HW #5

Q1) A vector field $\vec{A} = \hat{a}_\rho (3\cos\phi) - \hat{a}_\phi 2\rho + 5\hat{a}_z$ is expressed in cylindrical coordinates. Find its representation in rectangular Coordinate system.

Sol:

① $$\begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A_\rho \\ A_\phi \\ A_z \end{pmatrix} \Rightarrow$$

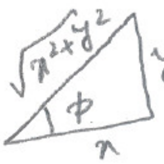
② $A_x = \cos\phi A_\rho - \sin\phi A_\phi = \cos\phi(3\cos\phi) - \sin\phi(-2\rho) = 3\cos^2\phi + 2\rho\sin\phi$

③ $A_y = \sin\phi A_\rho + \cos\phi A_\phi = \sin\phi(3\cos\phi) + \cos\phi(-2\rho) = 3\cos\phi\sin\phi - 2\rho\cos\phi$

④ $A_z = A_z = 5$

* Now we need to know how to convert th $\cos\phi$, $\sin\phi$ & ρ to rectangular coordinates. Recall

$x = \rho\cos\phi$, $y = \rho\sin\phi$ $\Leftrightarrow$ $x^2 + y^2 = \rho^2$, $\tan\phi = y/x$ $\Leftrightarrow$



$\Leftrightarrow$ $\cos\phi = \dfrac{x}{\sqrt{x^2+y^2}}$, $\sin\phi = \dfrac{y}{\sqrt{x^2+y^2}}$, $\rho = \sqrt{x^2+y^2}$

$$A_x = 3\cos^2\phi + 2\rho\sin\phi = 3\frac{x^2}{x^2+y^2} + 2\sqrt{x^2+y^2}\,\frac{y}{\sqrt{x^2+y^2}} \Rightarrow$$

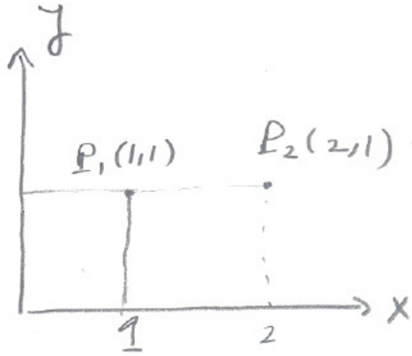
$$\boxed{A_x = \frac{3x^2}{x^2+y^2} + 2y}$$

$$A_y = 3\cos\phi\sin\phi - 2\rho\cos\phi = 3\frac{x}{\sqrt{x^2+y^2}}\,\frac{y}{\sqrt{x^2+y^2}} - 2\sqrt{x^2+y^2}\,\frac{x}{\sqrt{x^2+y^2}} \Rightarrow$$

$\boxed{A_y = \dfrac{3xy}{x^2+y^2} - 2x}$ & $\boxed{A_z = 5}$ then

$$\boxed{\vec{A} = A_x\hat{a}_x + A_y\hat{a}_y + A_z\hat{a}_z = \left(\frac{3x^2}{x^2+y^2} + 2y\right)\hat{a}_x + \left(\frac{3xy}{x^2+y^2} - 2x\right)\hat{a}_y + 5\hat{a}_z}$$

Q2: Consider the centeral force $\vec{F} = \frac{1}{r^2}\hat{a}_r$. calculate the work done in presence of the above f-ield in moving from $P_1(1,1)$ to $P_2(2,1)$ along the line P_1P_2



Sol: we convert the $\vec{F} = \frac{1}{r^2}\hat{a}_r$ to rectangular coordinates.

$$\begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{pmatrix} \begin{pmatrix} F_r \\ F_\theta = 0 \\ F_\phi = 0 \end{pmatrix} \Rightarrow$$

$F_x = \sin\theta\cos\phi\, F_r$
$F_y = \sin\theta\sin\phi\, F_r$
$F_z = \cos\theta\, F_r$

here since we are only in two-dimensions, i.e we are in x-y plane for which $\theta = \pi/2$ we have

$F_x = \cos\phi\, F_r = \cos\phi \frac{1}{r^2}$
$F_y = \sin\phi\, F_r = \sin\phi \frac{1}{r^2}$
$F_z = 0$

* The relation between (x,y,z) & (r,θ,ϕ) in 3-D is given by

$x = r\sin\theta\cos\phi$
$y = r\sin\theta\sin\phi$
$z = r\cos\theta$

In 2D (X-y plane) $\theta = \pi/2 \Rightarrow$

$x = r\cos\phi$
$y = r\sin\phi$
$z = 0$

$\Leftrightarrow$ $x^2 + y^2 = r^2$

$\cos\phi = \frac{x}{r} = \frac{x}{\sqrt{x^2+y^2}}$

$\sin\phi = \frac{y}{r} = \frac{y}{\sqrt{x^2+y^2}}$

Then

$$F_x = \cos\phi \frac{1}{r^2} = \frac{x}{\sqrt{x^2+y^2}} \frac{1}{x^2+y^2} = \frac{x}{(x^2+y^2)^{3/2}}$$

$$F_y = \sin\phi \frac{1}{r^2} = \frac{y}{\sqrt{x^2+y^2}} \frac{1}{x^2+y^2} = \frac{y}{(x^2+y^2)^{3/2}}$$

$$\boxed{\vec{F} = F_x \hat{a_x} + F_y \hat{a_y} = \frac{1}{(x^2+y^2)} \left[\frac{x}{\sqrt{x^2+y^2}} \hat{a_x} + \frac{y}{\sqrt{x^2+y^2}} \hat{a_y} \right]}$$

$$W = \int \vec{F} \cdot d\vec{l} = \int (F_x \hat{a_x} + F_y \hat{a_y}) \cdot (dx\, \hat{a_x} + dy\, \hat{a_y})$$

$$= \int F_x\, dx + F_y\, dy$$

but in going from P_1 to P_2 we move along the line $y = 1 \Rightarrow dy = 0 \Rightarrow$

$$W = \int F_x\, dx = \int_{x=1}^{x=2} \frac{x}{(x^2+y^2)^{3/2}} dx$$

we need to express y in terms of x. Note that in going from P_1 to P_2 y is constant & equal to $1 \Rightarrow$

$$W = \int_{x=1}^{x=2} \frac{x\, dx}{(x^2+1)^{3/2}}$$

let $u = x^2+1$
$du = 2x\, dx \Rightarrow \frac{1}{2} du = x\, dx \Rightarrow$

$$W = \int \frac{1}{2} \frac{du}{u^{3/2}} = \frac{1}{2} \left(\frac{u^{-1/2}}{-1/2} \right) =$$

$$-\left(\frac{1}{\sqrt{x^2+1}} \right) \Bigg|_{x=1}^{x=2} = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{5}}$$

Q3.

a) Let $\vec{r}$ be the position vector, pointing from origion to the point of observation P. show that

$$\nabla|\vec{r}| = \nabla r = \frac{\vec{r}}{|\vec{r}|} = \frac{\vec{r}}{r} = \hat{a}_r$$ where $\hat{a}_r$ is the unit normal

in the direction of $\vec{r}$.

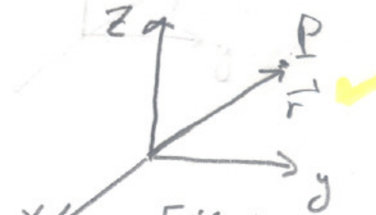


Fig.1

b) let function $g(r)$ depend only on the distance between the source points (Primed Coordinats) & the observation point (Un primed Coordinates), see Fig 2.

prove that $\nabla g(r) = \nabla f(R) = \hat{a}_R \dfrac{\partial f(R)}{\partial R}$. use this information to calculate the gradiant of $g(x,y,z) = f(R) = \sin\left(\pi \dfrac{R^2}{4}\right)$

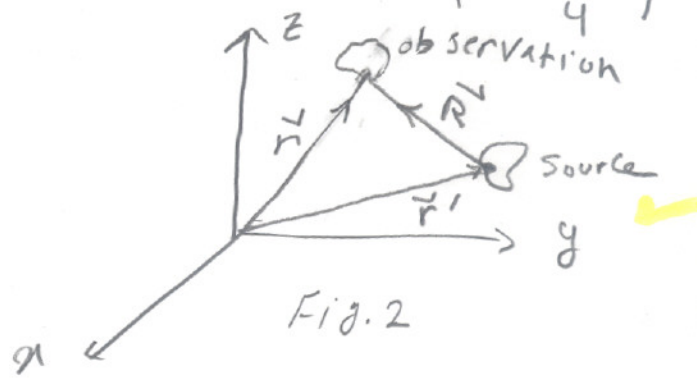


Fig.2

Sol:

a) $\vec{r} = x\hat{a}_x + y\hat{a}_y + z\hat{a}_z$

$|\vec{r}| = r = [x^2+y^2+z^2]^{1/2}$

$$\nabla|\vec{r}| = \nabla r = \frac{2x}{2\sqrt{x^2+y^2+z^2}}\hat{a}_x + \frac{2y}{2\sqrt{x^2+y^2+z^2}}\hat{a}_y + \frac{2z}{2\sqrt{x^2+y^2+z^2}}\hat{a}_z$$

$$= \frac{x\hat{a}_x + y\hat{a}_y + z\hat{a}_z}{\sqrt{x^2+y^2+z^2}} = \frac{\vec{r}}{|\vec{r}|} = \hat{a}_r$$

b) We note the following

$$\vec{R} = (x-x')\hat{a}_x + (y-y')\hat{a}_y + (z-z')\hat{a}_z \quad \&$$

$$R = |\vec{R}| = \left[(x-x')^2 + (y-y')^2 + (z-z')^2\right]^{1/2} \quad \text{then}$$

$$\nabla|\vec{R}| = \nabla R = \frac{(x-x')\hat{a}_x + (y-y')\hat{a}_y + (z-z')\hat{a}_z}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}}$$

$$= \frac{\vec{R}}{|\vec{R}|} = \hat{a}_R$$

Also from chain rule for differentiation

$$\nabla g(r) = \nabla f(R) = \frac{\partial f(R)}{\partial R}\nabla R = \frac{\partial f(R)}{\partial R}\hat{a}_R$$

Using the above to calculate

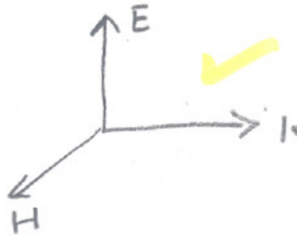
$$\nabla f(R) = \nabla\left[\sin\left(\pi \frac{R^2}{4}\right)\right] = \frac{\partial}{\partial R}\sin\left(\pi\frac{R^2}{4}\right)\hat{a}_R$$

$$= \frac{2\pi R}{4}\cos\left(\pi\frac{R^2}{4}\right)\hat{a}_R = \boxed{\frac{\pi R}{2}\cos\left(\pi\frac{R^2}{4}\right)\hat{a}_R}$$

Q4: Consider a plane wave $\vec{E} = \vec{E}_0 e^{-j\vec{k}\cdot\vec{r}+j\omega t}$ propagating in a homogeneous, lossless, source free region for which $\varepsilon > 0$ & $\mu > 0$ & $\vec{E}_0$ is constant.

a) show that $\vec{k} \perp \vec{E}$ & $\vec{k} \perp \vec{H}$

b) Show that $\vec{k}$, $\vec{E}$ & $\vec{H}$ form a right hand triplet (Hint $\vec{k} \times \vec{E} = \omega\mu\vec{H}$ & $\vec{k} \times \vec{H} = -\varepsilon\omega\vec{E}$



Sol: since source free (1) $\nabla\cdot\vec{E} = 0$ &

(2) $\nabla\cdot\vec{B} = 0$

From vector calculus (3) $\nabla\cdot(f\vec{F}) = (\nabla\cdot\vec{F})f + \nabla f\cdot\vec{F}$

then (4) $\nabla\cdot(\vec{E}_0 e^{-j\vec{k}\cdot\vec{r}+j\omega t}) = (\nabla\cdot\vec{E}_0)e^{-j\vec{k}\cdot\vec{r}+j\omega t} + \nabla(e^{-j\vec{k}\cdot\vec{r}+j\omega t})\cdot\vec{E}_0$

since $\vec{E}_0$ is constant (5) $\nabla\cdot\vec{E}_0 = 0 \Rightarrow$

(6) $\nabla\cdot(\vec{E}_0 e^{-j\vec{k}\cdot\vec{r}+j\omega t}) = \nabla(e^{-j\vec{k}\cdot\vec{r}+j\omega t})\cdot\vec{E}_0 =$

$-j\vec{k}\, e^{-j\vec{k}\cdot\vec{r}+j\omega t}\cdot\vec{E}_0 = -j(\vec{k}\cdot\vec{E}_0)e^{-j\vec{k}\cdot\vec{r}+j\omega t}$

but from (1) we must have

(7) $-j(\vec{k}\cdot\vec{E}_0)e^{-j\vec{k}\cdot\vec{r}+j\omega t} = 0 \Rightarrow$ (8) $\boxed{\vec{k}\cdot\vec{E}_0 = 0 \Rightarrow \vec{k}\perp\vec{E}_0 \Rightarrow \vec{k}\perp\vec{E}}$

* using equation (2) & similar argument, i.e

(9) $\nabla\cdot\vec{B} = 0 \Rightarrow \nabla\cdot\vec{H} = 0 \Rightarrow \nabla\cdot(\vec{H}_0 e^{-j\vec{k}\cdot\vec{r}+j\omega t}) = 0 \Rightarrow$

$\vec{k}\cdot\vec{H}_0 = 0 \Rightarrow \vec{k}\perp\vec{H}_0 \Rightarrow \boxed{\vec{k}\perp\vec{H}}$ (10)

b) since source free & simple medium (12) $\underline{\underline{7}}$

(11) $\nabla \times \vec{E} = -\frac{\partial B}{\partial t} \xrightarrow{\text{Time Harmonics}} \boxed{\nabla \times \vec{E} = -j\omega B = -j\omega\mu \vec{H}}$

(13) $\nabla \times \vec{H} = +\frac{\partial D}{\partial t} \xrightarrow{\text{Time Harmonics}} \boxed{\nabla \times \vec{H} = j\omega \vec{D} = j\omega \varepsilon \vec{E}}$ (14)

Let us calculate (15) $\nabla \times \vec{E} = \nabla \times (\vec{E_0}\, e^{-j\vec{K}\cdot\vec{r}+j\omega t})$

The following identity holds

(16) $\nabla \times (f\vec{F}) = \nabla f \times \vec{F} + f\, \nabla \times \vec{F}$

using (16) in (15)

$$\nabla \times \vec{E} = \nabla \times (\vec{E_0}\, e^{-j\vec{K}\cdot\vec{r}+j\omega t}) =$$

$$(\nabla e^{-j\vec{K}\cdot\vec{r}+j\omega t}) \times \vec{E_0} + e^{-j\vec{K}\cdot\vec{r}+j\omega t} \cancel{\nabla \times \vec{E_0}}^{\,0}$$

$$= -j\vec{K} e^{-j\vec{K}\cdot\vec{r}+j\omega t} \times \vec{E_0} = -j\,\vec{K} \times \vec{E_0}\, e^{-j\vec{K}\cdot\vec{r}+j\omega t} \Rightarrow$$

$\boxed{\nabla \times E = -j\vec{K} \times \vec{E}}$ (17)

sub (17) in (12) $\Rightarrow$ $\cancel{-j}\,\vec{K} \times \vec{E} = \cancel{-j}\,\omega\mu \vec{H} \Rightarrow$

(18) $\boxed{\vec{K} \times \vec{E} = \omega\mu\vec{H}}$

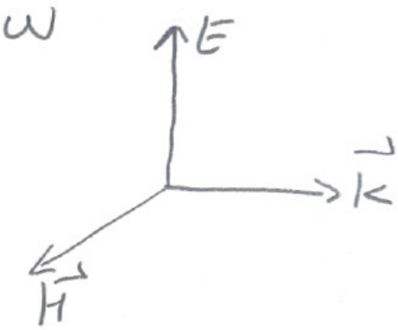
* We now use $\vec{H} = \vec{H_0}\, e^{-j\vec{K}\cdot\vec{r}+j\omega t}$ to calculate $\nabla \times \vec{H}$ & use the result in (14) to obtain the following

(19) $-j\vec{K}\times\vec{H} = j\omega\varepsilon\vec{E} \Rightarrow$ (20) $\boxed{\vec{K}\times\vec{H} = -\omega\varepsilon\vec{E}}$

To summarize we have shown

$\vec{K}\perp\vec{E}$, $K\perp\vec{H}$ &, $\vec{K}\times\vec{E} = \omega\mu\vec{H}$ & $\vec{K}\times\vec{H} = -\omega\varepsilon\vec{E}$

from these we see that $\vec{K}$, $\vec{E}$ & $\vec{H}$ Form a Right Hand triplet as shown below



Q5: Starting with differential form of the Maxwell's equations, find their integral form formulation.

sol:

* $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ Faraday's law

$$\iint_S \nabla \times \vec{E} \cdot \vec{ds} = -\frac{\partial}{\partial t} \iint_S \vec{B} \cdot \vec{ds}$$ use stokes's theorem

$$\boxed{\oint_C \vec{E} \cdot \vec{dl} = -\frac{\partial}{\partial t} \Phi_B}$$ where $\Phi_B = \iint_S \vec{B} \cdot ds$

* $\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{J}$ Ampere's law. use stokes's theorem

$$\iint_S \nabla \times \vec{H} \cdot ds = \frac{\partial}{\partial t} \iint_S \vec{D} \cdot \vec{ds} + \iint_{S'} \vec{J} \cdot \vec{ds} \Rightarrow$$

$$\boxed{\oint \vec{H} \cdot \vec{dl} = \frac{\partial}{\partial t} \iint_{S'} \vec{D} \cdot \vec{ds} + I}$$ where $I = \iint_S \vec{J} \cdot \vec{ds}$

* $\nabla \cdot \vec{D} = \rho_e$ Gauss law. use divergence theorem

$$\iiint_V \nabla \cdot \vec{D} \, dv = \iiint \rho_e \, dv \Rightarrow$$

$$\boxed{\oiint_S \vec{D} \cdot \vec{ds} = Q_{enc}}$$ where $Q_{en} = \iiint \rho_e \, dv$

* $\nabla \cdot \vec{B} = 0$ use divergence theorem

$$\iiint_V \nabla \cdot \vec{B} \, dv = 0 \Rightarrow \boxed{\oiint_S \vec{B} \cdot \vec{ds} = 0}$$