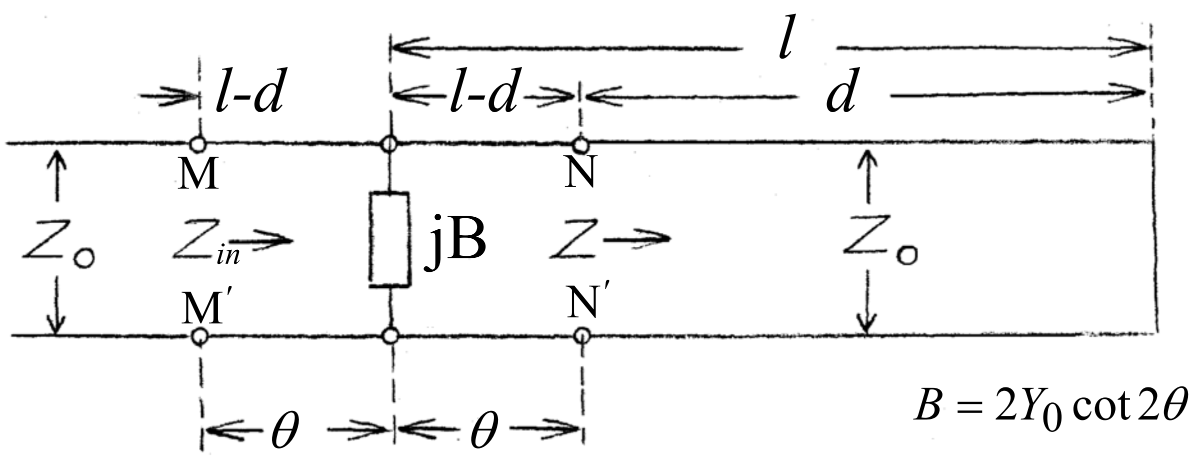


Figure 3. Equivalent circuit of transmission line cavity.

The amended circuit identifies convenient terminals of driving point impedance Z_{in} and allows one to employ standard transmission line circuit analysis, as will be carried out below.

The driving point impedance Z_{in} at terminals MM′ is related to the impedance Z at terminals NN′ by the relationship given in Equation 1 in Fig. 2,

$$Z_{in} = \frac{Z_0^2}{n^2}\frac{1}{Z} \tag{2a}$$

with $n = \cot\theta$ and $B = Y_0 2\cot 2\theta$.

But Z is the impedance of a short circuited section of length d and is $jZ_0 \tan \beta d$, where β is the propagation constant ω/u. Thus the driving point impedance Z_{in} is

$$Z_{in} = -j\frac{Z_0}{n^2}\frac{1}{\tan \beta d} \tag{2b}$$

We shall investigate the resonance behaviour of the circuit, i.e. when $Z_{in} = 0$ or ∞. The choice of the two extreme possibilities is suggested by the behaviour of conventional lossless series or parallel resonant circuits. We shall start the analysis of the circuit behaviour in the frequency range close to where $Z_{in} = \infty$, i.e. where $\cot \beta d = \infty$ or, where $\tan \beta d = 0$. This occurs whenever d is an integral multiple of half wavelength $\lambda/2$ i.e. where $d = m\lambda/2$. We shall investigate the lowest longitudinal mode $m = 1$, i.e. when $d = \lambda/2$.

We observe that in the vicinity of $\beta d = \pi$, i.e. $\frac{\omega}{u}d = \pi$ the approximate power series expansion of $\cot \beta d = 1/\tan \beta d$ is

$$\begin{aligned} \cot \beta d &= \frac{\cos \beta d}{\sin \beta d} \doteq \frac{-1}{\beta d - \pi} = \frac{-1}{\frac{\omega d}{u} - \pi} \\ \cot \beta d &\doteq -\frac{u/d}{\omega - \pi u/d} = -\frac{u/d}{\omega - \omega_0} 1 \end{aligned} \tag{3}$$

with $\omega_0 = \pi u/d$.

The expression for Z_{in} becomes

$$Z_{in} \doteq -j\frac{Z_0}{n^2}\frac{(u/d)}{\omega - \omega_0} = -j\frac{Z_0}{\pi n^2}\frac{\omega_0}{\omega - \omega_0} \tag{4}$$

When one observes that for a lossless parallel resonant circuit the input impedance near resonance is $-\frac{j}{2}\frac{1}{\omega_0 C}\frac{\omega_0}{\omega - \omega_0}$ the expression for Z_{in} derived in Eq. 4 can be considered to be the input impedance of equivalent circuit shown in Fig. 4.

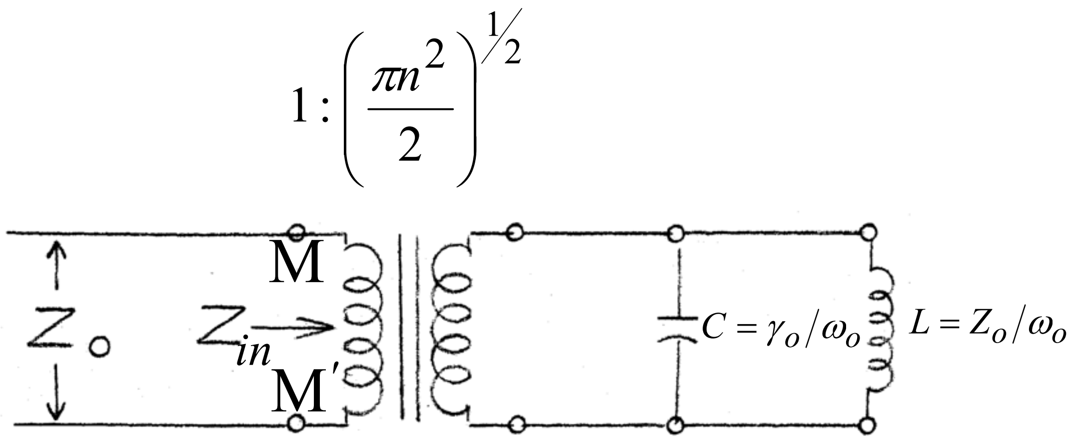


Figure 4. Lumped equivalent circuit of transmission line cavity near resonance.

We note that input impedance of a lossy parallel resonant circuit is $\frac{1}{\omega_0 C}\frac{1}{\frac{1}{Q}+\frac{2j(\omega-\omega_0)}{\omega_0}}$, describing the behaviour of circuit shown in Fig. 5.

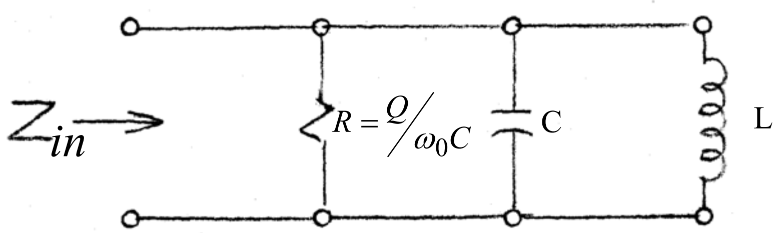


Figure 5. Lossy parallel resonant circuit.

Thus a good approximation of the impedance of a lossy transmission line cavity is, by analogy to the parallel resonant circuit the modification of the expression for Z_{in} in Equation 4

$$Z_{in} = \frac{2}{\pi n^2} Z_0 \frac{1}{\frac{1}{Q}+\frac{2j(\omega-\omega_0)}{\omega_0}} \tag{5}$$

leading to the equivalent circuit shown in Fig. 6.

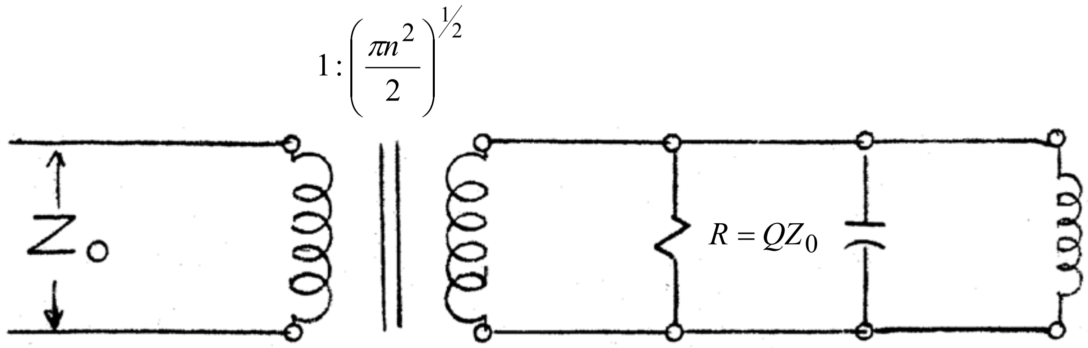


Figure 6. Equivalent circuit of a lossy transmission line in the vicinity of resonance.

In discussion to follow it will be necessary to consider frequencies substantially removed from resonance. Under these circumstances one is reminded that the exact expression for cavity input impedance is given in Equation 2, $Z_{in} = -j\frac{Z_0}{n^2}\cot\beta d = -j\frac{Z_0}{n^2}\cot\frac{\omega}{\omega_0}\pi$.

The input impedance of a resonant cavity at a frequency far removed from resonance is very nearly zero as viewed at terminals MM′, i.e. the cavity in these frequency ranges behaves as a short circuit. It is convenient to describe the frequency response of the cavity in terms of the SWR produced on the input line. The coordinates employed are shown in Fig. 3 and position of VSWR minimum will be designated S_i.

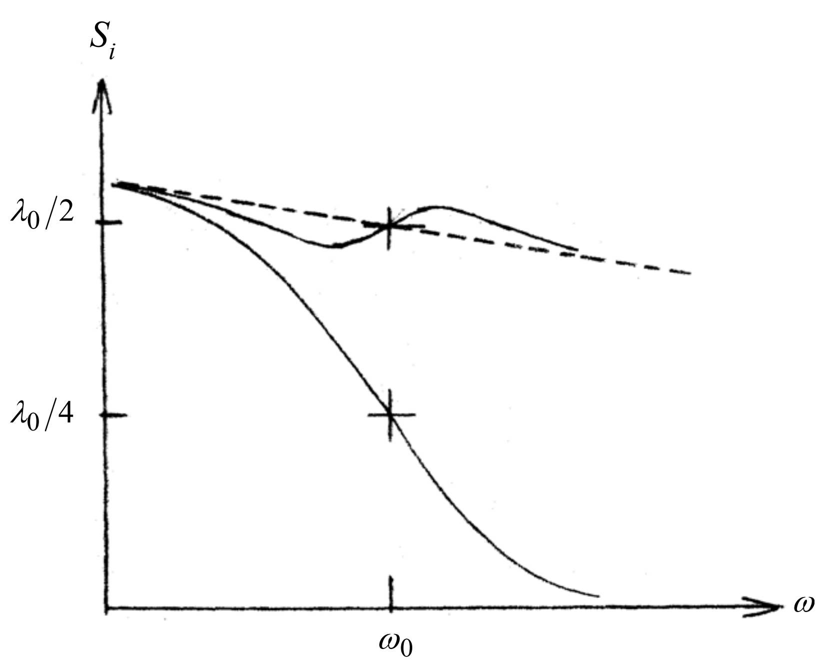


Figure 7. Location of VSWR minimum in the vicinity of resonance for undercoupled and overcoupled cavities.

The behaviour of the voltage minimum position as the frequency is swept through resonance and is conveniently visualized by comparing it with the position of voltage minimum produced by a short circuit located at observation terminals MM′ of Fig. 3. Plotted in Fig. 7 as a dotted line is the position of voltage minimum with reference to a point $\lambda/2$ away from the position of the short at MM′. As the frequency increases wavelength becomes shorter and the minimum moves closer to the position of the short, the process indicated by the slope of the dotted line.

When the short is replaced by the cavity, at frequencies sufficiently removed from resonance on the low side of the cavity impedance approximates zero as evident from Eq. 2 and the location of voltage minimum follows the dotted line of Fig. 7. As the frequency approaches resonance the conditions change. Cavity impedance below resonance is inductive as is apparent from equivalent circuit of Fig. 6 and the distance S_i of voltage minimum from terminals MM′ begins to drop faster than would be the case of short circuit termination. The conditions change when the frequency approaches resonance because of the effect of resistive term $R = QZ_0$ as evident from Fig. 7 and Fig. 6. At resonance the impedance of the cavity is purely resistive and its value is $Z_0 Q \frac{2}{\pi n^2} = Z_{in}(0)$.

Expression $\pi n^2/2$ will occur frequently is subsequent discussions and it will be convenient to introduce a symbol Q_e for it, i.e. $\pi n^2/2 = Q_e$ called the external Q. Thus $Z_{in} = Z_0 Q/Q_e$.

Depending on whether the resonant resistive impedance is smaller than, larger than, or equal to Z_0 there will obtain three different conditions as listed below.

(i) $Z_{in} = Z_0 Q/Q_e < Z_0$: in this case voltage minimum will occur at the same location as voltage null produced by short circuit termination.

(ii) $Z_{in} = Z_0 Q/Q_e > Z_0$: voltage maximum will occur at the null location produced by short circuit termination.

(iii) $Z_{in} = Z_0 Q/Q_e = Z_0$: the cavity is matched to the line, no standing wave pattern is present.

The three cases considered above are designated undercoupled for $Z_0 Q/Q_e < Z_0$, overcoupled for $Z_0 Q/Q_e > Z_0$, and critically coupled for $Z_0 Q/Q_e = Z_0$.

As the frequency is increased beyond resonance the input impedance acquires capacitive character. For the undercoupled case the distance of the observed minimum, moves initially away from the reference terminals. In high frequencies the input impedance begins to approximate short circuit and minimum approaches the location of selected minimum of the short-circuited termination as shown in Fig. 7.

The pattern of behaviour for overcoupled case is different in that at resonance, when input impedance Z_{in} is real, and is larger than Z_0 at reference terminals the voltage is maximum and minimum occurs $\lambda/4$ away. When the behaviour of voltage minima is traced in this case as the frequency is increased from its initial off resonance value the minimum moves towards the reference point but as the frequency approaches resonance it does not reverse its motion as was the case for undercoupled cavity, but stops at $\lambda/4$ distance from reference terminals, which become the location of voltage maximum as mentioned earlier. As the frequency is increased beyond resonance the minimum continues to move toward the location of short circuit minimum, but not the one from which it started but one $\lambda/2$ closer to reference terminals, as shown in Fig. 7.

External Q_e is thus seen as a parameter which quantifies the interaction of the inside of the cavity with external environment.

3. Loaded Transmission Line Cavity

In many instances the cavity has two input-output portals. The input portal connects the driver to the cavity while the output portal, usually an iris at the original short circuit wall of the cavity connects the inside of the cavity to the load which absorbs a portion of power supplied by the source, modified by interposition of the cavity. A common application of this nature is the use of the cavity as a bandpass filter.

When the short circuit wall of the cavity is replaced by an iris the equivalent circuit of the cavity as shown in Fig. 1 is modified to the configuration shown in Fig. 8.

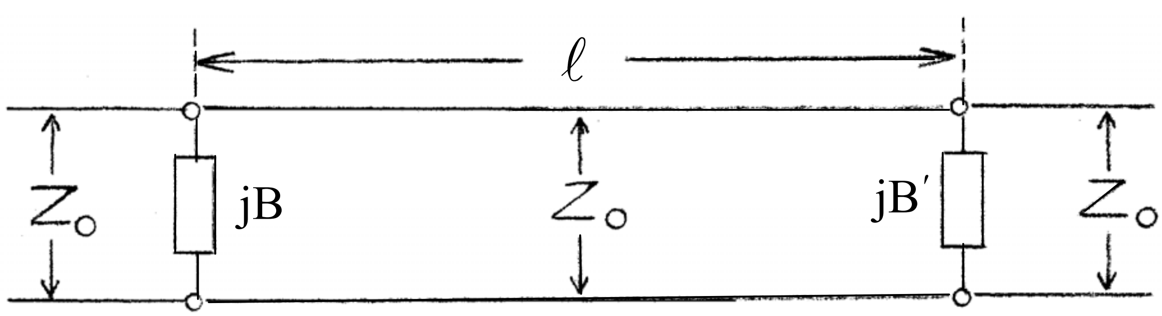


Figure 8. Loaded transmission line cavity.

The effect of the susceptance B′ can be conveniently evaluated by employing the impedance transformation of Equation 1 as shown in Fig. 2 and shown in Fig. 9.

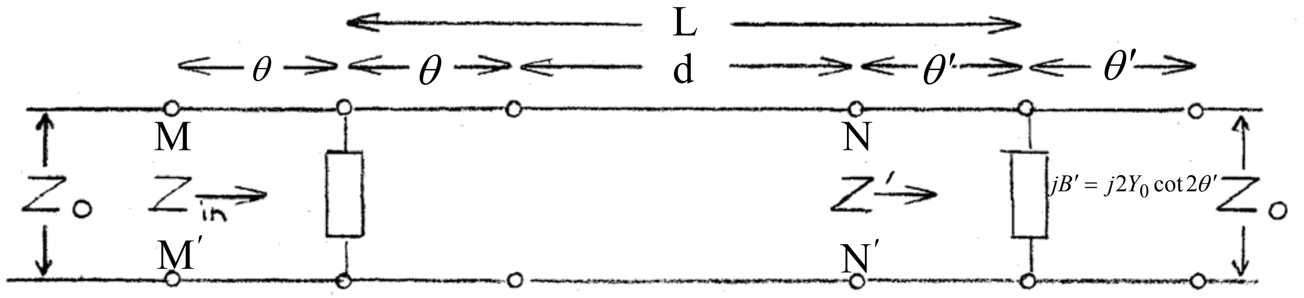


Figure 9. Application of inverter circuits to a transmission line cavity.

The resultant equivalent circuit of the loaded cavity is given in Fig. 10.

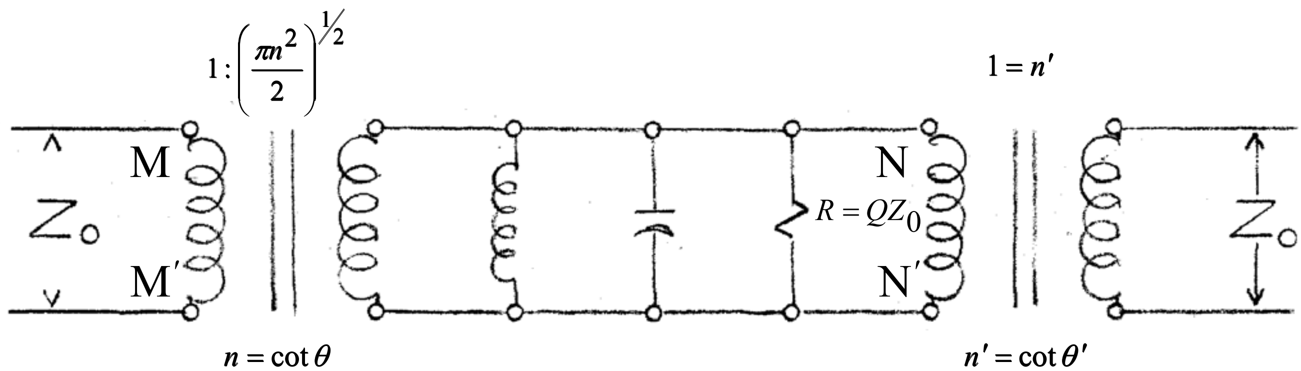


Figure 10. Lumped equivalent circuit of loaded transmission line cavity.

Analysis of the circuit of Fig. 9 is simplified if one expresses the terminal load $Z' = Z_0 / n'^2$ in the form

$$Z' = jZ_0 \tan \beta\delta' . \tag{8}$$

The cavity input impedance Z_{in} is then given by

$$\begin{aligned} Z_{in} &= \frac{Z_0}{n'^2} \frac{1}{j \tan \beta(d' + \delta')} \\ &= -j \frac{Z_0}{n'^2} \frac{1 - \tan \beta d' \tan \beta\delta'}{j \tan \beta d' + \tan \beta\delta'} \end{aligned} . \tag{9a}$$

Inasmuch as $\tan\beta d'$ in the vicinity of resonance is a small number and Z' is usually a small perturbation of the short circuit termination the expression for Z_{in} can be approximately reduced to

$$Z_{in} = \frac{Z_0}{n'^2}(-j)\frac{1}{\tan \beta d' + \tan \beta \delta'} \tag{9b}$$

Power series expansion of tangent function u in the vicinity of resonance reduces the expression for Z_{in}, in a manner analogous to that employed to develop Equation 4, to the form

$$Z_{in} = Z_0 \frac{1}{n'^2}\left(\frac{-j}{\pi}\right)\frac{1}{2j\frac{(\omega-\omega_0)}{\omega_0} + \frac{1}{n'^2}\frac{2}{\pi}}. \tag{10a}$$

Internal losses in the cavity can be accounted for by the addition of the term $1/Qi$ to the denominator. The terms of the form $n^2 \frac{\pi}{2}$ have been designated external Q'_e, Q_e. The final expression for Z_{in} then becomes

$$Z_{in} = Z_0 \frac{1}{Q_e}\frac{1}{2j\frac{(\omega-\omega_0)}{\omega_0} + \frac{1}{Q_i} + \frac{1}{Q'_e}} \tag{10b}$$

An equivalent circuit appropriate for the expression for Z_{in} as developed above is given in Fig. 11.

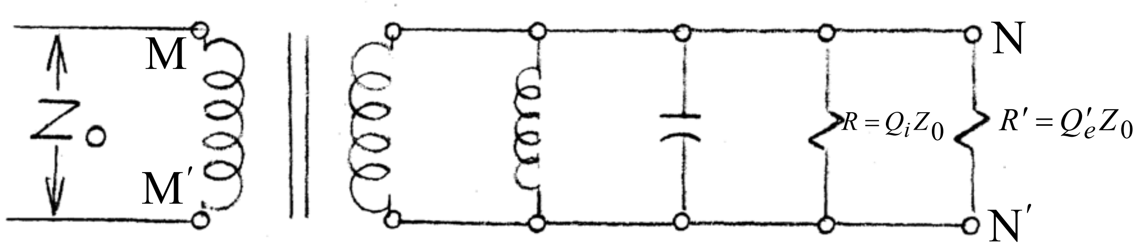


Figure 11. Reduced lumped equivalent circuit of a loaded transmission line cavity.

The reciprocal of the term $\frac{1}{Q_i} + \frac{1}{Q_e}$ in Equation 10b is commonly designated the loaded Q of the cavity and incorporates the effect of external loading on the performance of the cavity.

4. Frequency response of a resonant cavity.

It is often important to know the frequency response of a cavity. It may be defined as the ratio of power absorbed by the cavity at frequency ω, usually lying close to the resonant frequency ω_0, the power absorbed at resonance, the maximum power. A common measure of the effect is the spread of frequencies $\delta\omega$ in which the ratio is above $1/2$, the 3 dB bandwidth.

Power absorbed at frequency ω, $P(\omega)$ is equal to the incident power minus reflected power, so that with $\rho(\omega)$ the reflection coefficient at frequency ω,

$$P(\omega) = P_i\left[1 - |\rho(\omega)|^2\right], \tag{11}$$

with P_i the incident power.

The 3 dB bandwidth is therefore given by the relation

$$\frac{1 - \left|\rho\left(\omega_0 \pm \delta\omega/2\right)\right|}{1 - |\rho(\omega_0)|^2} = \frac{1}{2}. \tag{12}$$

The reflection coefficient $\rho(\omega)$ is

$$\rho(\omega) = \frac{Z_{in}(\omega) - Z_0}{Z_{in}(\omega) + Z_0} \tag{13a}$$

Substitution from Equation 10b yields

$$\rho(\omega) = \frac{\frac{1}{Q_e} - 2j\frac{(\omega - \omega_0)}{\omega_0} - \frac{1}{Q}}{\frac{1}{Q_e} + 2j\frac{(\omega - \omega_0)}{\omega_0} + \frac{1}{Q}}, \tag{13b}$$

where Q is the "loaded Q", the reciprocal of $\frac{1}{Q_i} + \frac{1}{Q_e'}$.

When the above expressions are introduced into Equation 12, it reduces to

$$1 - \frac{\left(\frac{1}{Q_e} - \frac{1}{Q'}\right)^2 + \left(\frac{\delta\omega}{\omega_0}\right)^2}{\left(\frac{1}{Q_e} - \frac{1}{Q'}\right)^2 + \left(\frac{\delta\omega}{\omega_0}\right)^2} = \frac{1}{2}\left\{1 - \frac{\left(\frac{1}{Q_e} - \frac{1}{Q'}\right)^2}{\left(\frac{1}{Q_e} + \frac{1}{Q'}\right)^2}\right\}. \tag{14}$$

The value of $\delta\omega/\omega_0$ obtained from the above is,

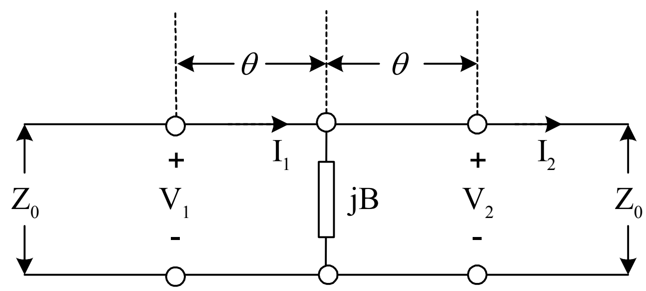
$$\frac{\delta\omega}{\omega_0} = \frac{1}{Q_e} + \frac{1}{Q} = \frac{1}{Q_e} + \frac{1}{Q_e'} + \frac{1}{Q_i} \tag{15}$$

It is apparent from the above equation for $\delta\omega/\omega_0$ that the power delivered to external source and load impedances as governed by external Qs has the same effect on the frequency response as the power delivered to internal loss mechanism.

Appendix

Transmission line inverter circuit.

Consider a transmission line network shown in the figure below:



The relationship between the column vectors $\begin{pmatrix} V_1 \\ I_1 \end{pmatrix}$ and $\begin{pmatrix} V_2 \\ I_2 \end{pmatrix}$ is given by the product of three component network matrices (θ), (jB) and (θ) so that,

$$\begin{pmatrix} V_1 \\ I_1 \end{pmatrix} = (\theta)(jB)(\theta)\begin{pmatrix} V_2 \\ I_2 \end{pmatrix} \tag{16}$$

The matrices are:

$$(\theta) = \begin{pmatrix} \cos\theta, & jZ_0 \sin\theta \\ jY_0 \sin\theta, & \cos\theta \end{pmatrix} \text{ and } (jB) = \begin{pmatrix} 1 & 0 \\ jB & 1 \end{pmatrix}.$$

Thus
$$\begin{pmatrix} V_1 \\ I_1 \end{pmatrix} = \begin{pmatrix} \cos\theta, & jZ_0 \sin\theta \\ jY_0 \sin\theta, & \cos\theta \end{pmatrix}\begin{pmatrix} 1, & 0 \\ jB, & 1 \end{pmatrix}\begin{pmatrix} \cos\theta, & jZ_0 \sin\theta \\ jY_0 \sin\theta, & \cos\theta \end{pmatrix}\begin{pmatrix} V_2 \\ I_2 \end{pmatrix}. \tag{17}$$

When the matrix multiplication is executed the relationship between the two sets of circuit variables becomes,

$$\begin{pmatrix} V_1 \\ I_1 \end{pmatrix} = \begin{pmatrix} \cos 2\theta - \frac{Z_0 B}{2}\sin 2\theta, & jZ_0\left(\sin 2\theta - BZ_0 \sin^2\theta\right) \\ jY_0\left(\sin 2\theta + BZ_0\cos^2\theta\right), & \cos 2\theta - \frac{BZ_0}{2}\sin 2\theta \end{pmatrix}\begin{pmatrix} V_2 \\ I_2 \end{pmatrix}. \tag{18}$$

For the case of $B = 2Y_0 \cot 2\theta$, the above equation reduces to

$$\begin{pmatrix} V_1 \\ I_1 \end{pmatrix} = \begin{pmatrix} 0, & jZ_0 \tan\theta \\ jY_0 \cot\theta, & 0 \end{pmatrix}\begin{pmatrix} V_2 \\ I_2 \end{pmatrix}, \tag{19}$$

or,

$$V_1 = jZ_0 I_2 \tan\theta,$$

$$I_1 = jY_0 V_2 \cot\theta. \tag{20a}$$

The resultant relation between the impedances $Z_2 = V_1/I_1$ and $Z_2 = V_2/I_2$ follows,

$$Z_1 Z_2 = Z_0^2 \tan^2\theta. \tag{20b}$$

Q.E.D.