Q1: Starting with general TL equations in time-domain, derive the TL wave equation for current $i(z,t)$

Sol: TL general equations are given by

(1) $$\frac{\partial v(z,t)}{\partial z} = -R\, i(z,t) - L \frac{\partial}{\partial t} i(z,t) \quad \&$$

(2) $$\frac{\partial i(z,t)}{\partial z} = -C \frac{\partial}{\partial t} v(z,t) - G\, v(z,t)$$

let $i(z,t) \equiv i$ (2)

(3) $v(z,t) \equiv v$

& take $\frac{\partial}{\partial z}$ of (2) $\Rightarrow$ (4) $$\frac{\partial^2 i}{\partial z^2} = -C \frac{\partial}{\partial t}\frac{\partial}{\partial z} v - G \frac{\partial}{\partial z} v$$

(5) where $\frac{\partial}{\partial z}\frac{\partial}{\partial t} = \frac{\partial}{\partial t}\frac{\partial}{\partial z}$

Sub (1) in (4) $\Rightarrow$

(6) $$\frac{\partial^2 i}{\partial z^2} = -C \frac{\partial}{\partial t}\left[-Ri - L\frac{\partial i}{\partial t}\right] - G\left[-Ri - L\frac{\partial}{\partial t} i\right]$$

(7) $$\frac{\partial^2 i}{\partial z^2} = CR\frac{\partial}{\partial t} i + CL\frac{\partial^2 i}{\partial t^2} + GRi + GL\frac{\partial i}{\partial t} \Rightarrow$$

$$\boxed{\frac{\partial^2 i(z,t)}{\partial z^2} = LC\frac{\partial^2}{\partial t^2} i(z,t) + (GL + RC)\frac{\partial i(z,t)}{\partial t} + GR\, i(z,t)}$$

which is given in your notes.

Q2: Derive the TL wave equation for current in sinusoidal steady state form.

Sol: From previous question we have

① $$\frac{\partial^2 i(z,t)}{\partial z^2} = LC\frac{\partial^2}{\partial t^2} i(z,t) + (GL+RC)\frac{\partial}{\partial t} i(z,t) + GR\, i(z,t)$$

Let's use harmonic fields: ② $i(z,t) = Re[\bar{I}(z)e^{j\omega t}] = Re[I e^{j\phi(z)} e^{j\omega t}]$

then (1) ⇒ ③

$$\frac{\partial^2}{\partial z} Re[\bar{I}(z)e^{j\omega t}] = LC\frac{\partial}{\partial t^2} Re[\bar{I}(z)e^{j\omega t}] + (GL+RC)\frac{\partial}{\partial t} Re[\bar{I}(z)e^{j\omega t}] + RG\, Re[\bar{I}(z)e^{j\omega t}] \Rightarrow$$

④

$$Re[e^{j\omega t}\frac{\partial}{\partial z^2}\bar{I}(z)] = LC\, Re[\bar{I}(z)\frac{\partial}{\partial t^2}e^{j\omega t}] + (GL+RC)\, Re[\bar{I}(z)\frac{\partial}{\partial t}e^{j\omega t}] + RG\, Re[\bar{I}(z)e^{j\omega t}] \Rightarrow$$

⑤

$$Re[e^{j\omega t}\frac{d^2}{dz^2}\bar{I}(z)] = LC\, Re[-\omega^2\bar{I}(z)e^{j\omega t}] + (GL+RC)\, Re[\bar{I}(z)\, j\omega\, e^{j\omega t}] + RG\, Re[\bar{I}(z)e^{j\omega t}] \Rightarrow$$

⑥

$$Re[e^{j\omega t}\frac{d^2}{dz^2}\bar{I}(z)] = Re[-\omega^2 LC\,\bar{I}(z)e^{j\omega t} + (GL+RC)\bar{I}(z)j\omega e^{j\omega t} + RG\,\bar{I}(z)e^{j\omega t}] \Rightarrow$$

From properties of Phasor we have

(1) $\cancel{e^{j\omega t}} \frac{d^2}{dz^2}\bar{I}(z) = -\omega^2 LC\,\bar{I}(z)\cancel{e^{j\omega t}} + (GL+RG)\,\bar{I}(z)\cancel{e^{j\omega t}}\, j\omega +$

$RG\,\bar{I}(z)\cancel{e^{j\omega t}} \Rightarrow$

(2) $\frac{d^2}{dz^2}\bar{I}(z) = \left[-\omega^2 LC + (GL+RG)j\omega + RG\right]\bar{I}(z)$

Note that (3) $\gamma^2 = (G+j\omega C)(R+j\omega L) = RG - \omega^2 LC + j(\omega CR + \omega LG)$

hence (2) can be written as

$\frac{d^2}{dz^2}\bar{I}(z) = \gamma^2\,\bar{I}(z)$ as given in your notes.

Q3: Prove that solution to (1) $\frac{d^2\bar{V}(z)}{dz^2} = \gamma^2 \bar{V}(z)$ is

given by (2) $\bar{V}(z) = V_o^+ e^{-\gamma z} + V_o^- e^{+\gamma z}$

Sol: To prove that (2) is a solution let us calculate the $\frac{d\bar{V}(z)}{dz}$ & hence $\frac{d^2\bar{V}(z)}{dz^2}$

(3) $\frac{d}{dz}\bar{V}(z) = -\gamma V_o^+ e^{-\gamma z} + \gamma V_o^- e^{\gamma z}$ &

(4) $\frac{d^2}{dz^2}\bar{V}(z) = \gamma^2 V_o^+ e^{-\gamma z} + \gamma^2 V_o^- e^{\gamma z} = \gamma^2 [V_o^+ e^{-\gamma z} + V_o^- e^{\gamma z}]$

(4) $\Rightarrow$ $\frac{d^2}{dz^2}\bar{V}(z) = \gamma^2 \bar{V}(z)$ which the desired result

Q4: For a distortionless line, find the propagation Constant, attenuation Constant, Phase constant, Phase velocity & Characteristic impedance

Sol: For distortionless line ① $\frac{R}{G} = \frac{L}{G} \Rightarrow$ ② $\frac{R}{L} = \frac{G}{G}$

Recall ② $\gamma = \sqrt{(R+j\omega L)(G+j\omega C)} \Rightarrow$ ③ $\gamma = \sqrt{L(\frac{R}{L}+j\omega)C(\frac{G}{C}+j\omega)} \Rightarrow$

④ $\gamma = \sqrt{LC(\frac{R}{L}+j\omega)(\frac{R}{L}+j\omega)} = \sqrt{LC}\,(\frac{R}{L}+j\omega) = R\sqrt{\frac{C}{L}} + j\omega\sqrt{LC}$

then ⑤ $\boxed{\beta = \omega\sqrt{LC}}$, ⑥ $\boxed{\alpha = R\sqrt{\frac{C}{L}}}$, ⑦ $\boxed{V_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{LC}}}$

also $Z_0 = \sqrt{\frac{R+j\omega L}{G+j\omega C}}$ For distortionless line $Z_0 = \sqrt{\frac{L(\frac{R}{L}+j\omega)}{C(\frac{G}{C}+j\omega)}} = \sqrt{\frac{L}{C}}$

$\Rightarrow Z_0 = R_0 = jX_0 = \sqrt{\frac{L}{C}} \Rightarrow \quad R_0 = \sqrt{\frac{L}{C}}, \quad X_0 = 0$

* Note that aside a non-vanishing attenuation constant (α) the other Parameters of a distortionless line is the same as the lossless line (i.e β, V_p, R_0, & X_0 are the same.)

25: Figure shows two charged conductors immersed in a lossy dielectric of conductivity σ & dielectric constant ε.

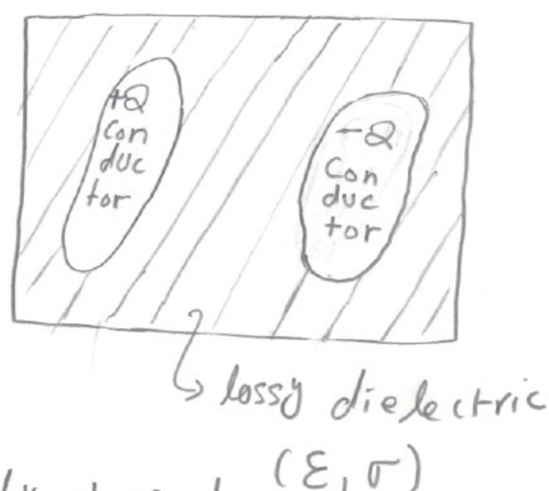
a) show that the following expression holds:

(1) $RC = \dfrac{C'}{G} = \dfrac{\varepsilon}{\sigma}$ where (2) $R = \dfrac{1}{G}$ is the resistance associated with the lossy dielectric, & C' the capacitance between the two conductors

b) From the result in part (a) find the leakage r[illegible] (conductance per unit length of a coaxial cable of inner radius a & outer radius b.

Sol:

a). The definition for capacitance between two conductor is given by



(3) $C = \dfrac{Q}{V} = \dfrac{\oiint_S \vec{D} \cdot \vec{ds}}{-\int_L \vec{E} \cdot \vec{dl}} =$

where $\oiint_S$ (surface integral) is carried over the surface enclosing the +tively charged conductor & $\int_L$ (line integral) is carried from -tively charged conductor (lower potential) to the +tively charged conductor (higher potential)

* When the dielectric between conductors is lossy (not a Perfect insulator) a current will flow from one conductor to the other. If medium is isotropic this current & electric field between the two conductors are co-linear since

$\vec{J} = \sigma \vec{E}$, where σ is the conductivity associated with our lossy (not perfect) dielectric. In other words, there will be a resistance between the two conductors, which is called the leakage resistance, its value is given by

(4) $$R = \frac{V}{I} = \frac{-\int_L \vec{E}\cdot d\vec{l}}{\oiint_{S'} \vec{J}\cdot d\vec{s}}$$

where $\int_L$ (line integral) & $\oiint_S$ (surface integral) are carried over the same Path & surface as before.

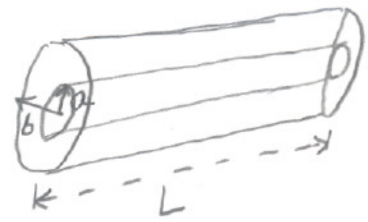
* multiply (4) & (3-P6) we have

(5) $$RC = \frac{\oiint_S \vec{D}\cdot d\vec{s}}{\cancel{-\int \vec{E}\cdot d\vec{l}}} \cdot \frac{\cancel{-\int_L \vec{E}\cdot d\vec{l}}}{\oiint_S \vec{J}\cdot d\vec{s}} = \frac{\oiint \varepsilon \vec{E}\cdot d\vec{s}}{\oiint \sigma \vec{E}\cdot d\vec{s}}$$ where (6) $\vec{D} = \varepsilon \vec{E}$ & (7) $\vec{J} = \sigma E$

If ε & σ are Position independent, i.e medium is homogeneous - then

(6) $\boxed{RC = \frac{\varepsilon}{\sigma}} \Rightarrow$ (7) $\boxed{\frac{C}{G} = \frac{\varepsilon}{\sigma}}$

b) For coaxial cable, we know from electrostatic $C' = \frac{2\pi\varepsilon L}{\ln(b/a)}$ ①



Where ε is the permitivity of medium between the two conductors.

Then capacitance per unit length $C = \frac{C'}{L}$ [F/m] is ②

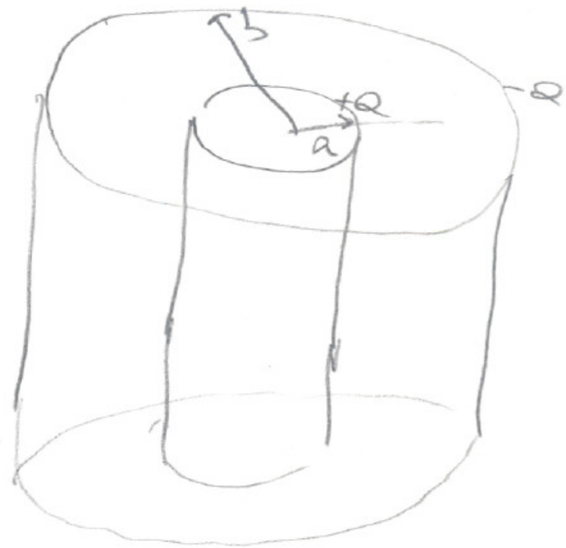
③ $C' = \frac{2\pi\varepsilon}{\ln(b/a)}$ [F/m]. From results in part (a) we have

④ $\frac{C}{G} = \frac{\varepsilon}{\sigma} \Rightarrow$ ⑤ $\frac{1}{G}\frac{2\pi\varepsilon}{\ln(b/a)} = \frac{\varepsilon}{\sigma} \Rightarrow$

$G = \frac{2\pi\sigma}{\ln(b/a)}$ $\left[\frac{1}{\Omega . m} = \frac{Sie}{m}\right]$

$$\oiint \vec{D}\cdot d\vec{s} = Q$$

$$\oiint \vec{E}\cdot d\vec{s} = Q/\varepsilon$$



$$E = E_\rho \hat{a_\rho}$$

$$d\vec{l} = d\rho\, \hat{a_\rho} + \rho\, d\phi\, \hat{a_\phi} + dz\, \hat{a_z}$$

ds here $= \rho\, d\phi\, dz\, \hat{a_\rho}$ then

$$\oiint \vec{E}\cdot d\vec{s} = \int_{\phi=0}^{2\pi}\int_{z=0}^{L} E_\rho \hat{a_\rho} \cdot \rho\, d\phi\, dz$$

$$= E_\rho\, \rho\, 2\pi L \Rightarrow \qquad E_\rho (\rho 2\pi L) = \frac{Q}{\varepsilon} \Rightarrow$$

$$\boxed{\vec{E} = \frac{Q}{2\pi\varepsilon L \rho}\hat{a_\rho}}$$

$$C' = Q/V$$

calculat V from lower to higher potential

$$E = -\int_{\rho=b}^{\rho=a} \vec{E}\cdot d\vec{l} = -\int_{\rho=b}^{\rho=a} \frac{Q\, \hat{a_\rho}}{2\pi\varepsilon L \rho}\cdot d\rho\, \hat{a_\rho}$$

$$= \frac{-Q}{2\pi\varepsilon L}\left[\ln \rho\right]_b^a = \frac{-Q}{2\pi\varepsilon L}\left[\ln(a) - \ln(b)\right]$$

$$= \frac{Q}{2\pi\varepsilon L}\left[\ln(b) - \ln(a)\right] = \frac{Q}{2\pi\varepsilon L}\ln\left(\frac{b}{a}\right)$$

$$C' = \frac{Q}{V} = \frac{\cancel{Q}\cdot 2\pi\varepsilon L}{\cancel{Q}\ln(b/a)} = \frac{2\pi\varepsilon L}{\ln(b/a)}$$

Q6: Figure shows a RG58C/U coaxial cable, idealized to solid outer conductor. Let us assume that cable is a low loss (high frequency limit) TL with characteristic impedance of $Z_0 = 50\,\Omega$.

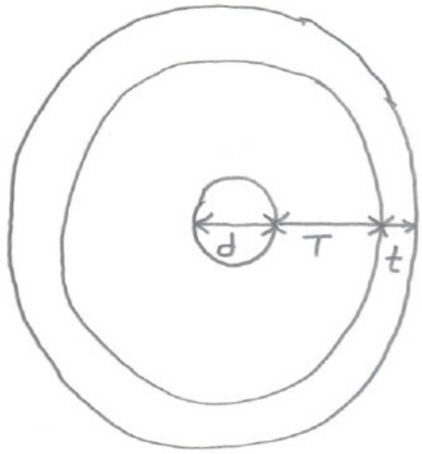
a) What is the TL capacitance?

b) What is the TL inductance?

c) What is the value of the TL resistance?

d) What is the TL conductance?

e) Find a frequency range for which the high frequency assumption used is valid.



Inner conductor: copper
$d = 0.9\,mm$
resistivity $= \rho_c = 1.7\times10^{-8}$ $[\Omega.m]$

Dielectric: Polyethylene
$T = 1.02\,mm$
resistivity $= \rho_d = 10^{14}$ $[\Omega.m]$
$\varepsilon_r = 2.3$

outer conductor: copper
$t = 1.\,mm$
resistivity $= \rho_c = 1.7\times10^{-8}$ $[\Omega.m]$

Sol:

a) From question (5) for coaxial cable we found

$$C' = \frac{2\pi\varepsilon}{\ln\left(\frac{T+d/2}{d/2}\right)} = \frac{2\pi\times8.85\times10^{-12}\times2.3}{\ln\left(\frac{1.02+0.9/2}{0.9/2}\right)} \Rightarrow$$

$$\boxed{C' = 1.08\times10^{-10} = 108.04\,[PF/m]}$$

b) For lowloss line $Z_0 = R_0 + jX_0 = \sqrt{\frac{L}{C}} + 0j$

$$\Rightarrow (Z_0)^2 = (R_0)^2 = \frac{L}{C} \Rightarrow \quad L = Z_0^2 C' = (50)^2 \times 108.04\times10^{-12} \Rightarrow$$

$$\boxed{L = 2.701\times10^{-7} = 270.1\,[nH/m]}$$

c) For a metalic rod of length L & area A we have

$$r = \rho \frac{L}{A} \Rightarrow R = \frac{r}{L} = \frac{\rho}{A}$$ where ρ is the rod resistivity

∴ For inner conductor

$$R_{in} = \frac{\rho_c}{\pi\left(\frac{d}{2}\right)^2} = \frac{1.7\times10^{-8}}{\pi\left(0.9\times10^{-3}/2\right)^2} = 0.0267\ [\Omega/m]$$

∴ For outer conductor

$$R_{out} = \frac{\rho_c}{\pi\left[\frac{d}{2}+T+t\right]^2 - \pi\left[\frac{d}{2}+T\right]^2} =$$

$$\frac{1.7\times10^{-8}}{\pi\left[\frac{0.9}{2}+1.02+1\right]^2\times10^{-6} - \pi\left[\frac{0.9}{2}+1.02\right]^2\times10^{-6}} = 0.0014\ [\Omega/m]$$

Then $\boxed{R_{total} = R_{in} + R_{out} = 0.0267 + 0.0014 = 0.0281\ \Omega/m}$

d) From previous Qustion $\frac{C}{G} = \frac{\varepsilon}{\sigma} \Rightarrow \boxed{G = \frac{C\sigma}{\varepsilon}}$ where

$\sigma = \frac{1}{\rho}$ this is the resistivity of our lossy dielectric

$$G = 108.04\times10^{-12} \times \frac{1}{10^{14}\times 8.85\times10^{-12}\times 2.3}$$

$$\boxed{G = 5.308\times10^{-14}\ \left[\frac{sie}{m}\right]}$$

e) In deriving the expressions for α, β, R_o & X_o in the case of low loss (high frequency limit) we have assumed

① $R << \omega L$ & ② $G << \omega C$. For the numbers we have calculated

we have (1) $\Rightarrow$ $0.0281 << \omega \times 270.1 \times 10^{-9}$ $\Rightarrow$ ③ $1.04 \times 10^{5} << \omega$

(2) $\Rightarrow$ $5.308 \times 10^{-14} << \omega \times 108.04 \times 10^{-12}$ $\Rightarrow$ ④ $0.0005 << \omega$

we note that at RF & microwave frequencies, the condition $\omega >> 0.005$ Hz is easily satisfied. As for the condition $\omega >> 1.04 \times 10^{5}$ we take $>>$ to mean at least 10 times & hence $\omega > 1.04 \times 10^{6}$ $\Rightarrow$

$\nu > 1.655 \times 10^{5} = 0.1655$ MHz should be o.k.

Q 7; The attenuation constant of a 50[Ω] distortionless line is 0.01[dB/m] & its capacitance is 0.1 [nF/m]. 12

a) What are the values of the line resistance, conductance & inductance per meter

b) What is the value of the phase velocity of the propagating waves.

c) What are the ratios of the magnitude of the propagating fields at 1 km & 5 km with respect to the starting point at $Z=0$

Sol: For a distortionless line ① $\frac{R}{L}=\frac{G}{G'}$ & ② $Z_0=R_0+jX_0 = \sqrt{\frac{L}{C}}+0j$ &

③ $\gamma=\alpha+j\beta=\sqrt{\frac{C}{L}}\,(R+j\omega L)$

a) From (2) $\boxed{L=(Z_0)^2G'=(50)^2\times 0.1\times 10^{-9}=2.5\times10^{-7}\ [H/m]}$

from (3) $\alpha=\sqrt{\frac{C}{L}}R \Rightarrow R=\alpha\sqrt{\frac{L}{C}}=\frac{0.01}{8.686}\times\sqrt{\frac{2.5\times10^{-7}}{0.1\times10^{-9}}} \Rightarrow$

$\boxed{R=0.0576\ [\Omega/m]}$

from (1) $\Rightarrow$ $G=\frac{G'R}{L}=\frac{0.1\times10^{-9}\times0.0576}{2.5\times10^{-7}}=2.303\times10^{-5}\ [\frac{sie}{m}]$

$\simeq 23.02\ [\frac{\mu sie}{m}]$

b) $V_p = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{2.5\times10^{-7} \times 0.1\times10^{-9}}} = 2\times10^{8}$ m/s

$= 0.667\,C$

where C is the speed of light in vacuum.

c) $\frac{V(Z=1000)}{V(Z=0)} = \frac{|V_0^+ e^{-\alpha Z} e^{-j\beta Z}|}{|V_0^+ e^{\alpha 0} e^{j0}|} = e^{-\alpha Z}$

$\alpha = \frac{0.01}{8.686} = 1.151\times10^{-3}$ [NP/m]

then $\frac{V(Z=1000m)}{V(Z=0)} = e^{-1.15\times10^{-3}\times10^{+3}} = e^{-1.15} = 0.316 =$ 32%

$\frac{V(Z=5000m)}{V(Z=0)} = e^{-1.15\times10^{-3}\times5\times10^{3}} = e^{-5.756} = 3.162\times10^{-3}$

$= 0.32$ %

Q8: Use the relation ① $\alpha = \frac{P_{dis}}{2P(z)}$ to find the attenuation constant for a low loss line (high frequency limit).

Sol: In notes we showed that average power is given by

② $P(z) = \frac{1}{2} \frac{|V_0|^2}{|Z_0|^2} R_0 e^{-2\alpha z}$

* we also note that time-averaged dissipated power is given by

③ $P_{diss} = \frac{1}{2} Re\left[|I(z)|^2 R + |V(z)|^2 G \right]$

$= \frac{1}{2} \left[|I(z)|^2 R + |V(z)|^2 G \right]$ since R & G are real.

* with ④ $V(z) = V_0 e^{-\alpha z} e^{-j\beta z} \Rightarrow$ ⑥ $|V(z)|^2 = |V_0|^2 e^{-2\alpha z}$

⑤ $I(z) = \frac{V_0}{Z_0} e^{-\alpha z} e^{-j\beta z} \Rightarrow$ ⑦ $|I(z)|^2 = \frac{|V_0|^2}{|Z_0|^2} e^{-2\alpha z}$

Then ⑥ $P_{diss} = \frac{1}{2} \left[\frac{|V_0|^2}{|Z_0|^2} e^{-2\alpha z} R + |V_0|^2 e^{-2\alpha z} G \right]$

$= \frac{1}{2} \frac{|V_0|^2}{|Z_0|^2} e^{-2\alpha z} \left[R + |Z_0|^2 G \right]$

Use (6) & (2) in (1) $\Rightarrow$

$$\alpha = \frac{\cancel{\frac{1}{2}} \frac{\cancel{|V_0|^2}}{\cancel{|Z_0|^2}} \cancel{e^{-2\alpha z}} \left[R + |Z_0|^2 G \right]}{2 \times \cancel{\frac{1}{2}} \frac{\cancel{|V_0|^2}}{\cancel{|Z_0|^2}} \cancel{e^{-2\alpha z}} R_0} = \frac{1}{2} \frac{\left[R + |Z_0|^2 G \right]}{R_0}$$

For low loss line $Z_0 = R_0 + j X_0 = \sqrt{\frac{L}{C}} + j0 \Rightarrow$

$Z_0 = R_0 = \sqrt{\frac{L}{C}}$ then

$$\alpha = \frac{1}{2} \frac{[R + \frac{L}{C} G]}{\sqrt{L/C}} = \frac{1}{2} \frac{\sqrt{C}}{\sqrt{L}} \left[R + G \frac{L}{C} \right] \Rightarrow$$

$$\boxed{\alpha = \frac{1}{2} \left[R \sqrt{\frac{C}{L}} + G \sqrt{\frac{L}{C}} \right]}$$ as befor