Question 1:

a) Starting with **time harmonic** (sinusoidal steady state) form of the Maxwell's equations find the expressions for the electric field intensity and magnetic flux density in terms of the vector and scalar potentials, $\vec{A}(r)$ and $V(r)$. Show all your work. [10 points]

b) Again, assuming time harmonic fields, derive the non-homogeneous wave equation for $\vec{A}(r)$ and $V(r)$ subject to the Lorentz gauge for a **simple medium**. Show all your work. [20 points]

a) From (1) $\nabla \cdot \vec{B} = 0$ & (2) div (curl of any vector) $= 0 \Rightarrow$

(3) $\boxed{\vec{B} = \nabla \times \vec{A}}$. From (4) $\nabla \times \vec{E} = -j\omega \vec{B}$ & (3) $\Rightarrow$

$\nabla \times \vec{E} = -j\omega(\nabla \times \vec{A}) \Rightarrow$ (4) $\nabla \times \vec{E} + j\omega \nabla \times \vec{A} = 0 \Rightarrow$

(5) $\nabla \times (\vec{E} + j\omega \vec{A}) = 0$ since (6) curl (grad of any scalar) $= 0 \Rightarrow$

$\vec{E} + j\omega \vec{A} = -\nabla V \Rightarrow$ (7) $\boxed{\vec{E} = -\nabla V - j\omega \vec{A}}$

b) From Faraday's law (8) $\nabla \times \vec{H} = \vec{J} + j\omega\varepsilon \vec{E}$ or since $\vec{H} = \frac{\vec{B}}{\mu}$ then

(9) $\nabla \times \vec{B} = \mu \vec{J} + j\omega\mu\varepsilon \vec{E}$ use (3) & (7) in (9) $\Rightarrow$

$\nabla \times (\nabla \times \vec{A}) = \mu \vec{J} + j\omega\mu\varepsilon(-\nabla V - j\omega \vec{A}) \Rightarrow$

$\nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A} = \mu \vec{J} - j\omega\mu\varepsilon \nabla V + \omega^2 \mu\varepsilon \vec{A} \Rightarrow$

$-\nabla^2 \vec{A} = \mu \vec{J} - j\omega\mu\varepsilon \nabla V - \nabla(\nabla \cdot \vec{A}) + \omega^2\mu\varepsilon \vec{A} \Rightarrow$

(10) $-\nabla^2 \vec{A} = \mu \vec{J} - \nabla(j\omega\mu\varepsilon V + \nabla \cdot \vec{A}) + \omega^2 \mu\varepsilon \vec{A}$

$(10) \Rightarrow$

(11) $$\boxed{-\nabla^2\vec{A} = \mu\vec{J} + \omega^2\mu\varepsilon\vec{A} - \nabla(j\omega\mu\varepsilon V + \nabla\cdot\vec{A})}$$

For Lorentz gauge $j\omega\mu\varepsilon V + \nabla\cdot\vec{A} = 0 \Rightarrow$

(12) $\boxed{\nabla\cdot\vec{A} = -j\omega\mu\varepsilon V}$ then

$(11) \Rightarrow$ (13) $-\nabla^2\vec{A} = \mu\vec{J} + \omega^2\mu\varepsilon\vec{A}$ or

(14) $$\boxed{\nabla^2\vec{A} + \omega^2\mu\varepsilon\vec{A} = -\mu\vec{J}}$$

We now use the Gauss law to find the wave Eq for V

(15) $\nabla\cdot\vec{D} = \rho \Rightarrow$ (16) $\nabla\cdot\vec{E} = \rho/\varepsilon$ use (7) in (16) $\Rightarrow$

$\nabla\cdot(-\nabla V - j\omega\vec{A}) = \rho/\varepsilon \Rightarrow$ (17) $-\nabla^2 V - j\omega\nabla\cdot\vec{A} = \rho/\varepsilon$

but $\nabla\cdot\vec{A}$ is given by (12) $\Rightarrow$

$-\nabla^2 V - j\omega(-j\omega\mu\varepsilon V) = \rho/\varepsilon \Rightarrow$

$-\nabla^2 V - \omega^2\mu\varepsilon V = \rho/\varepsilon \Rightarrow$

(18) $$\boxed{\nabla^2 V + \omega^2\mu\varepsilon V = -\rho/\varepsilon}$$

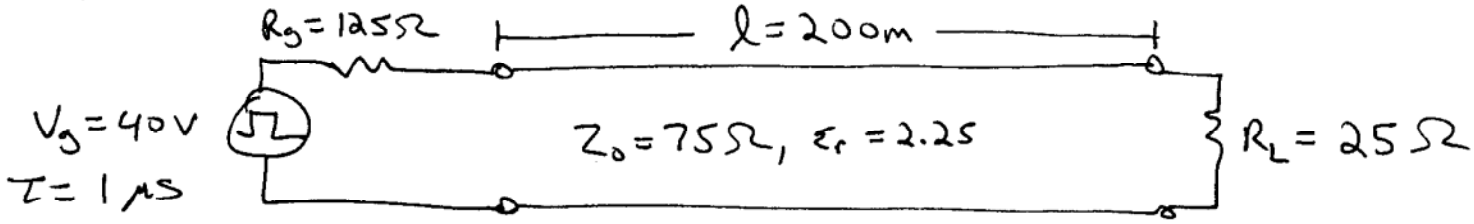
Question 2

A 75 Ω lossless transmission line, with $\varepsilon_r = 2.25$, is connected to a 40 V pulse generator with a source impedance of 125 Ω. The line is 200 m long and is terminated in a 25 Ω load. At time $t = 0$, a single rectangular pulse of width 1 μs is sent down the line.

(a) Draw a reflection diagram for $0 \le t \le 4$ μs. Be sure to include both the leading and trailing edges. Label each line with a numeric value. (15 marks)

(b) Plot $V(z = 100\text{ m}, t)$ for $0 \le t \le 4$ μs. Label all voltage levels and transition times with numeric values. (20 marks)

The TL circuit is



For this line, $v_p = \dfrac{c}{\sqrt{\varepsilon_r}} = \dfrac{3\times10^8}{\sqrt{2.25}} = 2\times10^8\ \text{m/s}$

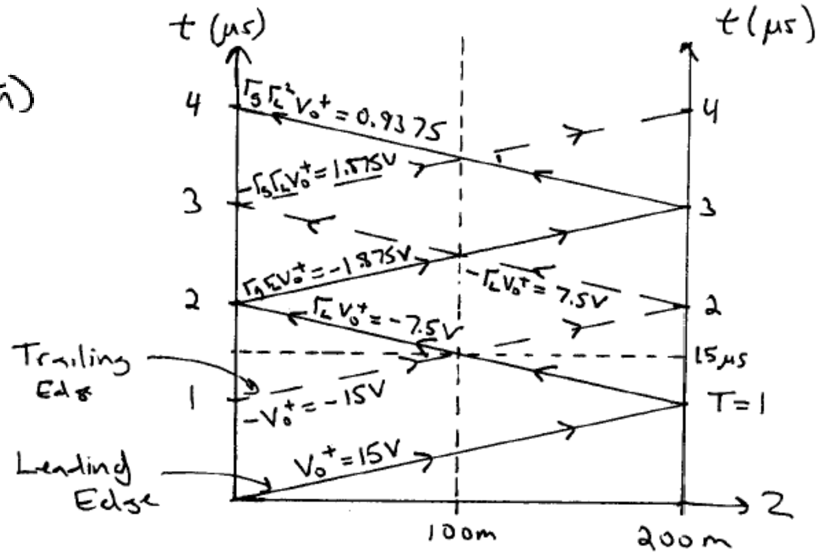
$$\therefore\ T = \frac{l}{v_p} = \frac{200}{2\times10^8} = \underline{\underline{1\,\mu s}}$$

$$V_0^+ = V_g\,\frac{Z_0}{Z_0 + R_g} = 40\,\frac{75}{75+125} = \underline{\underline{15\text{ V}}}$$

$$\Gamma_L = \frac{R_L - Z_0}{R_L + Z_0} = \frac{25-75}{25+75} = \underline{\underline{-\frac{1}{2}}}$$

$$\Gamma_g = \frac{R_g - Z_0}{R_g + Z_0} = \frac{125-75}{125+75} = \underline{\underline{\frac{1}{4}}}$$

(a)



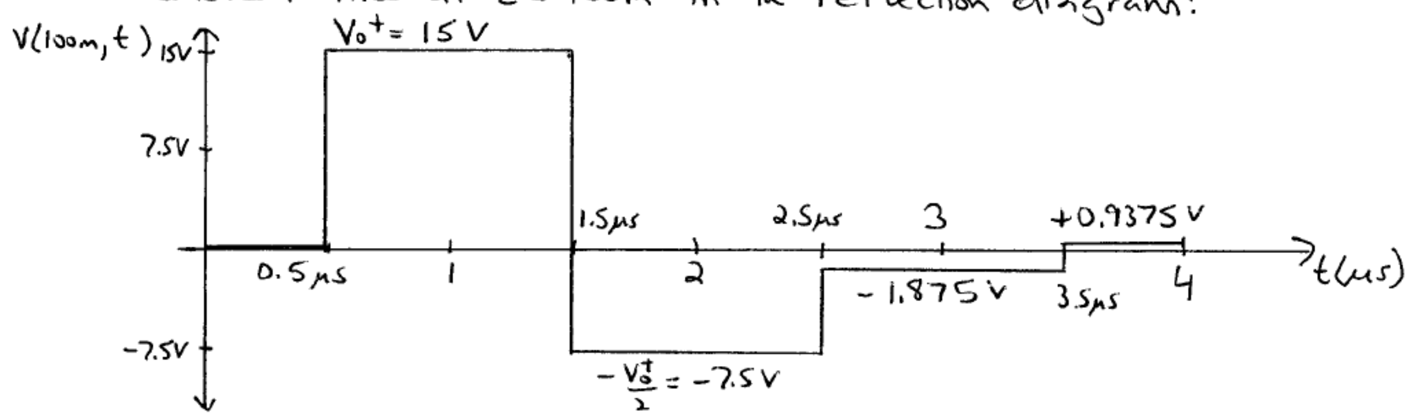
$V_0^+ = 15\,V$

$\Gamma_L V_0^+ = (-\frac{1}{2})(15) = -7.5\,V$

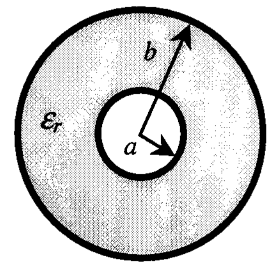
$\Gamma_g \Gamma_L V_0^+ = (\frac{1}{4})(-7.5) = -1.875\,V$

$\Gamma_g \Gamma_L^2 V_0^+ = (-\frac{1}{2})(-1.875) = 0.9375\,V$

(b) To plot the voltage versus time at z = 100m, follow dashed line at z = 100m in the reflection diagram:



Question 3



A coaxial cable lies along the z-axis, and has an inner conductor radius of $\rho = a = 0.25$ mm, and an outer conductor radius of $\rho = b = 0.75$ mm. The two conductors are separated by a non-magnetic ($\mu_r = 1$), lossless dielectric, with $\varepsilon_r = 2$. It is known, that the instantaneous electric field within this dielectric is given by

$$\overline{E} = \frac{200}{\rho}\sin(\omega t - \beta z)\mathbf{a}_\rho \text{ V/m} \quad \text{for } a < \rho < b$$

with $f = 30$ GHz (30×10^9 Hz).

(a) Find the instantaneous magnetic field, $\overline{H}$, within the cable, meaning for $a < \rho < b$. Show your work. (12 marks)

(b) Determine the expressions for the instantaneous surface charge density, ρ_S, and the instantaneous surface current density, J_S, on the outer conductor ($\rho = b$). You may assume that this is a perfect conductor. (10 marks)

(c) Describe the type of electromagnetic wave that these fields represent. Justify your answer. (4 marks)

(d) Find the numerical value of the propagation constant β. (4 marks)

(a) $\overline{H}$ can be found from Faraday's Law $\overline{\nabla} \times \overline{E} = -j\omega\mu_0\overline{H}$ (1)
but E must first be written in phasor form:

$$\overline{E}(\rho,z,t) = \frac{200}{\rho}\sin(\omega t - \beta z)\hat{a}_\rho = \frac{200}{\rho}\cos\left(\omega t - \beta z - \frac{\pi}{2}\right)\hat{a}_\rho$$

$$\therefore \overline{E}(\rho,z) = \frac{200}{\rho}e^{-j\beta z}\underbrace{e^{-j\frac{\pi}{2}}}_{-j}\hat{a}_\rho = -j\frac{200}{\rho}e^{-j\beta z}\hat{a}_\rho$$

$$\therefore \text{From (1)}, \quad \overline{H} = \frac{j}{\omega\mu_0}\overline{\nabla}\times\overline{E} = \left(\frac{j}{\omega\mu_0}\right)\left(\frac{\partial E_\rho}{\partial z}\hat{a}_\phi\right)$$

[The curl has been simplified in cylindrical coordinates for $\overline{E}(\rho,z)$]

$$= \left(\frac{j}{\omega\mu_0}\right)\frac{\partial}{\partial z}\left[-j\frac{200}{\rho}e^{-j\beta z}\right]\hat{a}_\phi$$

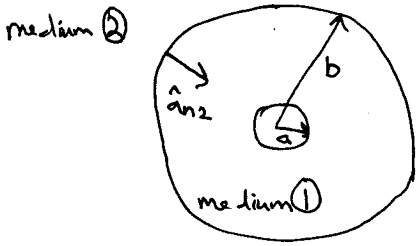
$$= \left(\frac{j}{\omega\mu_0}\right)\left(\frac{-j200}{\rho}\right)(-j\beta)\left(e^{-j\beta z}\right)\hat{a}_\phi$$

$$\overline{H}(\rho,z) = -j\frac{\beta}{\omega\mu_0}\left(\frac{200}{\rho}\right)e^{-j\beta z}\hat{a}_\phi = -\frac{j}{\eta}\left(\frac{200}{\rho}\right)e^{-j\beta z}\hat{a}_\phi$$

$\therefore$ The instantaneous $\overline{H}$ field is then: $\quad$ ↳ Since $\eta = \frac{\omega\mu_0}{\beta} = \sqrt{\frac{\mu}{\varepsilon}} = 266\ \Omega$

$$\overline{H}(\rho,z,t) = \text{Re}\left[\overline{H}(\rho,z)e^{j\omega t}\right] = \frac{1}{\eta}\frac{200}{\rho}\sin(\omega t - \beta z)\hat{a}_\phi = \underline{\underline{\frac{0.75}{\rho}\sin(\omega t - \beta z)\hat{a}_\phi \left[\frac{A}{m}\right]}}$$

(b) To solve for ρ_s & $\bar{J}_s$ we must use boundary conditions:



$$\Rightarrow \quad \hat{a}_{n2} \cdot (\bar{D}_1 - \bar{D}_2) = \rho_s \quad \text{and} \quad \hat{a}_{n2} \times (\bar{H}_1 - \bar{H}_2) = \bar{J}_s$$

* Our outward normal of medium 2 (Perfect conductor) (PEC) is $\hat{a}_{n2} = -\hat{a}_\rho$.

* Also $\bar{D}_2 = \bar{E}_2 = \bar{H}_2 = \bar{B}_2 = 0$ since it is a PEC.

$$\therefore \rho_s = -\hat{a}_\rho \cdot \bar{D}_1(\rho{=}b, z, t) = -\hat{a}_\rho \cdot \varepsilon_r \varepsilon_o \left[\frac{200}{0.75\,mm} \sin(\omega t - \beta z)\,\hat{a}_\rho\right]$$

$$= -\left(\frac{200}{0.75 \times 10^{-3}}\right)(2\varepsilon_o) \sin(\omega t - \beta z)$$

$$\underline{\underline{\rho_s = -4.7 \sin(\omega t - \beta z) \quad \mu C/m^2}}$$

For $\bar{J}_s$:

$$\bar{J}_s = -\hat{a}_\rho \times \bar{H}_1(\rho{=}b, z, t)$$

$$= -\hat{a}_\rho \times \left[\frac{0.75}{0.75 \times 10^{-3}} \sin(\omega t - \beta z)\,\hat{a}_\phi\right]$$

$$\underline{\underline{\bar{J}_s = -1000 \sin(\omega t - \beta z)\,\hat{a}_z \quad A/m^2}}$$

(c) These fields, $\bar{E}$ & $\bar{H}$, correspond to a TEM wave. This is because both $\bar{E}$ and $\bar{H}$ are perpendicular to the direction of propagation.

(d) For a TEM wave, $\beta = \omega\sqrt{\mu\varepsilon}$

$$\therefore \beta = 2\pi f \sqrt{\mu_o \varepsilon_r \varepsilon_o}$$

$$= 2\pi(30 \times 10^9)\sqrt{\mu_o (2) \varepsilon_o}$$

$$\underline{\underline{\beta = 889 \text{ rad/m}}}$$