Problem Set #8
ECE357 /ECE320
University of Toronto

Question 1) A uniform plane wave polarized along +*x*-direction is propagating in +*z*-direction in an unbounded lossy medium. The filed amplitude at *t=0* and *z=0* is E_0.
a) Give the appropriate expressions for the phasor and instantaneous electric field.
b) Give the appropriate expressions for the phasor and instantaneous magnetic field.
c) Give the appropriate expression for instantaneous and time average Poynting vector (power density).
d) Suppose that the frequency of operation is 3 [GHz], the field amplitude at *t=0* and *z=0* is 50 [V/m], the dielectric constant of the non-magnetic lossy medium is 2.5, and its loss tangent is 10^{-2}. Write the expression for the instantaneous magnetic field.

Question 2) Prove that
a) A circularly polarized plane wave can be obtained from a superposition of two oppositely directed elliptically polarized waves.
b) A linearly polarized wave can be obtained from the superposition of a left handed and right handed circularly polarized waves.

Question 3) The electrosatic energy associated with charge distribution ρ is given by $W_e = \frac{1}{2}\iiint_{v'} \rho\, V\, dv$ where V is the potential at the point where the volume charge density is ρ, and v' is the volume of the region where ρ exist. Show that this expression will simplify to $W_e = \frac{1}{2}\iiint_{v'} \varepsilon \left|\vec{E}\right|^2 dv$. Express the last expression in terms of the electric filed in intensity ($\vec{E}$) and electric filed flux density ($\vec{D}$).

Question 4) Make sure you understand Example 8-9, page 389 of the book by David Cheng

Question 5)
a) Show that for a dispersive medium the group velocity can be written as

$$V_g(\lambda_0) = \frac{c}{n(\lambda_0) - \lambda\, dn(\lambda_0)/d\lambda_0}$$

b) The dispersion in a certain material is described by its index of refraction as a function of frequency

$$n(\omega) = n_0 - \frac{a\,(\omega-\omega_0)}{a^2 + (\omega-\omega_0)^2}$$, where n_0, a, and ω_0 are constants.

What are the phase and group velocities in this medium?

Solution to Question 1)

* since lossy we have ① $\gamma = \alpha + j\beta$. The field is polarized along $\hat{ax}$ & propagates in +z-direction then

a) ② $\vec{E}(z) = E_0 e^{-\alpha z} e^{-j\beta z} e^{j\phi_0} \hat{ax}$ where ϕ_0 is the phase angle

or

③ $E(z,t) = E_0 e^{-\alpha z} \cos(\omega t - \beta z + \phi_0)\, \hat{ax}$

since ④ $|\vec{E}(z=0, t=0)| = E_0 \Rightarrow$ ⑤ $\phi_0 = 0$ & we have

⑥ $\boxed{\vec{E}(z) = E_0 e^{-\alpha z} e^{-j\beta z} \hat{ax}}$ In phasor form

⑦ $\boxed{\vec{E}(z,t) = E_0 e^{-\alpha z} \cos(\omega t - \beta z)\, \hat{ax}}$ The instantaneous field

b) magnetic field is obtained from

⑧ $\vec{H}(z) = \frac{1}{\eta} \hat{ak} \times \vec{E}(z)$ where ⑨ $\hat{ak} = \hat{az}$ &

⑩ $\eta = |\eta| e^{j\theta_\eta}$ since

⑪ $\eta = \sqrt{\frac{\mu}{\varepsilon_c}}$ where $\varepsilon_c = \varepsilon' - j\varepsilon''$

then

⑫ $\vec{H}(z) = \frac{1}{|\eta| e^{j\theta_\eta}} \hat{az} \times E_0 e^{-\alpha z} e^{-j\beta z} \hat{ax} \Rightarrow$

⑬ $\boxed{\vec{H}(z) = \frac{E_0}{|\eta|} \hat{ay}\, e^{-\alpha z} e^{-j\beta z} e^{-j\theta_\eta}}$ In phasor form

⑭ $\boxed{\vec{H}(z,t) = \hat{ay} \frac{E_0}{|\eta|} e^{-\alpha z} \cos(\omega t - \beta z - \theta_\eta)}$

(15)
c) $\vec{S}(z,t) = \frac{1}{2}Re[\vec{E}\times\vec{H}\, e^{2j\omega t}] + \frac{1}{2}Re[\vec{E}\times\vec{H}^*]$

which is the expression for instantaneous Power density

* From (6) & (13) we have

$$\vec{S}(z,t) = \frac{1}{2}Re\left[\frac{E_0^2}{|\eta|} e^{-2\alpha z} e^{-2j\beta z} e^{-j\theta_\eta} e^{2j\omega t} \hat{a}_z\right]$$

$$+\frac{1}{2}Re\left[\frac{E_0^2}{|\eta|} e^{-2\alpha z} e^{+j\theta_\eta} \hat{a}_z\right]$$

$$= \frac{1}{2}\frac{E_0^2}{|\eta|} e^{-2\alpha z} \cos(2\omega t - 2\beta z - \theta_\eta)\,\hat{a}_z +$$

$$\frac{1}{2}\frac{E_0^2}{|\eta|} e^{-2\alpha z} \cos(\theta_\eta)\,\hat{a}_z \Rightarrow$$

(16)
$$\boxed{\vec{S}(z,t) = \frac{1}{2}\frac{E_0^2}{|\eta|} e^{-2\alpha z}\left[\cos(2\omega t - 2\beta z - \theta_\eta) + \cos(\theta_\eta)\right]\hat{a}_z}$$

* The time averaged power density is given by

(17)
$$\boxed{\langle\vec{S}(z,t)\rangle = \frac{1}{2}Re[\vec{E}\times\vec{H}^*] = \frac{1}{2}\frac{E_0^2}{|\eta|} e^{-2\alpha z}\cos\theta_\eta\, \hat{a}_z}$$

d) The expression for $\vec{H}(z,t)$ was given in (14)

$$\vec{H}(z,t) = \hat{a}_y \frac{E_0}{|\eta|} e^{-\alpha z} \cos(\omega t - \beta z - \theta_\eta)$$

(18)
since $\tan\delta = \frac{\varepsilon''}{\varepsilon'} = 0.01 << 1$ this is the case of low loss dielectric

for which

(19) $\alpha \simeq \frac{\omega \varepsilon''}{2}\sqrt{\frac{\mu}{\varepsilon'}}$, (20) $\beta = \omega\sqrt{\mu\varepsilon'}\left[1+\frac{1}{8}\left(\frac{\varepsilon''}{\varepsilon'}\right)^2\right]$,

and (21) $\eta = \sqrt{\frac{\mu}{\varepsilon'}}\left(1+j\frac{\varepsilon''}{2\varepsilon'}\right)$,

Note the following that

(22) $\varepsilon = \varepsilon' - j\varepsilon'' = \varepsilon_0[\varepsilon_r' - j\varepsilon_r'']$ & (23) $\mu = \mu_0\mu_r$ where $\mu_r = 1$

* Here (24) $f = 3\times10^9 \Rightarrow \omega = 6\pi\times10^9$ [rad/s]

(25) $\tan\delta = \frac{\varepsilon''}{\varepsilon'} \Rightarrow$ (26) $0.01 = \frac{\varepsilon_0\varepsilon_r''}{\varepsilon_0\varepsilon_r'} \Rightarrow$ (27) $\varepsilon_r'' = \varepsilon_r'\times0.01$ & (28) $\varepsilon_r' = 2.5$

then (19) $\Rightarrow$ $\alpha = \frac{\omega\varepsilon''}{2}\sqrt{\frac{\mu}{\varepsilon'}} = \frac{\omega\varepsilon_0\varepsilon_r''}{2}\sqrt{\frac{\mu_0}{\varepsilon_0\varepsilon_r'}} = \omega\frac{\varepsilon_0}{2}\varepsilon_r''\sqrt{\frac{\mu_0}{\varepsilon_0}}\frac{1}{\sqrt{\varepsilon_r'}}$

use (27) $\Rightarrow$ $\alpha = \frac{\omega\varepsilon_0}{2}\varepsilon_r'\times0.01\times\eta_0\times\frac{1}{\sqrt{\varepsilon_r'}} = \frac{\omega\varepsilon_0\sqrt{\varepsilon_r'}}{2}\times0.01\times\eta_0$

$= \frac{6\pi\times10^9\times8.85\times10^{-12}}{2}\sqrt{2.5}\times0.01\times120\pi \Rightarrow$

(29) $\boxed{\alpha = 0.497\ [NP/m]}$

(20) $\Rightarrow$ $\beta = \omega\sqrt{\mu\varepsilon'}\left[1+\frac{1}{8}\left(\frac{\varepsilon''}{\varepsilon'}\right)^2\right] = \omega\sqrt{\mu_0\varepsilon_0}\sqrt{\varepsilon_r'}\left[1+\frac{1}{8}(\tan\delta)^2\right]$

$= \frac{\omega}{c}\sqrt{\varepsilon_r'}\left[1+\frac{1}{8}(\tan\delta)^2\right] = \frac{6\pi\times10^9}{3\times10^8}\sqrt{2.5}\left[1+\frac{1}{8}(0.01)^2\right] \Rightarrow$

(30) $\boxed{\beta = 31.62\pi\ \text{rad/s}}$

$$(21) \Rightarrow \eta = \sqrt{\frac{\mu}{\varepsilon'}}\left(1+j\frac{\varepsilon''}{2\varepsilon'}\right) = \sqrt{\frac{\mu_0}{\varepsilon_0\varepsilon_r'}}\left(1+j\frac{\tan\delta}{2}\right)$$

$$= \eta_0 \frac{1}{\sqrt{\varepsilon_r'}}\left(1+j\frac{\tan\delta}{2}\right) = 120\pi\frac{1}{\sqrt{2.5}}\left(1+j\frac{0.01}{2}\right)$$

(31) $$\boxed{\eta = 238.43 + j\,1.192 = 238.433\angle 0.287^\circ}$$

From (14) we have

$$\boxed{\vec{H}(z,t) = \hat{a}_y \frac{50}{238.433} e^{-0.497z} \cos(6\pi\times10^9 t - 31.62\pi z - 0.005)}$$

where $0.287^\circ = 0.005$ radian

Solution to question 2)

a) Let us consider the case of right handed circularly polarized wave first.

① $\vec{E_{rc}} = E_o(\hat{a_x} - j\hat{a_y})\, e^{-jkz} = E_x \hat{a_x} + E_y \hat{a_y}$

* Note that E_x & E_y are out of phase by $\pi/2$, but have equal amplitude - hence $\vec{E_{rc}}$ or (1) is a circularly polarized wave. Check the sense of rotation & see that it is indeed a right handed wave.

* Consider the following two elliptically polarized wave which have opposite sense of rotation

② $\vec{E_{1,elp}} = E_o\left(\frac{1}{2}\hat{a_x} - 2j\,\hat{a_y}\right) e^{-jkz}$

③ $\vec{E_{2,elp}} = E_o\left(\frac{1}{2}\hat{a_x} + j\,\hat{a_y}\right) e^{-jkz}$

* Note that components of (2) have unequal amplitude & are out of phase by $\pi/2 \Rightarrow$ elliptical polarization

* Note that components of (3) have unequal amplitudes & are out of phase by $\pi/2 \Rightarrow$ elliptical polarization

* Note that the sense of rotation for (2) & (3) are opposite of each other.

* Let us calculate $\vec{E_{1,elp}} + \vec{E_{2,elp}}$

④ $\vec{E}_{1,elp} + \vec{E}_{2,elp} = E_0(\hat{a_x} - j\,\hat{a_y})\,e^{-jkz}$

but (4) is an expression for a right handed circularly polarized wave as given by Eq. (1). Hence, we see that a right handed circularly polarized wave can be written as superposition of two oppositely directed elliptically polarized waves.

* If we start with a Left handed circularly polarized wave given by ⑤ $\vec{E}_{LC} = E_0\,(\hat{a_x} + j\,\hat{a_y})\,e^{-jkz} = E_x\,\hat{a_x} + E_y\,\hat{a_y}$

we then write for the elliptically polarized wave

⑥ $\vec{E}_{1,elp} = E_0(\frac{1}{2}\,\hat{a_x} + 2j\,\hat{a_y})\,e^{-jkz}$

⑦ $\vec{E}_{2,elp} = E_0(\frac{1}{2}\,\hat{a_x} - j\,\hat{a_y})\,e^{-jkz}$

We then note $\vec{E}_{1,elp} + \vec{E}_{2,elp} = E_0\,(\hat{a_x} + j\,\hat{a_y})\,e^{-jkz}$ which is the same as Eq. (5). Hence a Left handed circularly polarized wave can be written as the sum of two oppositely directed elliptically polarized waves.

b) A Linearly polarized wave can be written as

⑧ $\vec{E}_{Lin} = \hat{a_x}\,E_0\,e^{-jkZ}$

Consider the following Left handed & right handed circularly polarized waves-

(9) $$\vec{E}_{LC} = \frac{E_0}{2}(\hat{a}_x + j\hat{a}_y)e^{-jkz}$$

(10) $$\vec{E}_{RC} = \frac{E_0}{2}(\hat{a}_x - j\hat{a}_y)e^{-jkz}$$

* Note that $\vec{E}_{LC} + \vec{E}_{RC} = E_0 e^{-jkz}\hat{a}_z$, hence our linearly polarized wave of Eq (8) can be written as the superposition of a left hand & right hand circularly polarized waves.

solution to question 3

$$\text{①}\quad W_e = \frac{1}{2}\iiint_{v'} \rho V\,dv \;.\; \text{use Gauss law } \text{②}\; \nabla\cdot\vec{D} = \rho \text{ in (1)} \Rightarrow$$

$$\text{③}\quad W_e = \frac{1}{2}\iiint_{v'} (\nabla\cdot\vec{D})V\,dv$$

the following Vector Identity always holds

$$\text{④}\quad \nabla\cdot(V\vec{D}) = V\nabla\cdot\vec{D} + \vec{D}\cdot\nabla V \Rightarrow \text{⑤}\quad V\nabla\cdot\vec{D} = \nabla\cdot(V\vec{D}) - \vec{D}\cdot\nabla V$$

use (5) in (3) $\Rightarrow$

$$\text{⑤}\quad W_e = \frac{1}{2}\iiint_{v'} \left(\nabla\cdot(V\vec{D}) - \vec{D}\cdot\nabla V\right)dv$$

$$= \frac{1}{2}\oiint_{S'} V\vec{D}\cdot d\vec{s} - \frac{1}{2}\iiint_{v'} (\vec{D}\cdot\nabla V)\,dv$$

* In (5) we used the divergence theorem to transform the

$$\text{⑥}\quad \iiint_{v'} \nabla\cdot(V\vec{D})\,dv \text{ to } \oiint_{S'} V\vec{D}\cdot d\vec{s}$$

* In electrostatic we have ⑦ $\vec{E} = -\nabla V$ use (7) in (5) $\Rightarrow$

$$\text{⑧}\quad W_e = \frac{1}{2}\oiint_{S'} V\vec{D}\cdot d\vec{s} + \frac{1}{2}\iiint_{v'} (\vec{E}\cdot\vec{D})\,dv$$

* Now note the following: Potential $V \propto \frac{1}{r}$ where r is the distance, and $D \propto \frac{1}{r^2}$ hence $V\vec{D} \propto \frac{1}{r^3}$

Since the integral over the volume v' & surface enclosing the volume (i.e. s') can arbitarly be made large, as long as it contains the charges ρ, then we can consider the volume to be a sphere with $r \rightarrow \infty$. Now the surface of sphere is proportional to r^2, but the integrand in $\oint\!\!\oint_{s'} V\vec{D}\cdot d\vec{s}$ integral is proportional to $\frac{1}{r}\cdot\frac{1}{r^2} = \frac{1}{r^3}$, hence the integral

$\oint\!\!\oint_{s'} V\vec{D}\cdot d\vec{s} \rightarrow 0$ as $r \rightarrow \infty$, and the only term remaining in (8) is

$$W_e = \frac{1}{2}\iiint_{v'} \vec{E}\cdot\vec{D}\, dv = \frac{1}{2}\iiint_{v'} \varepsilon\, \vec{E}\cdot\vec{E}\, dv$$

$$= \frac{1}{2}\iiint \varepsilon\, |\vec{E}|^2\, dv$$

Question 4)

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EXAMPLE 8-9 A y-polarized uniform plane wave ($\mathbf{E}_i$, $\mathbf{H}_i$) with a frequency 100 (MHz) propagates in air in the $+x$ direction and impinges normally on a perfectly conducting plane at $x = 0$. Assuming the amplitude of $\mathbf{E}_i$ to be 6 (mV/m), write the phasor and instantaneous expressions for (a) $\mathbf{E}_i$ and $\mathbf{H}_i$ of the incident wave; (b) $\mathbf{E}_r$ and $\mathbf{H}_r$ of the reflected wave; and (c) $\mathbf{E}_1$ and $\mathbf{H}_1$ of the total wave in air. (d) Determine the location nearest to the conducting plane where E_1 is zero.

Solution At the given frequency 100 (MHz),

$$\omega = 2\pi f = 2\pi \times 10^8 \qquad \text{(rad/s)},$$

$$\beta_1 = k_0 = \frac{\omega}{c} = \frac{2\pi \times 10^8}{3 \times 10^8} = \frac{2\pi}{3} \qquad \text{(rad/m)},$$

$$\eta_1 = \eta_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} = 120\pi \qquad (\Omega).$$

a) *For the incident wave (a traveling wave):*

i) Phasor expressions:

$$\mathbf{E}_i(x) = \mathbf{a}_y 6 \times 10^{-3} e^{-j2\pi x/3} \qquad \text{(V/m)},$$

$$\mathbf{H}_i(x) = \frac{1}{\eta_1}\mathbf{a}_x \times \mathbf{E}_i(x) = \mathbf{a}_z \frac{10^{-4}}{2\pi} e^{-j2\pi x/3} \qquad \text{(A/m)}.$$

ii) Instantaneous expressions:

$$\mathbf{E}_i(x, t) = \mathscr{R}e\left[\mathbf{E}_i(x)e^{j\omega t}\right]$$

$$= \mathbf{a}_y 6 \times 10^{-3} \cos\left(2\pi \times 10^8 t - \frac{2\pi}{3}x\right) \qquad \text{(V/m)},$$

$$\mathbf{H}_i(x, t) = \mathbf{a}_z \frac{10^{-4}}{2\pi} \cos\left(2\pi \times 10^8 t - \frac{2\pi}{3}x\right) \qquad \text{(A/m)}.$$

b) *For the reflected wave (a traveling wave):*

i) Phasor expressions:

$$\mathbf{E}_r(x) = -\mathbf{a}_y 6 \times 10^{-3} e^{j2\pi x/3} \qquad \text{(V/m)},$$

$$\mathbf{H}_r(x) = \frac{1}{\eta_1}(-\mathbf{a}_x) \times \mathbf{E}_r(x) = \mathbf{a}_z \frac{10^{-4}}{2\pi} e^{j2\pi x/3} \qquad \text{(A/m)}.$$

ii) Instantaneous expressions:

$$\mathbf{E}_r(x, t) = \mathcal{R}e[\mathbf{E}_r(x)e^{j\omega t}] = -\mathbf{a}_y 6 \times 10^{-3} \cos\left(2\pi \times 10^8 t + \frac{2\pi}{3} x\right) \quad \text{(V/m)},$$

$$\mathbf{H}_r(x, t) = \mathbf{a}_z \frac{10^{-4}}{2\pi} \cos\left(2\pi \times 10^8 t + \frac{2\pi}{3} x\right) \quad \text{(A/m)}.$$

c) *For the total wave (a standing wave):*

i) Phasor expressions:

$$\mathbf{E}_1(x) = \mathbf{E}_i(x) + \mathbf{E}_r(x) = -\mathbf{a}_y j12 \times 10^{-3} \sin\left(\frac{2\pi}{3} x\right) \quad \text{(V/m)},$$

$$\mathbf{H}_1(x) = \mathbf{H}_i(x) + \mathbf{H}_r(x) = \mathbf{a}_z \frac{10^{-4}}{\pi} \cos\left(\frac{2\pi}{3} x\right) \quad \text{(A/m)}.$$

ii) Instantaneous expressions:

$$\mathbf{E}_1(x, t) = \mathcal{R}e[\mathbf{E}_1(x)e^{j\omega t}] = \mathbf{a}_y 12 \times 10^{-3} \sin\left(\frac{2\pi}{3} x\right) \sin(2\pi \times 10^8 t) \quad \text{(V/m)},$$

$$\mathbf{H}_1(x, t) = \mathbf{a}_z \frac{10^{-4}}{\pi} \cos\left(\frac{2\pi}{3} x\right) \cos(2\pi \times 10^8 t) \quad \text{(A/m)}.$$

d) The electric field vanishes at the surface of the conducting plane at $x = 0$. In medium 1 the first null occurs at

$$x = -\frac{\lambda_1}{2} = -\frac{\pi}{\beta_1} = -\frac{3}{2} \quad \text{(m)}.$$

Solution to question 5)

a) We have shown ① $V_g(\omega) = \dfrac{c}{n(\omega) + \omega \dfrac{dn(\omega)}{d\omega}} = \dfrac{c}{n_g(\omega)}$

& we want $V_g(\lambda_0)$. Recall that

② $\omega = \dfrac{2\pi c}{\lambda_0} \Rightarrow$ ③ $\lambda_0 = \dfrac{2\pi c}{\omega} \Rightarrow$ ④ $\dfrac{d\lambda_0}{d\omega} = -\dfrac{2\pi c}{\omega^2}$

then ⑤ $\dfrac{dn(\omega)}{d\omega} = \dfrac{dn(\lambda_0)}{d\lambda_0}\dfrac{d\lambda_0}{d\omega} = \dfrac{dn(\lambda_0)}{d\lambda_0}\left(-\dfrac{2\pi c}{\omega^2}\right)$

then (1) as a function λ_0 can be written as

$$V_g(\lambda_0) = \frac{c}{n(\lambda_0) + \cancel{\omega}\left(-\frac{2\pi c}{\omega^{\cancel{2}}}\right)\frac{dn(\lambda_0)}{d\lambda_0}} = \frac{c}{n(\lambda_0) - \frac{2\pi c}{\omega}\frac{dn(\lambda_0)}{d\lambda_0}}$$

$$= \boxed{\frac{c}{n(\lambda_0) - \lambda_0 \frac{dn(\lambda_0)}{d\lambda_0}}}$$

Remark: Note that in many books authors use λ to designate the wavelength in free space (what we have called λ_0 here). When reading a text make sure you understand the author's intention

b) $$V_p = \frac{c}{n} = \frac{c}{n_0 - \frac{a(\omega-\omega_0)}{a^2+(\omega-\omega_0)^2}} = \frac{c\left[a^2+(\omega-\omega_0)^2\right]}{n_0a^2+n_0(\omega-\omega_0)^2 - a(\omega-\omega_0)}$$

– For $k = \frac{\omega}{c} n(\omega)$ we have $V_g = \frac{c}{n + \omega\, dn/d\omega}$

$$\frac{dn}{d\omega} = \frac{d}{d\omega}\left[n_0 - \frac{a(\omega-\omega_0)}{a^2+(\omega-\omega_0)^2}\right] = -a\left\{\frac{a^2+(\omega-\omega_0)^2 - 2(\omega-\omega_0)a(\omega-\omega_0)}{\left[a^2+(\omega-\omega_0)^2\right]^2}\right\}$$

$$= -a\left\{\frac{a^2+(\omega-\omega_0)^2 - 2a(\omega-\omega_0)^2}{\left[a^2+(\omega-\omega_0)^2\right]^2}\right\}$$

$$= -a\left\{\frac{a^2+(\omega-\omega_0)^2(1-2a)}{\left[a^2+(\omega-\omega_0)^2\right]^2}\right\} = \frac{a(2a-1)(\omega-\omega_0)^2 - a^3}{\left[a^2+(\omega-\omega_0)^2\right]^2}$$

$$V_g = \frac{c}{n_0 - \frac{a(\omega-\omega_0)}{a^2+(\omega-\omega_0)^2} + \omega\frac{a(2a-1)(\omega-\omega_0)^2 - a3}{\left[a^2+(\omega-\omega_0)^2\right]^2}}$$

$$= \frac{c\left[a^2+(\omega-\omega_0)^2\right]^2}{n_0\left[a^2+(\omega-\omega_0)^2\right]^2 - a(\omega-\omega_0)\left[a^2+(\omega-\omega_0)^2\right] + \omega\left[a(2a-1)(\omega-\omega_0)^2 - a3\right]}$$