HW #7

Q1: prove that electric field given as —

$$\vec{E} = \hat{a}_x E_{10} \sin(\omega t - kz) + \hat{a}_y E_{20} \sin(\omega t - kz + \psi)$$

where E_{10}, E_{20} & ψ are arbitrary numbers is elliptically Polarized

sol: (1) let $\omega t - kz = \alpha$ then

(2) $\vec{E} = \hat{a}_x E_{10} \sin\alpha + \hat{a}_y E_{20} \sin(\alpha + \psi) = \hat{a}_x E_x + \hat{a}_y E_y \Rightarrow$

$E_x = E_{10} \sin\alpha \Rightarrow \boxed{\sin\alpha = \dfrac{E_x}{E_{10}}}$ (3) &

(4) $E_{20} \sin(\alpha + \psi) = E_y \Rightarrow$

(5) $\boxed{E_{20}\left[\sin\alpha \cos\psi + \cos\alpha \sin\psi\right] = E_y}$

use (3) in (5) $\Rightarrow$ $E_{20}\left[\dfrac{E_x}{E_{10}} \cos\psi + \sqrt{1 - \sin^2\alpha}\, \sin\psi\right] = E_y \Rightarrow$

(6) $E_{20}\left[\dfrac{E_x}{E_{10}} \cos\psi + \sqrt{1 - \left(\dfrac{E_x}{E_{10}}\right)^2}\, \sin\psi\right] = E_y$ where we again used (3)

(6) $\Rightarrow$ $\left[1 - \left(\dfrac{E_x}{E_{10}}\right)^2\right] \sin^2\psi = \left(\dfrac{E_y}{E_{20}} - \dfrac{E_x}{E_{10}} \cos\psi\right)^2 \Rightarrow$

$\left[1 - \left(\dfrac{E_x}{E_{10}}\right)^2\right] \sin^2\psi = \left(\dfrac{E_y}{E_{20}}\right)^2 + \left(\dfrac{E_x}{E_{10}} \cos\psi\right)^2 - \dfrac{2 E_x E_y}{E_{10} E_{20}} \cos\psi \Rightarrow$

(7) $1 - \left(\dfrac{E_x}{E_{10}}\right)^2 = \left(\dfrac{E_y}{E_{20} \sin\psi}\right)^2 + \left(\dfrac{E_x}{E_{10}} \dfrac{\cos\psi}{\sin\psi}\right)^2 - \dfrac{2 E_x E_y}{E_{10} E_{20}} \dfrac{\cos\psi}{\sin^2\psi}$

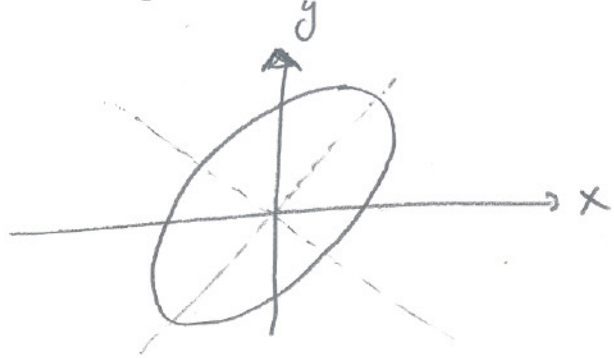
then (7) can further be simplified :-

$$1 = \left(\frac{E_y}{E_{20}\sin\varphi}\right)^2 + \left(\frac{E_x}{E_{10}}\right)^2\left[1 + \frac{\cos^2\varphi}{\sin^2\varphi}\right] - \frac{2E_xE_y}{E_{10}E_{20}}\frac{\cos\varphi}{\sin^2\varphi} \Rightarrow$$

(8)

$$\boxed{1 = \left(\frac{E_y}{E_{20}\sin\varphi}\right)^2 + \left(\frac{E_x}{E_{10}\sin\varphi}\right)^2 - \frac{2E_xE_y}{E_{10}E_{20}}\frac{\cos\varphi}{\sin^2\varphi}}$$

Eq(8) is an equation for ellipse.



Q2: Assume that the $z=0$ plane separates two loss less dielectric regions (no free surface charge or current densities) with $\varepsilon_{r1}=2$ & $\varepsilon_{r2}=3$. If we know that $\vec{E}_1$ in region 1 in phasor form is given by $\vec{E}_1 = [\hat{a}_x 2y - \hat{a}_y 3x + \hat{a}_z (5+z)] e^{j\omega t}$, what are $\vec{E}_2$ & $\vec{D}_2$ at the interface

Sol: The B.C. between two perfect dielectrics are given by

① $\vec{E}_{1t} = \vec{E}_{2t}$ &

② $D_{1n} = D_{2n}$

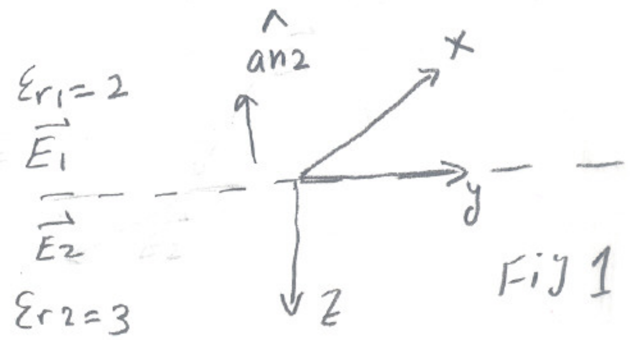


Fig 1

* From the fig & problem description it is clear that

③ $\vec{E}_{1t} = 2y\,\hat{a}_x - 3x\,\hat{a}_y$ & ④ $\vec{E}_{1n} = (5+z)\,\hat{a}_z$

From (1) & (3) we have

⑤ $2y\,\hat{a}_x - 3x\,\hat{a}_y = E_{2x}\hat{a}_x + E_{2y}\hat{a}_y \Rightarrow$ ⑥ $\boxed{E_{2x} = 2y}$ ⑦ $\boxed{E_{2y} = -3x}$

* From (2) at the interface ($z=0$) $D_{1n} = D_{2n} \Rightarrow \varepsilon_1 E_{1n} = \varepsilon_2 E_{2n}$ & note that E_{1n} & E_{2n} are along $\hat{a}_z$ then

⑧ $\left.\frac{\varepsilon_1}{\varepsilon_2} E_{1n}\right|_{z=0} = \left. E_{2n}\right|_{z=0} \Rightarrow \left.\frac{2}{3}(5+z)\right|_{z=0} = \left. E_{2z}\right|_{z=0}$

$\frac{10}{3} = E_{2z}(z=0)$

* From (1),(7),& (9) we can write

(9) $$\boxed{\vec{E}_2(z=0) = E_{2x}\hat{a}_x + E_{2y}\hat{a}_y + E_{2z}\hat{a}_z = 2y\,\hat{a}_x - 3x\,\hat{a}_y + \frac{10}{3}\hat{a}_z}$$

similarly

(10) $$\vec{D}_2(z=0) = \varepsilon_2 \vec{E}_2(z_0) = 3\varepsilon_0\left[2y\,\hat{a}_x - 3x\,\hat{a}_y + \frac{10}{3}\hat{a}_z\right]$$

Remark: Note that the application of $\hat{a}_{n2}\times(\vec{E}_1 - \vec{E}_2) = 0$ will correctly give us the condition on the continuity of tangential components at the interface.

(11) $\hat{a}_{n2} = -\hat{a}_z$ then $\cancel{-}\hat{a}_z\times\left[\hat{a}_x 2y - \hat{a}_y 3x + \hat{a}_z(5+z)\right] = \cancel{-}\hat{a}_z\times\left[E_{2x}\hat{a}_x + E_{2y}\hat{a}_y + E_{2z}\hat{a}_z\right] \Rightarrow$

$$\hat{a}_y 2y + \hat{a}_x 3x = \hat{a}_y E_{2x} - \hat{a}_x E_{2y} \Rightarrow$$

(12) $\boxed{2y = E_{2x}}$ & (13) $\boxed{-3x = E_{2y}}$ as you can see (12) & (13) are the same as (6) & (7) which we obtained by inspection.

However be carfule that $\hat{a}_{n2}\times\vec{E}_1$ or $\hat{a}_{n2}\times\vec{E}_2$ are not the tangential components of $\vec{E}_1$ or $\vec{E}_2$. For example from Figure 9 tangential components of $\vec{E}_1$ at the interface It is clear that the tangential components of $\vec{E}_1$ at the interface are (14) $\boxed{2y\,\hat{a}_x \;\&\; -3x\,\hat{a}_y}$. If we calculate $\hat{a}_{n2}\times\vec{E}_1$ we have

(15) $\hat{a}_{n2}\times\vec{E}_1 = -\hat{a}_z\times\vec{E}_1 = -3x\,\hat{a}_x - 2y\,\hat{a}_y$. Comparing (15) with (14) it is clear that $\hat{a}_{n2}\times\vec{E}_1$ is not the tangential component of $\vec{E}_1$

Q3: The $\vec{E}$-Field of a uniform plane wave propagating in a dielectric medium is given by Q3-1

$$E(t,z) = \hat{a}_x\, 2\cos(10^8 t - z/\sqrt{3}) - \hat{a}_y \sin(10^8 t - z/\sqrt{3})\ [V/m]$$

a) write the above expression for time harmonic fields.

b) Determine the frequency & the wavelength

c) Describe the polarization of the wave

d) Find the corresponding $\vec{H}$ field (express your result in both phasor & instantaneous form)

sol: a) for cosine based phasors we write $\vec{E}$ as

① $\vec{E} = \hat{a}_x\, 2\cos(10^8 t - z/\sqrt{3}) + \hat{a}_y \cos(\frac{\pi}{2} + 10^8 t - z/\sqrt{3})$ then

the time harmonic field is given by

$$\vec{E}(z) = \hat{a}_x\, 2\, e^{-jz/\sqrt{3}} + \hat{a}_y\, e^{j\frac{\pi}{2}}\, e^{-jz/\sqrt{3}} \Rightarrow$$

② $$\boxed{\vec{E}(z) = \hat{a}_x\, 2\, e^{-jz/\sqrt{3}} + \hat{a}_y\, j\, e^{-jz/\sqrt{3}}}\ [V/m]$$

b) $\omega = 10^8 \Rightarrow 2\pi f = 10^8 \Rightarrow f = \frac{10^8}{2\pi} = 1.5915\times10^7\ [HZ]$

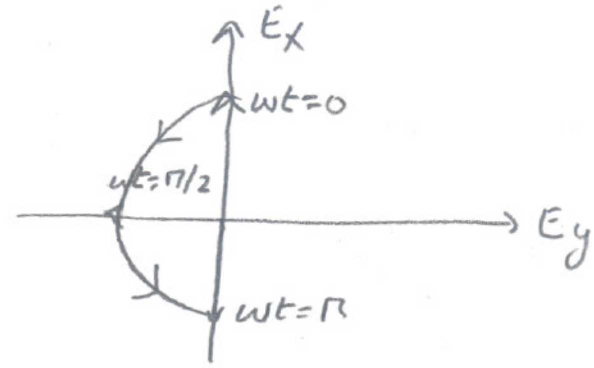
③ $\beta = \frac{1}{\sqrt{3}} \Rightarrow \lambda = \frac{2\pi}{\beta} = 2\pi\sqrt{3} = 10.8828\ [m]$

c) From (2) it is clear that E_x & E_y are out of phase by $\pi/2$ but their magnitude is not equal $\Rightarrow$ elliptical Polarization

As for the sense of rotation let us sketch the

(4) $\vec{E}(z=0,t) = \hat{a}_x\, 2\cos(\omega t) - \hat{a}_y \sin(\omega t)$

From figure we have CCW or LH sense of rotation



d) For plane wave

(5) $\vec{H} = \frac{1}{\eta}\, \hat{a}_k \times \vec{E}$ here (6) $\eta = \sqrt{\frac{\mu}{\varepsilon}} = \sqrt{\frac{\mu_0}{\varepsilon_0}}\sqrt{\frac{\mu_r}{\varepsilon_r}} = \sqrt{\frac{\mu_0}{\varepsilon_0}}\,\frac{1}{\sqrt{\varepsilon_r}}$

$$= 120\pi\,\frac{1}{\sqrt{\varepsilon_r}}$$

but what is ε_r. From part (a) $\beta = \frac{1}{\sqrt{3}}$

but $\beta = \frac{\omega}{c}\, n = \frac{\omega}{c}\sqrt{\varepsilon_r} \Rightarrow \frac{\beta c}{\omega} = \sqrt{\varepsilon_r} \Rightarrow$

(7) $\frac{1}{\sqrt{3}} \times \frac{3\times 10^8}{10^8} = \sqrt{\varepsilon_r} \Rightarrow \sqrt{3} = \sqrt{\varepsilon_r} \Rightarrow \boxed{3 = \varepsilon_r}$

then (8) $\boxed{\eta = \frac{120\pi}{\sqrt{\varepsilon_r}} = \frac{120\pi}{\sqrt{3}}}$

Also the propagation is along z-axis, hence $\hat{a}_k = \hat{a}_z$. Then from (5) we have

$$\vec{H} = \frac{\sqrt{3}}{120\pi}\,\hat{a}_z \times \left[\hat{a}_x\, 2\, e^{-jz/\sqrt{3}} + \hat{a}_y\, j\, e^{-jz/\sqrt{3}}\right]$$

$$\vec{H}(r) = \frac{\sqrt{3}}{120\pi}\left[\hat{a}_y\, 2\, e^{-jz/\sqrt{3}} - \hat{a}_x\, j\, e^{-jz/\sqrt{3}}\right]$$ Q3-3

or for time dependent field we have

$$\vec{H}(r,t) = \frac{\sqrt{3}}{120\pi}\, Re\left[\hat{a}_y\, 2\, e^{-jz/\sqrt{3}} e^{j\omega t} - \hat{a}_x\, e^{j\pi/2} e^{-jz/\sqrt{3}} e^{j\omega t}\right]$$

$$= \frac{\sqrt{3}}{120\pi}\, Re\left[\hat{a}_y\, 2\, e^{j(\omega t - z/\sqrt{3})} - \hat{a}_x\, e^{j[\frac{\pi}{2} + \omega t - z/\sqrt{3}]}\right]$$

$$= \frac{\sqrt{3}}{120\pi}\left[\hat{a}_y\, 2\cos(\omega t - \frac{z}{\sqrt{3}}) - \hat{a}_x \cos(\frac{\pi}{2} + \omega t - \frac{z}{\sqrt{3}})\right]$$

$$\boxed{= \frac{\sqrt{3}}{120\pi}\left[\hat{a}_y\, 2\cos(\omega t - \frac{z}{\sqrt{3}}) + \hat{a}_x \sin(\omega t - \frac{z}{\sqrt{3}})\right]}$$

oR

$$\boxed{\vec{H}(z,t) = \frac{\sqrt{3}}{120\pi}\left[\hat{a}_x \sin(10^8 t - \frac{z}{\sqrt{3}}) + \hat{a}_y\, 2\cos(10^8 t - z/\sqrt{3})\right]}$$

Q4: Prove the following relations between group velocity V_g & Phase velocity V_p in a dispersive medium

a) $V_g = V_p + K \frac{dV_p}{dK}$ ①

b) $V_g = V_p - \lambda \frac{dV_p}{d\lambda}$ ②

sol:

a) Recall that group velocity (in 1-D) is given by

④ $V_g = \frac{d\omega}{dK}$.

* Our dispersion relation $K = \frac{\omega}{c} n$ can be written as

⑤ $K = \frac{\omega}{c} n(\omega)$ or equally well as ⑥ $\omega = \frac{cK}{n(K)} = V_p K$

where $V_p = \frac{c}{n(K)} \equiv \frac{c}{n(\omega)}$ ⑥

* we use (6) to find V_g, i.e

⑦ $V_g = \frac{d\omega}{dK} = \frac{d}{dK}[V_p K]$ {note here that $V_p = \frac{c}{n(K)}$ is itself a function of K}

then (7) ⟹

⑧ $\boxed{V_g = \frac{d}{dK}[V_p K] = K \frac{dV_p}{dK} + V_p}$ by chain rule b)

* since $K = \frac{2\pi}{\lambda}$ Eq (8) can also be expressed in terms of wavelength (λ). From (8) we can write Q4-2

$$⑨ \quad V_g = K \frac{dV_p}{dK} + V_p = K \frac{dV_p}{d\lambda} \frac{d\lambda}{dK} + V_p$$

where we have used the chain rule. But what is $d\lambda/dK$.

since $\lambda = \frac{2\pi}{K}$ then $\frac{d\lambda}{dK} = \frac{-2\pi}{K^2}$ ⑩

sub (10) in (9) & we have

$$V_g = \cancel{K} \frac{dV_p}{d\lambda}\left(\frac{-2\pi}{K^{\cancel{2}}}\right) + V_p = -\frac{2\pi}{K} \frac{dV_p}{d\lambda} + V_p$$

$$= -\lambda \frac{dV_p}{d\lambda} + V_p = \boxed{V_p - \lambda \frac{dV_p}{d\lambda}} \quad ⑪$$

* In (11) it is understood that V_p is a function of wavelength i.e. $V_p(\lambda)$, & hence V_g is also a function of wavelength [$V_g(\lambda)$]