Assignment #1

ECE320F - Fields and Waves
Fall 2003 / University of Toronto

1. Show that $e^{j\theta} = \cos\theta + j\sin\theta$.

Recall that $e^{j\theta} = 1 + (j\theta) + \frac{(j\theta)^2}{2!} + \frac{(j\theta)^3}{3!} + \frac{(j\theta)^4}{4!} + \ldots$

So $e^{j\theta} = \left[1 + \frac{(j\theta)^2}{2!} + \frac{(j\theta)^4}{4!} + \ldots\right] + \left[(j\theta) + \frac{(j\theta)^3}{3!} + \frac{(j\theta)^5}{5!} + \ldots\right]$

$= \left[1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \ldots\right] + j\left[\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!}\right]$

$= \cos\theta + j\sin\theta.$

2. The current through a circuit is found to be $\vec{I} = 6\angle\text{-}40°$. What is the expression for the time dependent current with $\upsilon = 60$ Hz?

$$i(t) = Re\{\vec{I} e^{j\omega t}\}$$

$$= Re\{6 \angle -40° \, e^{j\omega t}\}$$

$$= Re\{6 e^{-j40°} e^{j\omega t}\}$$

$$= Re\{6 e^{j(\omega t - 40°)}\}$$

$$= 6\cos(\omega t - 40°)$$

$$= 6\cos(2\pi \upsilon t - 40°)$$

$$= 6\cos(2\pi (60\ Hz) t - 40°)$$

$$\boxed{i(t) = 6\cos(120\pi t - 40°)}$$

Remember that the degree sign is important. If it is not present then units in radians are assumed. In this example an equivalent answer is

$$\boxed{i(t) = 6\cos\left(120\pi t - \frac{40}{180}\pi\right)}$$

3. Prove that $\mathrm{Re}\{A\} = \dfrac{A + A^*}{2}$, where $A \in C^2$.

Let $A = a_1 + j a_2$.

Then $\mathrm{Re}\{A\} = a_1$.

$$RHS = \frac{A + A^*}{2} = \frac{a_1 + \cancel{ja_2} + a_1 - \cancel{ja_2}}{2} = a_1$$

$LHS = RHS$.

4. Prove that for a sinusoidal function $a(z,t)$ where z is the space coordinate and t is the time, there exists a phasor $\vec{A}(z)$ such that

   a. $\frac{\partial}{\partial t}a(z,t) = \frac{\partial}{\partial t}\mathrm{Re}\{\vec{A}(z)e^{j\omega t}\}$ 1

   b. $\frac{\partial}{\partial t}a(z,t) = \mathrm{Re}\{j\omega\vec{A}(z)e^{j\omega t}\}.$ 2

a). Let $a(z,t) = A\cos(\omega t + \varphi(z))$.

Then $a(z,t) = \mathrm{Re}\{A\cos(\omega t + \varphi(z)) + jA\sin(\omega t + \varphi(z))\}$

$$= \mathrm{Re}\{A e^{j(\omega t + \varphi(z))}\}$$

$$= \mathrm{Re}\{A e^{j\varphi(z)} e^{j\omega t}\}$$

$$a(z,t) = \mathrm{Re}\{\bar{A}(z) e^{j\omega t}\}.$$

Take derivative with respect to time of both sides

$$\boxed{\frac{\partial}{\partial t} a(z,t) = \frac{\partial}{\partial t}\mathrm{Re}\{\bar{A}(z)e^{j\omega t}\}}$$

b). From LHS of 2

$$\frac{\partial}{\partial t} a(z,t) = \frac{\partial}{\partial t}[A\cos(\omega t + \varphi(z))]$$

$$= -A\omega\sin(\omega t + \varphi(z)).$$

From the RHS of 2

$$\mathrm{Re}\{j\omega\bar{A}(z)e^{j\omega t}\} = \mathrm{Re}\{j\omega A e^{j\varphi(z)} e^{j\omega t}\}$$

$$= \mathrm{Re}\{j\omega A e^{j(\omega t + \varphi(z))}\}$$

$$= Re\{ j\omega A[\cos(\omega t + \varphi(z)) + j\sin(\omega t + \varphi(z))]\}$$

$$= -\omega A \sin(\omega t + \varphi(z)).$$

Thus LHS = RHS and part (b) is proven.

This shows that the $\frac{\partial}{\partial t}$ operator can be moved inside the $Re\{\}$, and that $\frac{\partial}{\partial t}$ can be substituted with $j\omega$ when only a single frequency ω is considered. $\left(\frac{\partial}{\partial t} \rightarrow j\omega\right)$

5. Show that $\frac{\partial^2}{\partial t^2} a(z,t) = \text{Re}\left\{-\omega^2 \vec{A}(z) e^{j\omega t}\right\}$.

Let $a(z,t) = A\cos(\omega t + \varphi(z))$.

$= \text{Re}\left\{ A e^{j\varphi(z)} e^{j\omega t} \right\}$.

Then $\frac{\partial^2}{\partial t^2} a(z,t) = \frac{\partial^2}{\partial t^2}\left[\text{Re}\left\{ A e^{j\varphi(z)} e^{j\omega t} \right\} \right]$

$= \text{Re}\left\{ A e^{j\varphi(z)} \frac{\partial^2}{\partial t^2} e^{j\omega t} \right\}$

$= \text{Re}\left\{ -\omega^2 A e^{j\varphi(z)} e^{j\omega t} \right\}$

$= \text{Re}\left\{ -\omega^2 \bar{A}(z) e^{j\omega t} \right\}$.

6. Prove that if $\text{Re}\{\vec{A}(z)e^{j\omega t}\} = \text{Re}\{\vec{B}(z)e^{j\omega t}\}$, then $\vec{A}(z) = \vec{B}(z)$. This means that the Re{ } operator can be removed on phasors of the same frequency.

Since $\text{Re}\{\bar{A}(z)e^{j\omega t}\} = \text{Re}\{\bar{B}(z)e^{j\omega t}\}$ is true for all t, it is certainly true for $t=0$.

$\Rightarrow$ Set $t=0$.

$$\text{Re}\{\bar{A}(z)\} = \text{Re}\{\bar{B}(z)\}.$$

Let $\bar{A}(z) = A e^{j\varphi_A(z)} = A_r(z) + jA_i(z)$

$\bar{B}(z) = B e^{j\varphi_B(z)} = B_r(z) + jB_i(z)$

Thus $\boxed{A_r(z) = B_r(z)}$.

Now consider another specific time, $\omega t = \frac{\pi}{2}$. $\left(t = \frac{\pi}{2\omega}\right)$.

$\left(e^{j\frac{\pi}{2}} = j\right)$

$$\text{Re}\{\bar{A}(z)e^{j\frac{\pi}{2}}\} = \text{Re}\{\bar{B}(z)e^{j\frac{\pi}{2}}\}$$

$$\text{Re}\{(A_r(z) + jA_i(z))j\} = \text{Re}\{(B_r(z) + jB_i(z))j\}$$

$$\text{Re}\{-A_i(z) + jA_r(z)\} = \text{Re}\{-B_i(z) + jB_r(z)\}$$

$$-A_i(z) = -B_i(z)$$

$$\boxed{A_i(z) = B_i(z)}$$

Thus $A_r(z) + jA_i(z) = B_r(z) + jB_i(z)$

$$\boxed{\bar{A}(z) = \bar{B}(z)}.$$

7. Show that $\mathrm{Re}\left\{\sum_{n=1}^{M}\overrightarrow{A_n}(z)\right\} = \sum_{n=1}^{M}\mathrm{Re}\left\{\overrightarrow{A_n}(z)\right\}$, i.e. that the real part of the sum of phasors is the sum of the real parts.

Let $\bar{A}_n(z) = A_{n,r}(z) + jA_{n,i}(z)$

$$A_{n,r}(z) \in \mathbb{R}$$

$$A_{n,i}(z) \in \mathbb{R}.$$

Then $$\mathrm{Re}\left\{\sum_{n=1}^{M}\bar{A}_n(z)\right\} = \mathrm{Re}\left\{\sum_{n=1}^{M}\left[A_{n,r}(z) + jA_{n,i}(z)\right]\right\}$$

$$= \mathrm{Re}\left\{\sum_{n=1}^{M}A_{n,r}(z) + j\sum_{n=1}^{M}A_{n,i}(z)\right\}$$

$$= \mathrm{Re}\left\{\sum_{n=1}^{M}A_{n,r}(z)\right\}$$

$$= \sum_{n=1}^{M}\mathrm{Re}\left\{A_{n,r}(z)\right\}$$

$$= \sum_{n=1}^{M}\mathrm{Re}\left\{A_{n,r}(z) + jA_{n,i}(z)\right\}$$

$$= \sum_{n=1}^{M}\mathrm{Re}\left\{\bar{A}_n(z)\right\}.$$

8. Show that $\mathrm{Re}\{\vec{A}(z)\}\mathrm{Re}\{\vec{B}(z)\} = \dfrac{\mathrm{Re}\{\vec{A}(z)\vec{B}(z)\}}{2} + \dfrac{\mathrm{Re}\{\vec{A}(z)\vec{B}^*(z)\}}{2}$.

Start with RHS.

$$\frac{\mathrm{Re}\{\bar{A}(z)\bar{B}(z)\}}{2} + \frac{\mathrm{Re}\{\bar{A}(z)\bar{B}^*(z)\}}{2}$$

$$= \frac{\mathrm{Re}\{[A_r(z)+jA_i(z)][B_r(z)+jB_i(z)]\}}{2} + \frac{\mathrm{Re}\{[A_r(z)+jA_i(z)][B_r(z)-jB_i(z)]\}}{2}$$

$$= \frac{\mathrm{Re}\{[A_r(z)B_r(z)-A_i(z)B_i(z)]+j[A_r(z)B_i(z)+A_i(z)B_r(z)]\}}{2}$$

$$+ \frac{\mathrm{Re}\{[A_r(z)B_r(z)+A_i(z)B_i(z)]+j[-A_r(z)B_i(z)+A_i(z)B_r(z)]\}}{2}$$

$$= \frac{A_r(z)B_r(z)-\cancel{A_i(z)B_i(z)}}{2} + \frac{A_r(z)B_r(z)+\cancel{A_i(z)B_i(z)}}{2}$$

$$= A_r(z)B_r(z)$$

$$= \mathrm{Re}\{\bar{A}(z)\}\cdot\mathrm{Re}\{\bar{B}(z)\}.$$

Note that this proves that $\mathrm{Re}\{\bar{A}(z)\bar{B}(z)\} \neq \mathrm{Re}\{\bar{A}(z)\}\mathrm{Re}\{\bar{B}(z)\}$.

9. Show that $\left\langle \mathrm{Re}\left\{\vec{A}(z)e^{j\omega t}\right\}\mathrm{Re}\left\{\vec{B}(z)e^{j\omega t}\right\}\right\rangle_{timeavg} = \frac{1}{2}\mathrm{Re}\left\{\vec{A}(z)\vec{B}^*(z)\right\}$.

Note that $\langle f(t)\rangle = \frac{1}{T_2 - T_1}\int_{T_1}^{T_2} f(t)\,dt$

where $[T_1, T_2]$ is the interval over which the time average is taken.

If $f(t)$ is periodic with period $T = T_2 - T_1$, we can take any interval, one of which is

$$\langle f(t)\rangle = \frac{1}{T}\int_0^T f(t)\,dt.$$

$$\langle \mathrm{Re}\{\bar{A}(z)e^{j\omega t}\}\mathrm{Re}\{\bar{B}(z)e^{j\omega t}\}\rangle = \frac{1}{T}\int_0^T \mathrm{Re}\{\bar{A}(z)e^{j\omega t}\}\cdot\mathrm{Re}\{\bar{B}(z)e^{j\omega t}\}\,dt$$

From the results of Problem 8,

$$= \frac{1}{T}\int_0^T \left[\frac{1}{2}\mathrm{Re}\{\bar{A}(z)\bar{B}(z)e^{2j\omega t}\} + \frac{1}{2}\mathrm{Re}\{\bar{A}(z)\bar{B}(z)\cancel{e^{j\omega t}}\,\cancel{e^{-j\omega t}}\}\right]dt$$

$$= \frac{1}{T}\int_0^T \frac{1}{2}\mathrm{Re}\{\bar{A}(z)\bar{B}(z)e^{j2\omega t}\}\,dt + \frac{1}{T}\int_0^T \frac{1}{2}\mathrm{Re}\{\bar{A}(z)\bar{B}^*(z)\}\,dt$$

$$= \frac{1}{T}\frac{1}{2}\mathrm{Re}\left\{\bar{A}(z)\bar{B}(z)\cancel{\int_0^T e^{j2\omega t}\,dt}^{\,0}\right\} + \frac{1}{T}\frac{1}{2}\mathrm{Re}\left\{\bar{A}(z)\bar{B}^*(z)\int_0^T dt\right\}$$

$$= \frac{1}{\cancel{T}}\,\frac{1}{2}\mathrm{Re}\{\bar{A}(z)\bar{B}^*(z)\}\cdot\cancel{T}$$

$$= \frac{1}{2}\mathrm{Re}\{\bar{A}(z)\bar{B}^*(z)\}.$$

Note that this gives

$\langle \text{Power}\rangle = \langle v(t)\,i(t)\rangle = \frac{1}{2}\mathrm{Re}\{\bar{V}\cdot\bar{I}^*\}.$

10. The voltage across and current through a device are given by $v(t) = V_m \cos(\omega t + \theta_v)$ and $i(t) = I_m \cos(\omega t + \theta_i)$. What is the average power dissipated in the device?

$$v(t) = V_m \cos(\omega t + \theta_v) = Re\{V_m e^{j\theta_v} e^{j\omega t}\} = Re\{\bar{V} e^{j\omega t}\}$$

$$i(t) = I_m \cos(\omega t + \theta_i) = Re\{I_m e^{j\theta_i} e^{j\omega t}\} = Re\{\bar{I} e^{j\omega t}\}$$

Then the average power dissapated is

$$\langle P(t)\rangle = \langle v(t)\, i(t)\rangle = \langle Re\{\bar{V} e^{j\omega t}\} Re\{\bar{I} e^{j\omega t}\}\rangle$$

$$= \frac{1}{2} Re\{\bar{V} \cdot \bar{I}^*\} \quad \text{(from Problem \#9)}$$

$$= \frac{1}{2} Re\{V_m e^{j\theta_v} I_m e^{-j\theta_i}\}$$

$$= \frac{1}{2} Re\{V_m I_m e^{j(\theta_v - \theta_i)}\}$$

$$\boxed{\langle P(t)\rangle = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i)}.$$

Resistors

$$v(t) = R\, i(t)$$

$$\bar{V} = R\bar{I} \Rightarrow \theta_v - \theta_i = 0$$

$$\langle P(t)\rangle = \frac{1}{2} V_m I_m = \frac{1}{2}\frac{V_m^2}{R} = \frac{1}{2} I_m^2 R$$

Note that this is diffenent than $P_{DC} = VI = \frac{V^2}{R} = I^2 R$.

To make a correspondence Let $V_{eff} = \frac{V_m}{\sqrt{2}}$, $I_{eff} = \frac{I_m}{\sqrt{2}}$

Then $\langle P(t)\rangle = P_{ave} = \frac{V_m}{\sqrt{2}}\frac{I_m}{\sqrt{2}} = V_{eff} I_{eff}$.

V_{eff} also known as V_{RMS} (Root-Mean-Square).

Inductor

$$v(t) = L \frac{di(t)}{dt}$$

$$\bar{V} = Lj\omega \bar{I}$$

$$V_m e^{j\theta_v} = L e^{j\frac{\pi}{2}} \omega I_m e^{j\theta_i}$$

$$\theta_v = \frac{\pi}{2} + \theta_i \Rightarrow \theta_v - \theta_i = \frac{\pi}{2}$$

$$\langle P(t) \rangle = \langle v(t)\, i(t) \rangle = \frac{1}{2} V_m I_m \cos\left(\frac{\pi}{2}\right) = 0.$$

Capacitor

$$i(t) = C \frac{dv(t)}{dt}$$

$$\bar{I} = Cj\omega \bar{V}$$

$$I_m e^{j\theta_i} = C e^{j\frac{\pi}{2}} \omega V_m e^{j\theta_v}$$

$$\theta_i = \frac{\pi}{2} + \theta_v \Rightarrow \theta_v - \theta_i = -\frac{\pi}{2}$$

$$\langle P(t) \rangle = \langle v(t)\, i(t) \rangle = \frac{1}{2} V_m I_m \cos\left(-\frac{\pi}{2}\right) = 0.$$

Therefore the average power dissapated in an inductor or capacitor is zero. Note that the instantaneous power $p(t)$ is not in general zero, however.

11. A voltage source is connected to resistor R as shown in Fig. 1. The source produces the voltage waveform shown if Fig. 2.

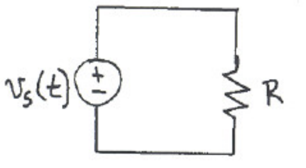

Figure 1.

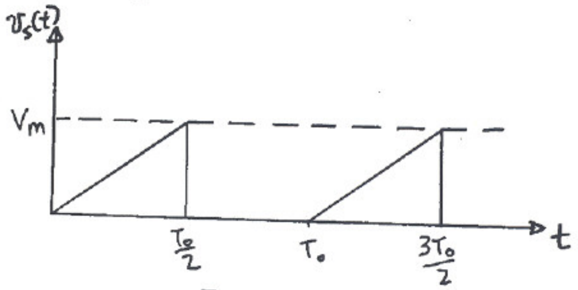

Figure 2.

a. Find the instantaneous power delivered to R.

b. What is the average power delivered to R?

$$v_s(t) = \begin{cases} \dfrac{2V_m}{T_0}\, t , & 0 \le t \le \dfrac{T_0}{2} \\ 0 , & \dfrac{T_0}{2} < t < T_0 \end{cases}$$

$$v_s(t + T_0) = v_s(t).$$

a) $$p(t) = \frac{v^2(t)}{R} = \begin{cases} \dfrac{1}{R}\left(\dfrac{2V_m}{T_0}\, t\right)^2 , & 0 \le t \le \dfrac{T_0}{2} \\ 0 , & \dfrac{T_0}{2} < t < T_0 \end{cases}$$

$$p(t+T_0) = p(t)$$

b) $$P_{ave} = \langle p(t) \rangle = \frac{1}{T_0}\int_0^{T_0} p(t)\, dt$$

$$= \frac{1}{T_0}\int_0^{\frac{T_0}{2}} \frac{1}{R}\left(\frac{2V_m}{T_0}\, t\right)^2 dt$$

$$= \frac{1}{R}\,\frac{1}{T_0^3}\, 4V_m^2\, \frac{1}{3} t^3 \Big|_0^{\frac{T_0}{2}}$$

$$= \frac{4V_m^2}{3R\cancel{T_0}}\,\frac{\cancel{T_0^3}}{8} = \boxed{\frac{V_m^2}{6R}}.$$

12. Show that the RMS (or effective value) for a sinusoidal voltage $v(t) = V_m \cos(\omega t + \theta)$ is $\frac{V_m}{\sqrt{2}}$.

The RMS value is defined by

$$V_{RMS} = \sqrt{\frac{1}{T_o}\int_0^{T_o} v^2(t)\,dt}$$

$$= \sqrt{\frac{1}{T_o}\int_0^{T_o} V_m^2 \cos^2(\omega t + \theta)\,dt}$$

$$= \sqrt{\frac{1}{T_o}\int_0^{T_o} \left\{\frac{V_m^2}{2} + \frac{V_m^2}{2}\cos[2(\omega t + \theta)]\right\} dt} \qquad \left(\text{since } \cos^2 x = \frac{1}{2}(1 + \cos 2x)\right)$$

$$= \sqrt{\frac{1}{T_o}\left[\int_0^{T_o} \frac{V_m^2}{2}\,dt + \cancel{\int_0^{T_o} \frac{V_m^2}{2}\cos[2(\omega t + \theta)]\,dt}^{\,0}\right]}$$

$$= \sqrt{\frac{1}{T_o}\frac{V_m^2}{2}\int_0^{T_o} dt}$$

$$= \sqrt{\frac{1}{\cancel{T_o}}\frac{V_m^2}{2}\cancel{T_o}}$$

$$\boxed{V_{RMS} = \frac{V_m}{\sqrt{2}}}.$$