HW#9

1)

a) Show that Snell's laws for $\vec{E}$ parallel polarization hold.

b) Obtain the filed transmission and reflection coefficients for the $\vec{E}$ parallel polarization.

c) Obtain an expression for the Brewster angle and the critical angle for the $\vec{E}$ parallel polarization.

d) What is the Brewster angle for nonmagnetic media in the case of $\vec{E}$ parallel polarization?

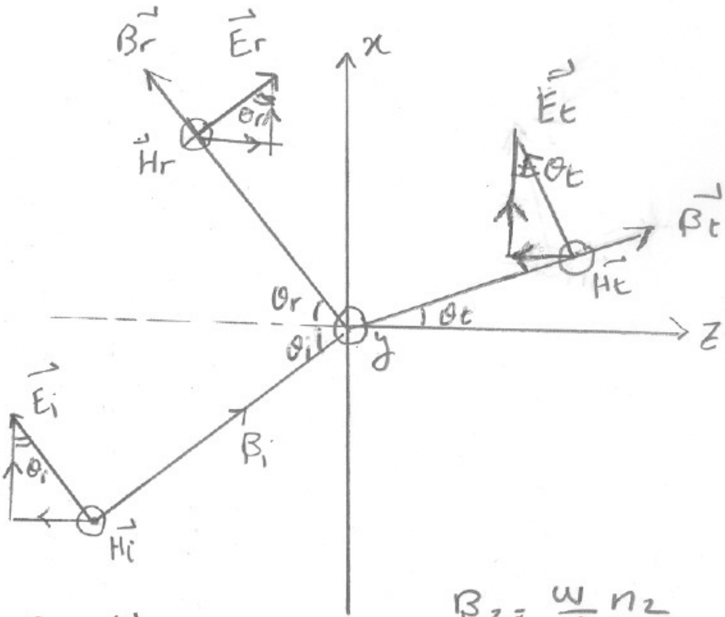
The solutions for the above can be found in Cheng's text book, pages 414-416.

a)

Incident wave:

① $\vec{E_i} = E_{i0}(\cos\theta_i \hat{a_x} - \sin\theta_i \hat{a_y})\, e^{-j\vec{\beta_i}\cdot\vec{r}}$

② $\vec{\beta_i} = (\sin\theta_i \hat{a_x} + \cos\theta_i \hat{a_z})\beta_1$



$\beta_1 = \frac{\omega}{c} n_1$ $\beta_2 = \frac{\omega}{c} n_2$

then

③ $$\boxed{\vec{E_i} = E_{i0}(\cos\theta_i \hat{a_x} - \sin\theta_i \hat{a_z})\, e^{-j(\sin\theta_i x + \cos\theta_i z)\beta_1}}$$

④ $\vec{H_i} = \frac{E_{i0}}{\eta_1} e^{-j\vec{\beta_i}\cdot\vec{r}} \hat{a_y}$

⑤ $$\boxed{\vec{H_i} = \frac{E_{i0}}{\eta_1} e^{-j(\sin\theta_i x + \cos\theta_i z)\beta_1} \hat{a_y}}$$

Reflected wave

⑥ $\vec{E_r} = E_{r0}(\cos\theta_r \hat{a_x} + \sin\theta_r \hat{a_z})\, e^{-j\vec{\beta_r}\cdot\vec{r}}$

⑦ $\vec{\beta_r} = (\sin\theta_r \hat{a_x} - \cos\theta_r \hat{a_z})\beta_1$

⑧ $$\boxed{\vec{E_r} = E_{r0}(\cos\theta_r \hat{a_x} + \sin\theta_r \hat{a_z})\, e^{-j(\sin\theta_r x - \cos\theta_r z)\beta_1}}$$

⑨ $$\boxed{\vec{H_r} = -\frac{E_{r0}}{\eta_1} e^{-j(\sin\theta_r x - \cos\theta_r z)\beta_1} \hat{a_y}}$$

(10) $\vec{E}_t = E_{t0}(\cos\theta_t \hat{a}_x - \sin\theta_t \hat{a}_z)\, e^{-j\vec{\beta}_t \cdot \vec{r}}$

(11) $\beta_t = \beta_2(\sin\theta_t \hat{a}_x + \cos\theta_t \hat{a}_z) \Rightarrow$

(12) $$\boxed{\vec{E}_t = E_{t0}(\cos\theta_t \hat{a}_x - \sin\theta_t \hat{a}_z)\, e^{-j[\sin\theta_t x + \cos\theta_t z]\beta_2}}$$

$$\boxed{\vec{H}_t = \frac{E_{t0}}{\eta_2} e^{-j[\sin\theta_t x + \cos\theta_t z]\beta_2} \hat{a}_y}$$

(13)

Continuity of tangential $\vec{E}$ & $\vec{H}$ at $z=0$ implies –

(14) $E_{i0}\cos\theta_i\, e^{-j\sin\theta_i x\beta_1} + E_{r0}\cos\theta_r\, e^{-j\sin\theta_r x\beta_1} = E_{t0}\cos\theta_t\, e^{-j\sin\theta_t x\beta_2}$

&

(15) $\frac{E_{i0}}{\eta_1} e^{-j\sin\theta_i x\beta_1} - \frac{E_{r0}}{\eta_1} e^{-j\sin\theta_r x\beta_1} = \frac{E_{t0}}{\eta_2} e^{-j\sin\theta_t x\beta_2}$

since (14) & (15) must hold for all x we require

(16) $\sin\theta_i x\beta_1 = \sin\theta_r x\beta_1 = \sin\theta_t x\beta_2 \Rightarrow$

$\sin\theta_i = \sin\theta_r \Rightarrow$ (17) $\theta_i = \theta_r$ &

$\sin\theta_r\beta_1 = \sin\theta_t\beta_2 \Rightarrow$ (18) $\sin\theta_i\beta_1 = \sin\theta_t\beta_2 \Rightarrow$

(19) $$\frac{\sin\theta_t}{\sin\theta_i} = \frac{\beta_1}{\beta_2} = \frac{\frac{\omega}{c}n_1}{\frac{\omega}{c}n_2} = \frac{n_1}{n_2} = \frac{\sqrt{\mu_1\varepsilon_1}}{\sqrt{\mu_2\varepsilon_2}} =$$

b)

using (17) & (18) in (14) & (15) we have

(20) $E_{io}\cos\theta_i + E_{ro}\cos\theta_i = E_{to}\cos\theta_t$

(21) $\frac{E_{io}}{\eta_1} - \frac{E_{ro}}{\eta_1} = \frac{E_{to}}{\eta_2}$

(20) & (21) can be solved to give

(22) $E_{ro} = \frac{\eta_2\cos\theta_t - \eta_1\cos\theta_i}{\eta_2\cos\theta_t + \eta_1\cos\theta_i} E_{io} \Rightarrow$ (23) $\boxed{\Gamma_{\parallel} = \frac{E_{ro}}{E_{io}} = \frac{\eta_2\cos\theta_t - \eta_1\cos\theta_i}{\eta_2\cos\theta_t + \eta_1\cos\theta_i}}$

(24) & $E_{to} = \frac{2\eta_2\cos\theta_i}{\eta_2\cos\theta_t + \eta_1\cos\theta_i} E_{io} \Rightarrow$ $\boxed{T_{\parallel} = \frac{E_{to}}{E_{io}} = \frac{2\eta_2\cos\theta_i}{\eta_2\cos\theta_t + \eta_1\cos\theta_i}}$ (25)

c) For Brewster angle we require $\Gamma_{\parallel} = 0 \Rightarrow$

(26) $\eta_2\cos\theta_t = \eta_1\cos\theta_B \Rightarrow$ (27) $\eta_2^2\cos^2\theta_t = \eta_1^2\cos^2\theta_B \Rightarrow$

(28) $\eta_2^2(1-\sin^2\theta_t) = \eta_1^2(1-\sin^2\theta_B)$

From Snell's law $n_1\sin\theta_i = n_2\sin\theta_t \Rightarrow$ for $\theta_i = \theta_B$ we have

$n_1^2\sin^2\theta_B = n_2^2\sin^2\theta_t \Rightarrow$ (30) $\frac{n_1^2}{n_2^2}\sin^2\theta_B = \sin^2\theta_t$

Use (30) in (28) $\Rightarrow$

$\eta_2^2\left(1 - \frac{n_1^2}{n_2^2}\sin^2\theta_B\right) = \eta_1^2(1-\sin^2\theta_B) \Rightarrow$

$\left(\eta_1^2 - \eta_2^2\frac{n_1^2}{n_2^2}\right)\sin^2\theta_B = \eta_1^2 - \eta_2^2 \Rightarrow \sin^2\theta_B = \frac{\eta_1^2 - \eta_2^2}{\eta_1^2 - \eta_2^2\frac{n_1^2}{n_2^2}}$ (31)

* Then since $\eta = \sqrt{\frac{\mu}{\varepsilon}}$ & $n = \sqrt{\mu_r \varepsilon_r}$ we have

$$\sin^2\theta_B = \frac{\frac{\mu_1}{\varepsilon_1} - \frac{\mu_2}{\varepsilon_2}}{\frac{\mu_1}{\varepsilon_1} - \frac{\mu_2}{\varepsilon_2}\frac{\mu_{r1}\varepsilon_{r1}}{\mu_{r2}\varepsilon_{r2}}\frac{\mu_0\varepsilon_0}{\mu_0\varepsilon_0}} = \frac{\frac{\mu_1}{\varepsilon_1}\left[1 - \frac{\mu_2\varepsilon_1}{\varepsilon_2\mu_1}\right]}{\frac{\mu_1}{\varepsilon_1} - \frac{\cancel{\mu_2}}{\varepsilon_2}\frac{\mu_1\varepsilon_1}{\cancel{\mu_2}\varepsilon_2}} =$$

$$\frac{\cancel{\frac{\mu_1}{\varepsilon_1}}\left[1 - \frac{\mu_2\varepsilon_1}{\varepsilon_2\mu_1}\right]}{\cancel{\frac{\mu_1}{\varepsilon_1}}\left[1 - \frac{\varepsilon_1^2}{\varepsilon_2^2}\right]} \Rightarrow \boxed{\sin^2\theta_B = \frac{1 - \frac{\mu_2\varepsilon_1}{\mu_1\varepsilon_2}}{1 - \frac{\varepsilon_1^2}{\varepsilon_2^2}}} \quad (32)$$

* For critical angle $\theta_i = \theta_c$ we have

$$n_1 \sin\theta_i\Big|_{\theta_i=\theta_c} = n_2 \sin\theta_t\Big|_{\theta_t=\pi/2} \Rightarrow \quad (33)$$

$$n_1 \sin\theta_c = n_2 \Rightarrow \boxed{\sin\theta_c = \frac{n_2}{n_1}}$$

as for the $\vec{E}\perp$ case

d) For non-magnetic material $\mu_1 = \mu_2$ & (32) $\Rightarrow$

$$\sin^2\theta_B = \frac{1 - \varepsilon_1/\varepsilon_2}{1 - \frac{\varepsilon_1^2}{\varepsilon_2^2}} = \frac{\frac{\varepsilon_2 - \varepsilon_1}{\varepsilon_2}}{\frac{\varepsilon_2^2 - \varepsilon_1^2}{\varepsilon_2^{\cancel{2}}}} = \frac{\varepsilon_2\,\cancel{(\varepsilon_2 - \varepsilon_1)}}{(\varepsilon_2 + \varepsilon_1)\cancel{(\varepsilon_2 - \varepsilon_1)}} = \frac{\varepsilon_2}{\varepsilon_2 + \varepsilon_1}$$

$$= \frac{1}{1 + \varepsilon_1/\varepsilon_2} \Rightarrow \boxed{\sin\theta_B = \sqrt{\frac{1}{1 + \varepsilon_1/\varepsilon_2}}} \quad (34)$$

Note that for $\vec{E}\parallel$ polarization we can always find a Brewster angle given by (34) even though medium 1 & medium 2 are non-magnetic, i.e $\mu_1 = \mu_2$. This was not the case for $\vec{E}\perp$ polarization.