Last Name:	First name:
Student number	Signature

Faculty of Applied Science and Engineering

ECE357 Electromagnetic Fields

Final Exam, April 14, 2005
Examination Time 14:00-16:30

Examiners – M. Mojahedi

Examination Type D: Every student is allowed to bring a single aid-sheet (8.5" by 11") to the examination for his or her personal use only. Student can write on both sides of the single page aid-sheet any equation, expression, text, etc. that he or she deems necessary.
*** Only Calculators approved by the Registrar are allowed**
*** Student may bring ruler, compass, protractor and/or additional pens, pencils, and erasers.**
*** Answer the questions in the spaces provided or on the facing page**
*** A complete paper consists of answers to all questions**
*** For numerical answers specify units**

DO NOT REMOVE STAPLE

Do not write in these spaces

1	2	3	4	5	TOTAL

$\varepsilon_0 = 8.854 \times 10^{-12}\ [F/m], \quad \mu_0 = 4\pi \times 10^{-7}\ [H/m], \quad c = 3 \times 10^{8}\ [m/s]$

Question 1: For the transmission line shown below

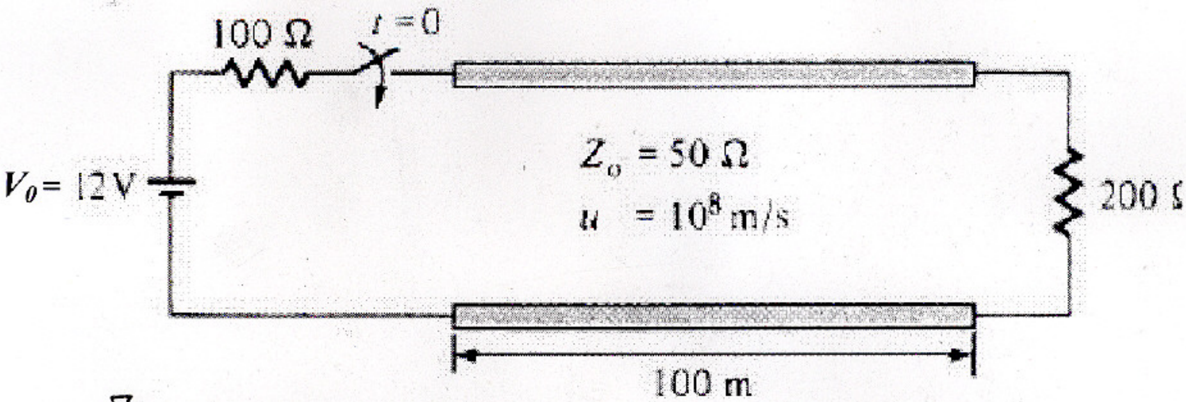
a) Draw the voltage reflection diagram. **(5 pts)**

b) Plot the voltage at the source for $0 < t < 6\ \mu s$. What is the final voltage value? **(5 pts)**

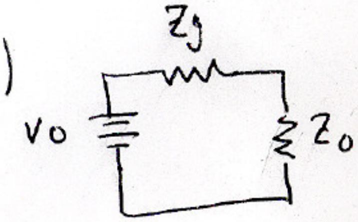
c) Plot the voltage at the load for $0 < t < 6\ \mu s$. **(5 pts)**

d) Plot the current at the load for $0 < t < 6\ \mu s$. **(5 pts)**

Note: In all your plots clearly mark the values on the figures.



a)

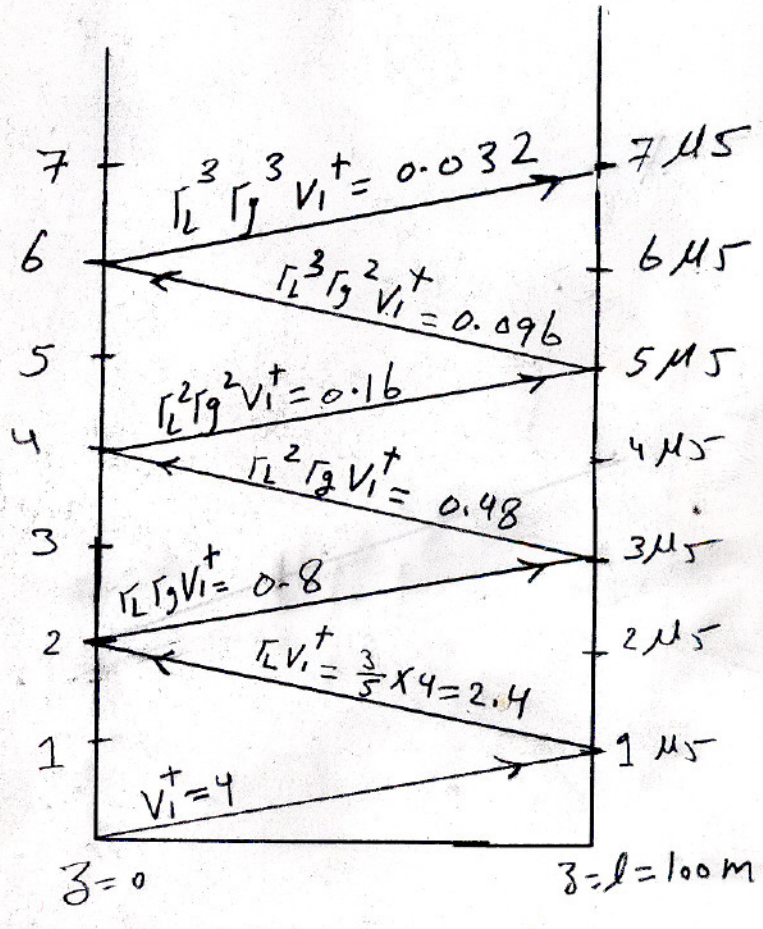


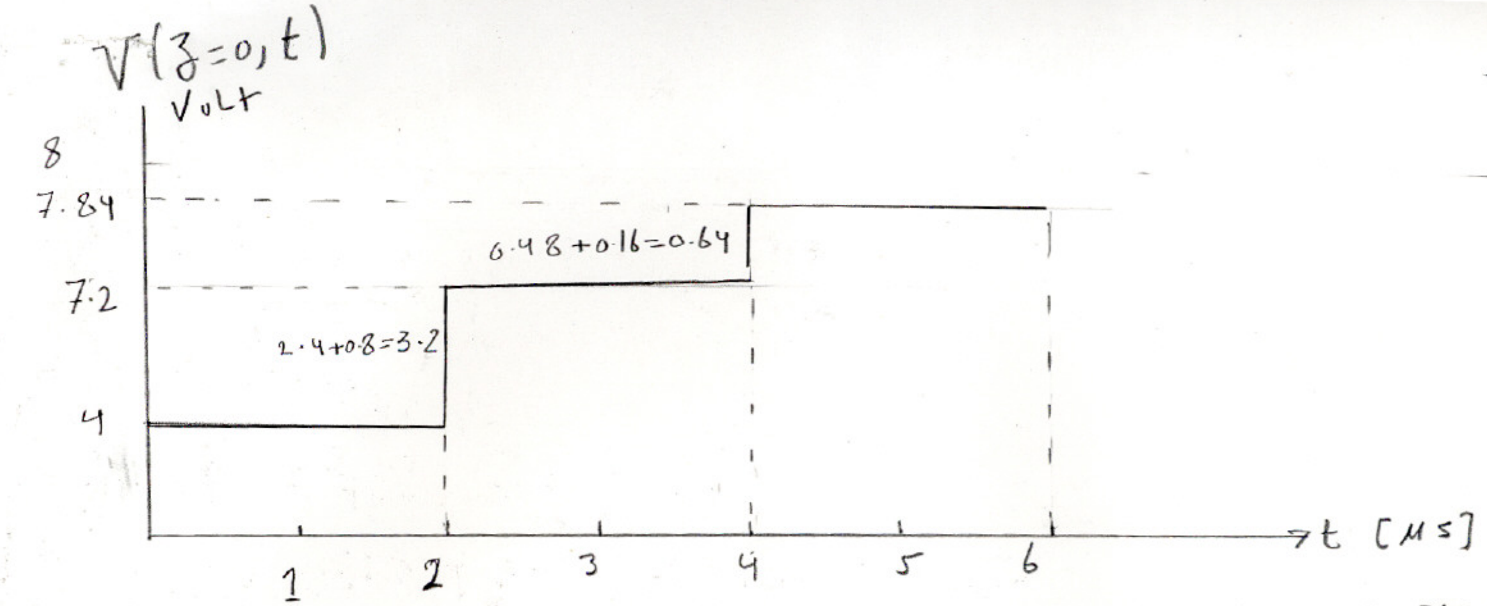
$$V_1^+ = V_0 \frac{Z_0}{Z_0 + Z_g} = 12 \frac{50}{100+50} = 12 \times \frac{5}{15} = \boxed{4 \text{ volt}}$$

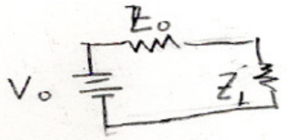
$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{200-50}{200+50} = \boxed{3/5}$$

$$\Gamma_g = \frac{Z_g - Z_0}{Z_g + Z_0} = \frac{100-50}{100+50} = \boxed{1/3}$$

$$T = \frac{l}{u} = \frac{100}{10^8} = 1\ \mu s$$

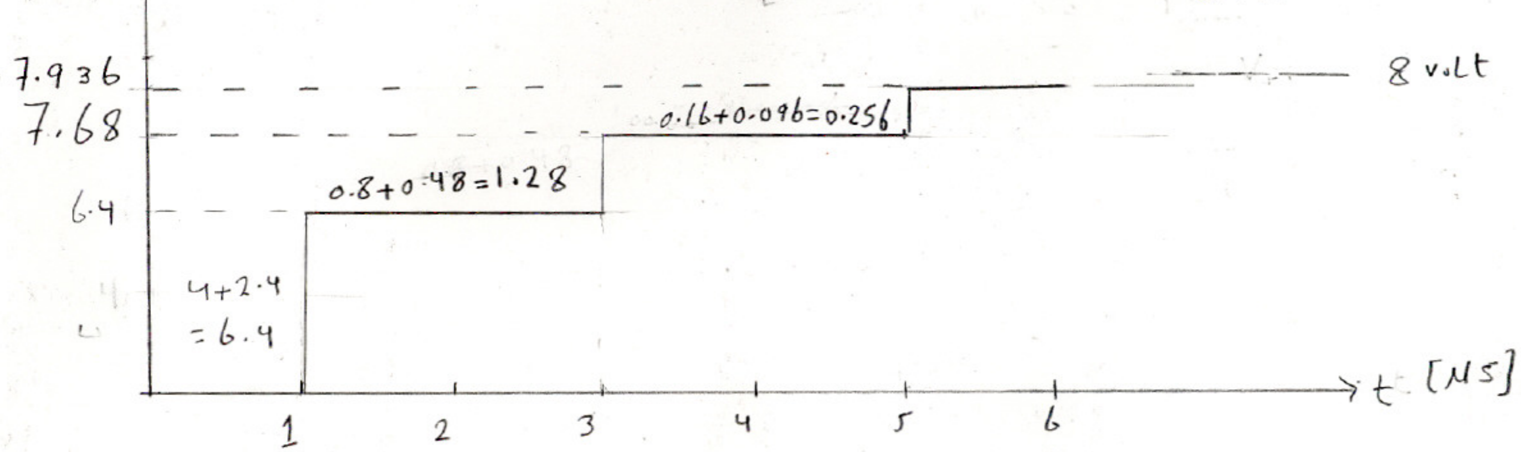




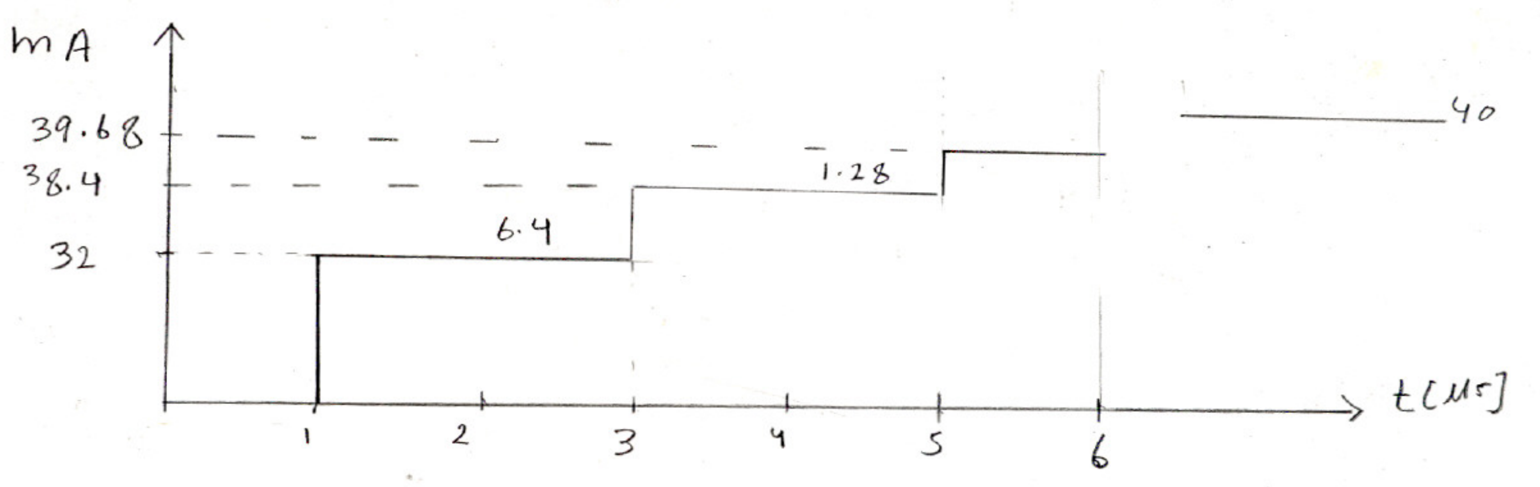


$$V_{final} = V_1^+ \left(\frac{1+\Gamma_L}{1-\Gamma_g\Gamma_L}\right) = 4\left(\frac{1+3/5}{1-\frac{3}{5}\times\frac{1}{3}}\right) = 8$$

OR $$V_{final} = V_0 \frac{Z_L}{Z_L+Z_0} = 12\,\frac{200}{200+100} = 8$$



$I(z=l,t) = V(z=l,t)/R_L$ when $R_L = 200$



Question 2: A segment of lossy transmission line of length l, characteristic impedance Z_0, and propagation constant γ is shown in Fig. 1, where (1, 1′) and (2, 2′) are the input and output terminals respectively. The transmission line can also be modeled as a symmetric 2-port T-network as shown in Fig. 2.

a) Find a relation between Z_1, Y_2, and $Cosh(\gamma l)$.

b) Find a relation between Z_1, Y_2, Z_0, and $Sinh(\gamma l)$.

c) Find a relation between Y_2, Z_0, and $Sinh(\gamma l)$.

Hint: You may begin by formulating the problem in terms of a 2-port Network, relating voltages and currents according to

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix}$$

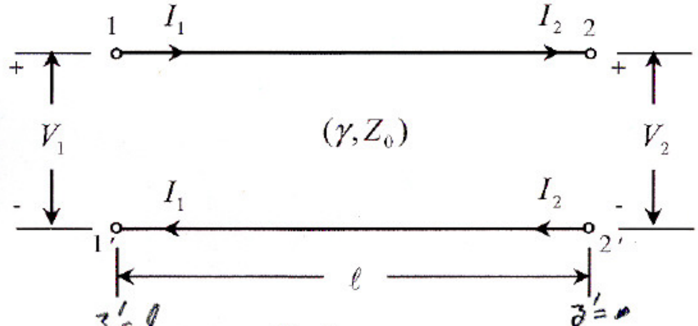


Fig. 1

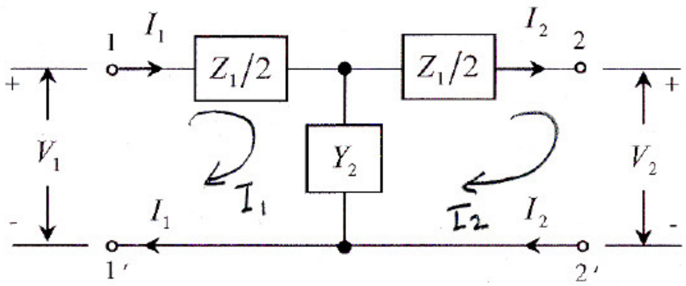


Fig. 2

From TL Eqs

① $V(z') = I_L Z_L \cosh \gamma z' + I_L Z_0 \sinh \gamma z'$

② $I(z') = \dfrac{I_L Z_L}{Z_0} \sinh \gamma z' + I_L \cosh \gamma z'$

For $z' = \ell$ ⇒

③ $V(z'=\ell) = V_L \cosh \gamma \ell + I_L Z_0 \sinh \gamma \ell$

④ $I(z'=\ell) = \dfrac{V_L}{Z_0} \sinh \gamma \ell + I_L \cosh \gamma \ell$

using (1,1′) for input & (2,2′) for output

$$V_1 = V_2 \cosh(\gamma \ell) + I_2 Z_0 \sinh(\gamma \ell)$$

$$I_1 = \frac{V_2}{Z_0} \sinh(\gamma \ell) + I_2 \cosh(\gamma \ell)$$

$$\begin{pmatrix} V_1 \\ I_1 \end{pmatrix} = \begin{pmatrix} \cosh(\gamma \ell) & Z_0 \sinh(\gamma \ell) \\ \dfrac{\sinh(\gamma \ell)}{Z_0} & \cosh(\gamma \ell) \end{pmatrix} \begin{pmatrix} V_2 \\ I_2 \end{pmatrix}$$

where $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ are evident.

(1)

For 2-Port network

loop 1: ① $V_1 = I_1 \frac{Z_1}{2} + (I_1 - I_2)\frac{1}{Y_2} \Rightarrow V_1 = I_1\left(\frac{Z_1}{2} + \frac{1}{Y_2}\right) - \frac{I_2}{Y_2}$ ②

loop 2: ③ $I_2 \frac{Z_1}{2} + (I_2 - I_1)\frac{1}{Y_2} + V_2 = 0 \Rightarrow$

④ $I_2\left(\frac{Z_1}{2} + \frac{1}{Y_2}\right) + V_2 = I_1 \frac{1}{Y_2} \Rightarrow$

⑤ $\boxed{I_1 = V_2 Y_2 + \left(\frac{Z_1 Y_2}{2} + 1\right) I_2}$

use (5) in (2) $\Rightarrow$ ⑥ $V_1 = \left[V_2 Y_2 + \left(\frac{Z_1 Y_2}{2} + 1\right) I_2\right]\left(\frac{Z_1}{2} + \frac{1}{Y_2}\right) - \frac{I_2}{Y_2} \Rightarrow$

⑦ $V_1 = V_2 (Y_2)\left(\frac{Z_1}{2} + \frac{1}{Y_2}\right) + I_2 \left\{\left(\frac{Z_1 Y_2}{2} + 1\right)\left(\frac{Z_1}{2} + \frac{1}{Y_2}\right) - \frac{1}{Y_2}\right\}$

⑧ $V_1 = V_2\left(\frac{Z_1 Y_2}{2} + 1\right) + I_2\left(\frac{Z_1^2 Y_2}{4} + \frac{Z_1}{2} + \frac{Z_1}{2} + \cancel{\frac{1}{Y_2}} - \cancel{\frac{1}{Y_2}}\right) \Rightarrow$

⑨ $\boxed{V_1 = V_2\left(1 + \frac{Z_1 Y_2}{2}\right) + I_2 \, Z_1\left(1 + \frac{Z_1 Y_2}{4}\right)}$ = ⑩

From (5) & (9) $\Rightarrow$ $\boxed{\begin{pmatrix} V_1 \\ I_1 \end{pmatrix} = \begin{pmatrix} 1 + \frac{Z_1 Y_2}{2} & Z_1\left(1 + \frac{Z_1 Y_2}{4}\right) \\ Y_2 & 1 + \frac{Z_1 Y_2}{2} \end{pmatrix}\begin{pmatrix} V_2 \\ I_2 \end{pmatrix}}$

Compare (10) with (1) (previous page) $\Rightarrow$

$\boxed{1 + \frac{Z_1 Y_2}{2} = \cosh(\gamma l)}$

$\boxed{Z_1\left(1 + \frac{Z_1 Y_2}{4}\right) = Z_0 \sinh(\gamma l)}$

$\boxed{Y_2 = \frac{\sinh(\gamma l)}{Z_0}}$

Question 3: A uniform plane wave is traveling downward in +z-direction in seawater, with the x-y plane denoting the sea surface and $z=0$ denoting a point just below the surface. The constitutive parameters of seawater are $\varepsilon_r = 80$, $\mu_r = 1$, and $\sigma = 4\,[S/m]$. If the magnetic field at $z=0$ is given by $\vec{H}(0,t) = \hat{a}_y\, 100\cos(2\pi\times10^3\, t + 15^o)\ \ [mA/m]$.

a) Obtain expression for $\vec{E}(z,t)$ and $\vec{H}(z,t)$.

b) Determine the depth at which the amplitude of $\vec{E}$ is 1% of its value at $z=0$.

Let us calculate $\frac{\sigma}{\omega\varepsilon}$ to see if seawater is a good conductor or good dielectric. From $\vec{H}(0,t) = \hat{a}_y\, 100\cos(2\pi\times10^3 t + 15^o) \Rightarrow$

① $\omega = 2\pi\nu = 2\pi\times10^3$ then

② $$\frac{\sigma}{\omega\varepsilon} = \frac{4}{2\pi\times10^3\times8.85\times10^{-12}\times80} = 8.992\times10^5 \gg 1 \Rightarrow$$

Good conductor

In General
$$\vec{E}(z) = E_o\, e^{-\alpha z}\, e^{-j\beta z}\, \hat{a}_x = |E_o|\, e^{j\phi_o}\, e^{-\alpha z}\, e^{-j\beta z}\, \hat{a}_x \quad ③$$
$$\vec{H}(z) = \frac{E_o}{\eta}\, e^{-\alpha z}\, e^{-j\beta z}\, \hat{a}_y = \frac{|E_o|}{\eta}\, e^{j\phi_o}\, e^{-\alpha z}\, e^{-j\beta z}\, \hat{a}_y \quad ④$$

[or if you like $\vec{H} = \frac{E_o}{\eta}\, e^{-\alpha z}\, e^{-j\beta z}\,\hat{a}_y$ & $\vec{E} = -\eta\,\hat{a}_\beta\times\vec{H} = -\eta\frac{E_o}{\eta}\, e^{-\alpha z}\, e^{-j\beta z}\,\hat{a}_z\times\hat{a}_y$

$\Rightarrow \vec{E} = E_o\, e^{-\alpha z}\, e^{-j\beta z}\,\hat{a}_x$ as given in (3)]

⑤ $$\alpha = \sqrt{\frac{\omega\mu\sigma}{2}} = \sqrt{\frac{2\pi\nu\mu\sigma}{2}} = \sqrt{\frac{2\pi\times10^3\times4\pi\times10^{-7}\times4}{2}} = 0.1257\ [NP/m]$$

⑥ $$\boxed{\beta = \sqrt{\frac{\omega\mu\sigma}{2}} = \sqrt{\frac{2\pi\nu\mu\sigma}{2}} = 0.1257\ [rad/m]}$$

⑦ $$\eta = \sqrt{\frac{\omega\mu}{2\sigma}}\,(1+j) = \sqrt{\frac{2\pi\nu\mu}{2\sigma}}\,(1+j) =$$
$$\sqrt{\frac{2\pi\times10^3\times4\pi\times10^{-7}}{2\times4}}\,(1+j) = 3.1416(1+j) = \boxed{0.044\, e^{j45^o}}\ \Omega$$

From (4) ⑧ $\vec{H}(z) = \frac{|E_0| e^{j\phi_0}}{0.044 e^{j45^\circ}} e^{-0.1257z} e^{-j0.1257z} \hat{a_y} \Rightarrow$

⑨ $\vec{H}(z,t) = \frac{|E_0|}{0.044} e^{-0.1257z} \cos(2\pi \times 10^3 t + \phi_0 - 45^\circ - 0.1257z)\, \hat{a_y}$

$\Rightarrow H(z=0,t) = \frac{|E_0|}{0.044} \cos(2\pi \times 10^3 t + \phi_0 - 45^\circ)\, \hat{a_y}$ ⑩

Compare (10) with given $\vec{H}(0,t)$ in problem statement, i.e

⑪ $\frac{|E_0|}{0.044} \cos(2\pi \times 10^3 t + \phi_0 - 45^\circ) = 100 \cos(2\pi \times 10^3 t + 15^\circ) \Rightarrow$

From (11) it is clear that

⑫ $\frac{|E_0|}{0.044} = 100 \Rightarrow \quad |E_0| = 4.4 \; [mV/m]$

⑬ & $\boxed{\phi_0 = 60^\circ}$ then

⑭ $\vec{E}(z) = 4.4\, e^{j60^\circ} e^{-0.1257z} e^{-j0.1257z} \hat{a_x} \; [mV/m]$

or

⑮ $\boxed{\vec{E}(z,t) = 4.4\, e^{-0.1257z} \cos(2\pi \times 10^3 t - 0.1257z + 60^\circ)\, \hat{a_x}}$ $[mV/m]$

16 $\vec{H}(z) = 4.4\, e^{+j60^\circ} \frac{1}{0.044\, e^{j45^\circ}} e^{-0.1257z} e^{-j0.1257z} \hat{a_y}$ OR

$\boxed{\vec{H}(z,t) = 100\, e^{-0.1257z} \cos(2\pi \times 10^3 t - 0.1257z + 15^\circ)\, \hat{a_y}}$

b) At $z=0$ $|\vec{E}| = 4.4$, 1% of $4.4 = \frac{4.4}{100}$ then we want z' for which

$\frac{4.4}{100} = 4.4\, e^{-0.1257z'} \Rightarrow \quad \frac{1}{100} = e^{-0.1257z'} \Rightarrow$

$z' = \frac{\ln(0.01)}{-0.1257} = 36.636\,[m]$

Question 4: In class we saw that for ionized gas $v_p\, v_g = c^2$; where v_g is the group velocity, v_p is the phase velocity and c is the speed of light in vacuum. Prove that in general $v_p\, v_g = c^2$ implies a hyperbolic dispersion relation (k vs. ω). Recall that the equation for hyperbola is given by $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. For simplicity only consider the one-dimensional propagation case.

(1) $V_p = \frac{\omega}{k}$, (2) $V_g = \frac{d\omega}{dk}$ then

(3) $V_p V_g = c^2 \Rightarrow$ (4) $\frac{\omega}{k}\frac{d\omega}{dk} = c^2 \Rightarrow$ (5) $\omega\, d\omega = c^2\, k\, dk \Rightarrow$

(6) $\int \omega\, d\omega = c^2 \int k\, dk \Rightarrow$ (7) $\frac{\omega^2}{2} + A_0 = c^2\left[\frac{k^2}{2} + A_1\right]$ where

A_0 & A_1 are constants

(7) $\Rightarrow$ $A_0 - c^2 A_1 = c^2\frac{k^2}{2} - \frac{\omega^2}{2} \Rightarrow$

let $A_2 = A_0 - c^2 A_1$ where A_2 is a constant then

(8) $\frac{c^2k^2}{2} - \frac{\omega^2}{2} = A_2 \Rightarrow$ (9) $\frac{k^2}{2A_2/c^2} - \frac{\omega^2}{2A_2} = 1$

let $a^2 = 2A_2/c^2$ & $b^2 = 2A_2$ then

(9) $\Rightarrow$ $\boxed{\frac{k^2}{a^2} - \frac{\omega^2}{b^2} = 1}$ a hyperbola

OR

$$V_p V_g = c^2$$

$$V_p = \frac{c}{n} \qquad V_g = \frac{1}{dk/d\omega} \quad \& \quad k = \frac{\omega}{c} n \Rightarrow$$

$$V_g = \frac{c}{\frac{d}{d\omega}\omega n} \quad \text{then}$$

$$V_g V_p = c^2 \Rightarrow \frac{c}{n}\frac{c}{d\omega n/d\omega} = c^2 \Rightarrow$$

$$n\frac{d\omega n}{d\omega} = 1 \quad \text{let } z = \omega n \Rightarrow n = \frac{z}{\omega}$$

then

$$\frac{z}{\omega}\frac{dz}{d\omega} = 1 \Rightarrow z\,dz = \omega\,d\omega \Rightarrow \int z\,dz = \int \omega\,d\omega$$

$$\frac{z^2}{2} = \frac{\omega^2}{2} + A \quad A \text{ is a constant}$$

Recall $z^2 = \omega^2 n^2 = c^2 k^2$ then

$$\frac{c^2k^2}{2} = \frac{\omega^2}{2} + A \Rightarrow \frac{k^2}{(2/c^2)} - \frac{\omega^2}{2} = A \Rightarrow$$

$$\boxed{\frac{k^2}{(2A/c^2)} - \frac{\omega^2}{2A} = 1} \Rightarrow \boxed{\frac{k^2}{(\sqrt{2A}/c)^2} - \frac{\omega^2}{(\sqrt{2A})^2} = 1} \equiv$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Question 5: Figure shows a uniform plane wave obliquely incident on the interface between two perfect dielectrics.
For this configuration, the so called parallel polarization, where electric field is on the plane of incidence, answer the following questions.

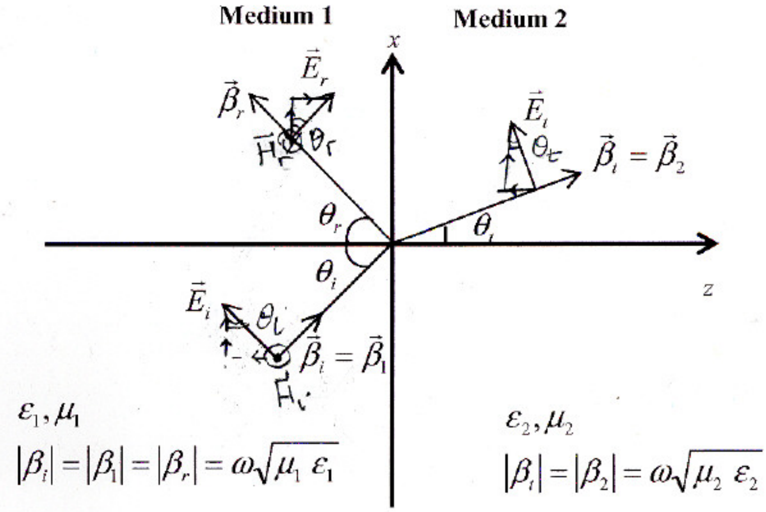
a) What are the time harmonic field expressions for the incident electric and magnetic fields? **(4 Pts)**

b) What are the time harmonic field expressions for the reflected electric and magnetic fields? **(4 Pts)**

c) What are the time harmonic field expressions for the transmitted electric and magnetic fields? **(4 Pts)**

d) Describe the procedure by which the Fresnel transmission and reflection coefficients can be obtained. **(3 Pts)**

e) Give the expression for the Fresnel reflection coefficient in terms of the propagation constant β_{1z} and β_{2z}, (i.e., the projection of $\vec{\beta}_1$ and $\vec{\beta}_2$ along the z-direction.) **(5 pts)**



a)

1 $\vec{E}_i = E_{io}(\cos\theta_i \hat{a}_x - \sin\theta_i \hat{a}_z)\, e^{-j\vec{\beta}_i\cdot\vec{r}} \Rightarrow$

2 $\boxed{\vec{E}_i = E_{io}(\cos\theta_i \hat{a}_x - \sin\theta_i \hat{a}_z)\, e^{-j(\sin\theta_i x + \cos\theta_i z)\beta_1}}$

$\vec{\beta}_i = \beta_1(\sin\theta_i \hat{a}_x + \cos\theta_i \hat{a}_z)$ 5

$\vec{\beta}_r = \beta_1(\sin\theta_r \hat{a}_x - \cos\theta_r \hat{a}_z)$ 6

$\vec{\beta}_t = \beta_2(\sin\theta_t \hat{a}_x + \cos\theta_t \hat{a}_z)$ 7

3 $\vec{H}_i = \frac{E_{io}}{\eta_1} e^{-j\vec{\beta}_i\cdot\vec{r}}\, \hat{a}_y \Rightarrow$

4 $\boxed{\vec{H}_i = \frac{E_{io}}{\eta_1} e^{-j(\sin\theta_i x + \cos\theta_i z)\beta_1}\, \hat{a}_y}$

b)

8 $\vec{E}_r = E_{ro}(\cos\theta_r \hat{a}_x + \sin\theta_r \hat{a}_z)\, e^{-j\vec{\beta}_r\cdot\vec{r}} \Rightarrow$

9 $\boxed{\vec{E}_r = E_{ro}(\cos\theta_r \hat{a}_x + \sin\theta_r \hat{a}_z)\, e^{-j(\sin\theta_r x - \cos\theta_r z)\beta_1}}$

$E_{ro} = \Gamma \cdot E_{io}$

10 $\vec{H}_r = -\frac{E_{ro}}{\eta_1} e^{-j\vec{\beta}_r\cdot\vec{r}}\, \hat{a}_y \Rightarrow$

11 $\boxed{\vec{H}_r = -\frac{E_{ro}}{\eta_1} e^{-j(\sin\theta_r x - \cos\theta_r z)\beta_1}\, \hat{a}_y}$

c) (12) $\vec{E_t} = E_{t0}(\cos\theta_t\,\hat{a}_x - \sin\theta_t\,\hat{a}_z)\,e^{-j\vec{\beta_t}\cdot\vec{r}} \Rightarrow$

(13) $$\boxed{\vec{E_t} = E_{t0}(\cos\theta_t\,\hat{a}_x - \sin\theta_t\,\hat{a}_z)\,e^{-j(\sin\theta_t\,x + \cos\theta_t\,z)\beta_2}}$$

(14) $$\boxed{\vec{H_t} = \frac{E_{t0}}{\eta_2}\,e^{-j(\sin\theta_t\,x + \cos\theta_t\,z)\beta_2}\,\hat{a}_y}$$ $E_{t0} = \tau E_{i0}$

d) we apply B.C at $z = 0$ plane which require tangential $\vec{E}$ & $\vec{H}$ to be continuous across the boundary

(15) $$E_{i0}\cos\theta_i\,e^{-j\sin\theta_i\,x\beta_1} + E_{r0}\cos\theta_r\,e^{-j\sin\theta_r\,x\beta_1} = E_{t0}\cos\theta_t\,e^{-j\sin\theta_t\,x\beta_2}$$

&

(16) $$\frac{E_{i0}}{\eta_1}e^{-j\sin\theta_i\,x\beta_1} - \frac{E_{r0}}{\eta_1}e^{-j\sin\theta_r\,x\beta_1} = \frac{E_{t0}}{\eta_2}e^{-j\sin\theta_t\,x\beta_2}$$

(15) & (16) $\Rightarrow$ (17) $\boxed{\theta_i = \theta_r}$ & (18) $\boxed{\beta_1\sin\theta_i = \beta_2\sin\theta_t}$

Snell's laws

e) $$\Gamma_{\parallel} = \frac{\eta_2\cos\theta_t - \eta_1\cos\theta_i}{\eta_2\cos\theta_t + \eta_1\cos\theta_i} = \frac{\sqrt{\frac{\mu_2}{\varepsilon_2}}\cos\theta_t - \sqrt{\frac{\mu_1}{\varepsilon_1}}\cos\theta_i}{\sqrt{\frac{\mu_2}{\varepsilon_2}}\cos\theta_t + \sqrt{\frac{\mu_1}{\varepsilon_1}}\cos\theta_i}$$

$$= \frac{\varepsilon_1\varepsilon_2}{\varepsilon_1\varepsilon_2}\,\frac{\sqrt{\frac{\mu_2}{\varepsilon_2}}\cos\theta_t - \sqrt{\frac{\mu_1}{\varepsilon_1}}\cos\theta_i}{\sqrt{\frac{\mu_2}{\varepsilon_2}}\cos\theta_t + \sqrt{\frac{\mu_1}{\varepsilon_1}}\cos\theta_i} = \frac{\varepsilon_1\sqrt{\varepsilon_2\mu_2}\cos\theta_t - \varepsilon_2\sqrt{\varepsilon_1\mu_1}\cos\theta_i}{\varepsilon_1\sqrt{\varepsilon_2\mu_2}\cos\theta_t + \varepsilon_2\sqrt{\varepsilon_1\mu_1}\cos\theta_i}$$

$$= \frac{\varepsilon_1\omega\sqrt{\varepsilon_2\mu_2}\cos\theta_t - \varepsilon_2\omega\sqrt{\varepsilon_1\mu_1}\cos\theta_i}{\varepsilon_1\omega\sqrt{\varepsilon_2\mu_2}\cos\theta_t + \varepsilon_2\omega\sqrt{\varepsilon_1\mu_1}\cos\theta_i} = \boxed{\frac{\varepsilon_1\beta_{2z} - \varepsilon_2\beta_{1z}}{\varepsilon_1\beta_{2z} + \varepsilon_2\beta_{1z}}}$$ where

$$\boxed{\beta_{1z} = \omega\sqrt{\varepsilon_1\mu_1}\cos\theta_i = \beta_1\cos\theta_i} \quad \& \quad \boxed{\beta_{2z} = \omega\sqrt{\varepsilon_2\mu_2}\cos\theta_t = \beta_2\cos\theta_t}$$