Family Name:	Given name:
Student number	Signature

Faculty of Applied Science and Engineering

ECE320 Fields and Waves

Second Test, November 20, 2006

Examiners – M. Mojahedi

Only Calculators approved by Registrar allowed
Answer the questions in the spaces provided or on the facing page
A complete paper consists of answers to all questions
For numerical answers specify units

DO NOT REMOVE STAPLE

Do not write in these spaces

1	2	3	TOTAL

$$\varepsilon_0 = 8.854\times10^{-12}\ [F/m],\quad \mu_0 = 4\pi\times10^{-7}\ [H/m],\quad c = 1\big/\sqrt{\varepsilon_0\ \mu_0} = 3\times10^{8}\ [m/s]$$

$$\iiint_v \nabla\cdot\vec{A}\ dv = \oiint_s \vec{A}\cdot\vec{ds}\ ,\ \iint_s \left(\nabla\times\vec{A}\right)\cdot\vec{ds} = \oint_c \vec{A}\cdot\vec{dl}$$

Problem 1: This question has 5 parts (a through e). In the following we are concerned with electro-dynamical fields in their instantaneous form.

a) Give Faraday's law in its differential form and derive its integral form. Identify the fields (symbols) in your equation by their names and units in SI system. **[7 pts]**

$$\text{FL: } \nabla \times \vec{E}(\vec{r},t) = -\frac{\partial}{\partial t}\vec{B}(\vec{r},t) \Rightarrow \iint_S \nabla \times \vec{E}(\vec{r},t)\cdot \vec{ds} = -\frac{\partial}{\partial t}\iint_S \vec{B}(\vec{r},t)\cdot \vec{ds}$$

$$\Rightarrow \oint_C \vec{E}\cdot \vec{dl} = -\frac{\partial}{\partial t}\iint_S \vec{B}\cdot \vec{ds} = -\frac{\partial}{\partial t}\Phi_B \text{ where } \Phi_B = \iint_S \vec{B}(\vec{r},t)\cdot \vec{ds}$$

$\vec{E} \triangleq$ electric field intensity $[V/m]$

$\vec{B} \triangleq$ magnetic flux density $\left[T = \frac{\text{Weber}}{m^2} = \frac{H\cdot A}{m^2} = \frac{V.s}{m^2}\right]$

b) Give Ampere's law in its differential form and derive its integral form. Identify the fields (symbols) in your equation by their names and units in SI system. **[7 Pts]**

$$\nabla \times \vec{H}(\vec{r},t) = \vec{J}(\vec{r},t) + \frac{\partial}{\partial t}\vec{D}(\vec{r},t)$$

$$\iint_S \nabla \times \vec{H}(\vec{r},t)\cdot \vec{ds} = \iint_S \vec{J}(\vec{r},t)\cdot \vec{ds} + \frac{\partial}{\partial t}\iint_S \vec{D}(\vec{r},t)\cdot \vec{ds} \Rightarrow$$

$$\oint_C \vec{H}\cdot \vec{dl} = \iint_S \vec{J}(\vec{r},t)\cdot \vec{ds} + \frac{\partial}{\partial t}\iint_S \vec{D}(\vec{r},t)\cdot \vec{ds} \quad \text{OR}$$

$$\oint_c \vec{H}\cdot \vec{dl} = I + \frac{\partial}{\partial t}\iint_S \vec{D}(\vec{r},t)\cdot \vec{ds} \quad \text{where } I = \iint \vec{J}(\vec{r},t)\cdot \vec{ds} \text{ is the}$$

conduction current

you may also write $\vec{J} = \sigma \vec{E}$

$\vec{H} \triangleq$ magnetic field intensity [A/m]

$J \triangleq$ Conduction current density [A/m²]

$\vec{D} \triangleq$ electric flux density or electric displacement [C/m²]

$\sigma \triangleq$ Conductivity $\left[\frac{1}{\Omega \cdot m}\right]$

c) Give Gauss's law for electric charges in its differential form and derive its integral form. Identify the fields (symbols) in your equation by their names and units in SI system. [7 pts]

$$\nabla \cdot \vec{D}(\vec{r},t) = \rho_v(\vec{r},t) \Rightarrow \iiint_v \nabla \cdot \vec{D}(\vec{r},t)\, dv = \iiint_v \rho_v(\vec{r},t)\, dv$$

$$\Rightarrow \oiint_S \vec{D} \cdot d\vec{s} = Q \quad \text{where} \quad Q = \iiint_v \rho_v(\vec{r},t)\, dv$$

$\vec{D} \triangleq$ is defined above

$\rho_v \triangleq$ volume charge density [C/m³]

$Q \triangleq$ total charge [C]

d) Show that the Gauss law for magnetic fields (the so called non-existence of magnetic monopoles) can be derived from one of the above laws. [7 pts]

We start with (1) $\nabla \times \vec{E}(\vec{r},t) = -\frac{\partial}{\partial t}\vec{B}(\vec{r},t) \Rightarrow$

(2) $\nabla \cdot (\nabla \times \vec{E}(\vec{r},t)) = -\frac{\partial}{\partial t}\nabla \cdot \vec{B}(\vec{r},t) \Rightarrow$ (3) $0 = -\frac{\partial}{\partial t}\nabla \cdot \vec{B}(\vec{r},t)$

since we require $\vec{B}$ to be a function of time (electrodynamical fields) the only way (3) can be true is for us to have

(4) $\boxed{\nabla \cdot \vec{B}(\vec{r},t) = 0}$

e) Derive the current continuity equation from Maxwell's equations. **[7 pts]**

We start with Ampere's law

(1) $\nabla \times \vec{H}(\vec{r},t) = \vec{J}(\vec{r},t) + \frac{\partial}{\partial t}\vec{D}(\vec{r},t) \Rightarrow$

(2) $\nabla \cdot (\nabla \times \vec{H}(\vec{r},t)) = \nabla \cdot \vec{J}(\vec{r},t) + \frac{\partial}{\partial t} \nabla \cdot \vec{D}(\vec{r},t) \Rightarrow$

(3) $0 = \nabla \cdot \vec{J}(\vec{r},t) + \frac{\partial}{\partial t} \nabla \cdot \vec{D}(\vec{r},t)$ but from

Gauss law (4) $\nabla \cdot \vec{D}(\vec{r},t) = \rho_v(\vec{r},t)$. use (4) in (3) $\Rightarrow$

(5) $\boxed{0 = \nabla \cdot \vec{J}(\vec{r},t) + \frac{\partial}{\partial t} \rho_v(\vec{r},t)}$

Problem 2: Consider a lossless transmission line.

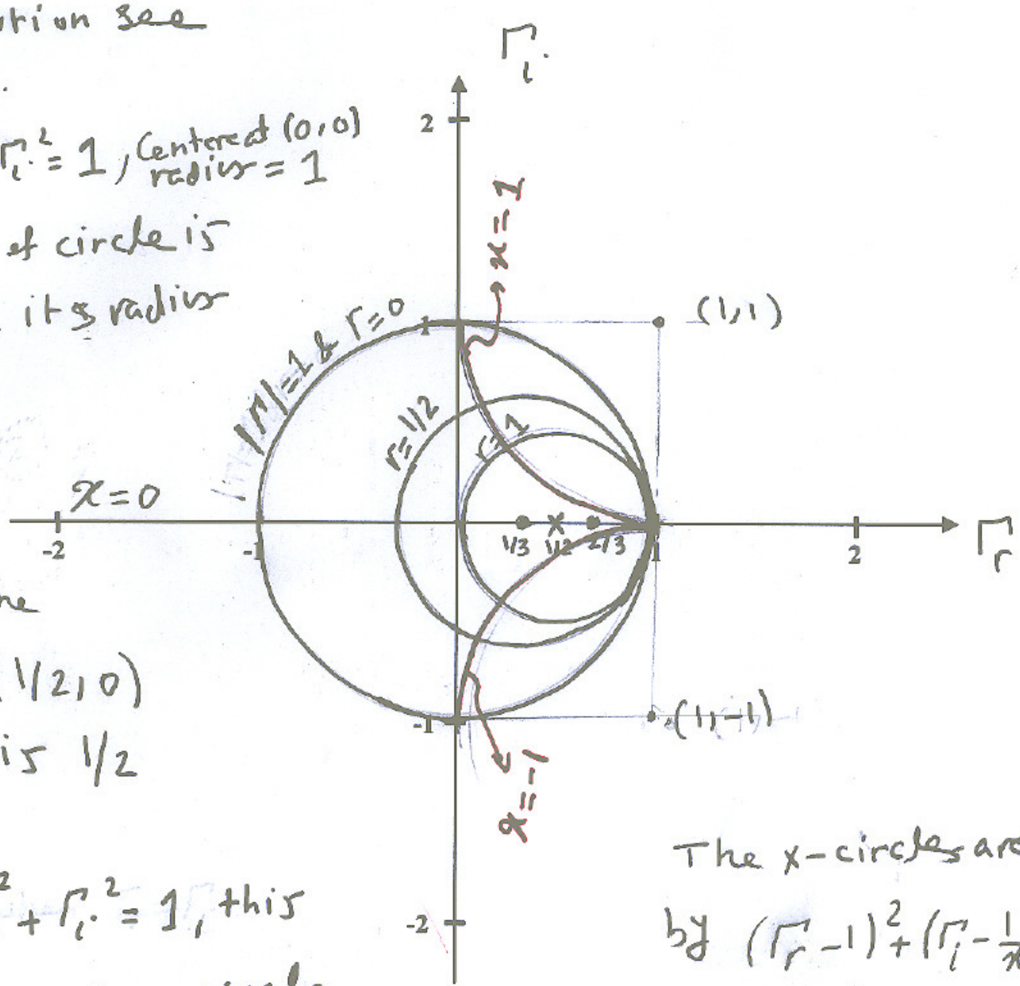
a) Derive the equation for *r-circles* in the Smith's chart. Identify the *r-circles'* center and radius. Show all your work. **[15 Pts]**

b) In the figure below draw and identify the following:

- $r = 0$ circle, $r = 0.5$ circle, and $r = 1$ circle . **[6 pts]**
- $|\Gamma| = 1$ circle. **[3 Pts]**
- $x = 0$ circle, $x = 1$ circle, and $x = -1$ circle. **[6 pts]**

For Part (a) solution see next page.

- For $r=0 \Rightarrow \Gamma_r^2 + \Gamma_i^2 = 1$, Centered $(0,0)$ radius $= 1$
- For $r = 1/2$ center of circle is at $(1/3, 0)$ & its radius is $2/3$
- For $r = 1$ the center of the circle is at $(1/2, 0)$ & its radius is $1/2$
- $|\Gamma| = 1 \Rightarrow \Gamma_r^2 + \Gamma_i^2 = 1$, this is the same as $r = 0$ circle



The x-circles are given by $(\Gamma_r - 1)^2 + (\Gamma_i - \frac{1}{x})^2 = (1/x)^2$ & are centered at $(1, 1/x)$ with radius $1/x$. Then

* $x = 0$ circle is the Γ_r-axis
* $x = 1$ is centered at $(1,1)$ with radius 1.
* $x = -1$ is centered at $(1,-1)$ & has radius of 1

The starting point is ① $\Gamma' = \dfrac{Z_L - R_0}{Z_L + R_0} \Rightarrow$ ② $Z_L = R_L + jX_L = R_0 \dfrac{1+\Gamma}{1-\Gamma'}$

(2) ⇒ ③ $\dfrac{Z_L}{R_0} = \dfrac{R_L}{R_0} + j\dfrac{X_L}{R_0} = r + jx = \dfrac{1 + \Gamma_r' + j\Gamma_i'}{1 - \Gamma_r' - j\Gamma_i'}$ where ④ $\Gamma' = \Gamma_r' + j\Gamma_i'$

(3) ⇒

$$r + jx = \frac{[(1+\Gamma_r') + j\Gamma_i'][(1-\Gamma_r') + j\Gamma_i']}{[(1-\Gamma_r') - j\Gamma_i'][(1-\Gamma_r') + j\Gamma_i']} \Rightarrow$$

$$r + jx = \frac{(1+\Gamma_r')(1-\Gamma_r') - \Gamma_i'^2 + j[\Gamma_i'(1-\Gamma_r') + \Gamma_i'(1+\Gamma_r')]}{(1-\Gamma_r')^2 + \Gamma_i'^2} \Rightarrow$$

⑤ $\boxed{r = \dfrac{1 - \Gamma_r'^2 - \Gamma_i'^2}{(1-\Gamma_r')^2 + \Gamma_i'^2}}$ & ⑥ $\boxed{x = \dfrac{2\Gamma_i'}{(1-\Gamma_r')^2 + \Gamma_i'^2}}$

we expand (5) ⇒

⑦ $r + r\Gamma_r'^2 - 2r\Gamma_r' + r\Gamma_i'^2 = 1 - \Gamma_r'^2 - \Gamma_i'^2 \Rightarrow$

⑧ $r\Gamma_r'^2 + \Gamma_r'^2 - 2r\Gamma_r' + r + r\Gamma_i'^2 + \Gamma_i'^2 = 1 \Rightarrow$

$\Gamma_r'^2(1+r) - 2r\Gamma_r' + r + \Gamma_i'^2(r+1) = 1 \Rightarrow$

$\Gamma_r'^2 - \dfrac{2r\Gamma_r'}{1+r} + \dfrac{r}{1+r} + \Gamma_i'^2 = \dfrac{1}{1+r}$ we add & subtract $\left(\dfrac{r}{1+r}\right)^2 \Rightarrow$

$\Gamma_r'^2 - \dfrac{2r}{1+r}\Gamma_r' + \left(\dfrac{r}{1+r}\right)^2 - \left(\dfrac{r}{1+r}\right)^2 + \dfrac{r}{1+r} + \Gamma_i'^2 = \dfrac{1}{1+r} \Rightarrow$

$\left(\Gamma_r' - \dfrac{r}{1+r}\right)^2 + \Gamma_i'^2 = \dfrac{1}{1+r} - \dfrac{r}{1+r} + \left(\dfrac{r}{1+r}\right)^2 \Rightarrow$

$$\left(\Gamma_r - \frac{r}{1+r}\right)^2 + \Gamma_i^2 = \frac{1-r}{1+r} + \frac{r^2}{(1+r)^2}$$

$$= \frac{1-r^2+r}{(1+r)^2}$$

$$= \frac{1}{(1+r)^2} \Rightarrow$$

(9)

$$\boxed{\left(\Gamma_r - \frac{r}{1+r}\right)^2 + \Gamma_i^2 = \frac{1}{(1+r)^2}}$$ Eq of r-circles

Center is at $\left(\frac{r}{1+r}, 0\right)$

radius is $\frac{1}{1+r}$

Problem 3) The electric field of an electromagnetic wave in vacuum given by

$$\vec{E}(\vec{r},t) = \hat{a}_x\ E_0 \cos\left[10^8\ \pi\left(t - \frac{z}{c}\right) + \theta\right],$$

is the sum of

$$\vec{E}_1(\vec{r},t) = \hat{a}_x\ \left(\sqrt{22} - 3\right)\sin\left[10^8\ \pi\left(t - \frac{z}{c}\right)\right],$$

and

$$\vec{E}_2(\vec{r},t) = \hat{a}_x\ 2\sqrt{3}\cos\left[10^8\ \pi\left(t - \frac{z}{c}\right) - \frac{\pi}{3}\right],$$

where c is the speed of light in vacuum.

a) What are the numerical values of E_0 and θ? **[15 pts]**

b) Find the expression for $\vec{B}(\vec{r},t)$. (Show all of your work) **[15 pts]**

c) What is the expression for $\vec{H}(\vec{r},t)$? **[5 pts]**

In phasor form $\vec{E}_1(\vec{r})$, $\vec{E}_2(\vec{r})$, and $\vec{E}(\vec{r})$ are given by

$$\vec{E}_1(\vec{r}) = \hat{a}_x (\sqrt{22} - 3)\, e^{-j10^8\pi\frac{z}{c}}\, e^{-j\pi/2} \quad ① \ \&$$

$$\vec{E}_2(\vec{r}) = \hat{a}_x\, 2\sqrt{3}\, e^{-j10^8\pi\frac{z}{c}}\, e^{-j\pi/3} \quad ②$$

$$\vec{E}(\vec{r}) = \hat{a}_x\, E_0\, e^{-j10^8\pi\frac{z}{c}}\, e^{j\theta} \quad ③$$

since $\vec{E}(\vec{r}) = \vec{E}_1(\vec{r}) + \vec{E}_2(\vec{r})$ ④ then

$$⑤ \quad E_0 \cancel{e^{-j10^8\pi\frac{z}{c}}} e^{j\theta} = (\sqrt{22} - 3)\cancel{e^{-j10^8\pi\frac{z}{3}}} e^{-j\pi/2} + 2\sqrt{3}\cancel{e^{-j10^8\pi\frac{z}{c}}} e^{-j\pi/3} \Rightarrow$$

$$⑥ \quad E_0 e^{j\theta} = (\sqrt{22} - 3)\, e^{-j\pi/2} + 2\sqrt{3}\, e^{-j\pi/3} \Rightarrow$$

$$= (\sqrt{22} - 3)[\cos(-\pi/2) - j\sin(\pi/2)] + 2\sqrt{3}[\cos(\pi/3) - j\sin(\pi/3)]$$

$$\Rightarrow E_0 e^{j\theta} = -j(\sqrt{22} - 3) + 2\sqrt{3}\left[\frac{1}{2} - j\frac{\sqrt{3}}{2}\right] \quad ⑦$$

$$= -j(\sqrt{22} - 3) + \sqrt{3} - j3$$

$$= -j\sqrt{22} + \cancel{3j} + \sqrt{3} - \cancel{3j}$$

$$E_0 e^{j\theta} = \sqrt{3} - j\sqrt{22} \quad ⑧$$

$\underline{\underline{2}}$

(9) $E_0 e^{j\theta} = \sqrt{(\sqrt{3})^2+(\sqrt{22})^2}\; e^{j\tan^{-1}\frac{-\sqrt{22}}{\sqrt{3}}} \Rightarrow$

$E_0 e^{j\theta} = \sqrt{25}\; e^{j\tan^{-1} -\sqrt{\frac{22}{3}}} \Rightarrow$

(10) $E_0 e^{j\theta} = 5\, e^{-j69.73^\circ} \Rightarrow$ (11) $\boxed{E_0=5}$ (12) $\boxed{\theta=-69.73^\circ}$

b) From (13) $\nabla\times\vec{E} = -j\omega\vec{B} \Rightarrow$ (14) $\vec{B}(\vec{r}) = \dfrac{\nabla\times\vec{E}}{-j\omega} \Rightarrow$

(15) $\vec{B}(\vec{r}) = \dfrac{1}{-j\omega}\begin{vmatrix} \hat{a_x} & \hat{a_y} & \hat{a_z}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ E_x & 0 & 0\end{vmatrix}$ where $E_x = E_0 e^{-j10^8\pi\frac{z}{c}} e^{j\theta}$
$= 5 e^{-j10^8\pi\frac{z}{c}} e^{-j69.73^\circ}$

then (16)

$\vec{B}(\vec{r}) = \dfrac{1}{-j\omega}\left\{\hat{a_y}\frac{\partial}{\partial z}E_x - \hat{a_z}\cancelto{0}{\frac{\partial}{\partial y}E_x}\right\} = \dfrac{\hat{a_y}}{-j\omega}\frac{\partial}{\partial z}E_x \Rightarrow$

(17) $\vec{B}(\vec{r}) = \dfrac{\hat{a_y}}{\cancel{-}j\omega}\left[E_0(\cancel{-}j10^8\pi)\frac{1}{c}e^{-j10^8\pi\frac{z}{c}}e^{j\theta}\right] \Rightarrow$

(18) $\vec{B}(\vec{r}) = \hat{a_y}\dfrac{E_0 10^8\pi}{\omega c} e^{-j10^8\pi\frac{z}{c}} e^{j\theta} = \hat{a_y}\dfrac{5\times10^8\pi}{\omega c} e^{-j10^8\pi\frac{z}{c}} e^{-j69.73^\circ}$

From definition of $\vec{E}$ or $\vec{E_1}$ or $\vec{E_2}$ it is clear that

(19) $\vec{B}(\vec{r},t) = Re\left[\hat{a_y}\dfrac{5\times10^8\pi}{\omega c} e^{-j10^8\pi\frac{z}{c}} e^{-j69.73^\circ} e^{j\omega t}\right] \Rightarrow$

(20) $\vec{B}(\vec{r},t) = \hat{a_y}\dfrac{5\times10^8\pi}{\omega c}\cos\left(\omega t - 10^8\pi\frac{z}{c} - 69.73^\circ\right)$

From definition of $\vec{E}$, $\vec{E_1}$ & $\vec{E_2}$ it is clear that

$\boxed{\omega = 10^8\pi \text{ rad/m}}$ (21)

using (21) in (20)

$$\vec{B}(\vec{r},t)=\hat{a}_y \frac{5\times\cancel{10^8\pi}}{\cancel{10^8\pi}\, c}\cos\left(10^8\pi t-10^8\pi\frac{3}{c}-69.73^\circ\right)$$

$$\boxed{\begin{array}{l}\vec{B}(\vec{r},t)=\hat{a}_y\frac{5}{c}\cos\left(10^8\pi\left(t-\frac{3}{c}\right)-69.73^\circ\right)\\ \vec{B}(\vec{r},t)=\hat{a}_y\, 1.667\times10^{-8}\cos\left(10^8\pi\left(t-\frac{3}{3\times10^8}\right)-69.73^\circ\right)\end{array}} \quad (22)$$

$c=3\times10^8$ m/s

c) $\vec{B}(\vec{r},t)=\mu\vec{H}(\vec{r},t) \Rightarrow \vec{H}(\vec{r},t)=\frac{\vec{B}(\vec{r},t)}{\mu}$ (23) For vacuum $\mu=\mu_0 \Rightarrow$

$$\boxed{\begin{array}{l}\vec{H}(\vec{r},t)=\hat{a}_y\frac{5}{\mu_0 c}\cos\left(10^8\pi\left(t-\frac{3}{c}\right)-69.73^\circ\right)\\ \vec{H}(\vec{r},t)=\hat{a}_y\, 0.013\cos\left(10^8\pi\left(t-\frac{3}{3\times10^8}\right)-69.73^\circ\right)\end{array}} \quad (24)$$

$\mu_0=4\pi\times10^{-7}$ H/m

• Note we could have found $\vec{H}$ from $\vec{H}=\frac{1}{\eta_0}\hat{a}_k\times\vec{E}$ where $\hat{a}_k=\hat{a}_z$ then

(25) $\vec{H}(\vec{r},t)=\frac{\sqrt{\epsilon_0}}{\sqrt{\mu_0}}\hat{a}_z\times\hat{a}_x E_0\cos\left(10^8\pi\left(t-\frac{3}{c}\right)+\theta\right)=\frac{\sqrt{\epsilon_0}\sqrt{\mu_0}}{\mu_0}\hat{a}_y E_0\cos\left(10^8\pi\left(t-\frac{3}{c}\right)+\theta\right)$

$$\boxed{\vec{H}(\vec{r},t)=\frac{\hat{a}_y}{\mu_0 c}5\cos\left(10^8\pi\left(t-\frac{3}{c}\right)-69.73^\circ\right)} \quad (26)$$ which is the same as (24).

Consequently, $\vec{B}=\mu_0\vec{H}\Rightarrow \vec{B}(\vec{r},t)=\frac{\hat{a}_y\mu_0}{\mu_0 c}5\cos\left(10^8\pi\left(t-\frac{3}{c}\right)-69.73^\circ\right)\Rightarrow$

$$\boxed{\vec{B}(\vec{r},t)=\hat{a}_y\frac{5}{c}\cos\left(10^8\pi\left(t-\frac{3}{c}\right)-69.73^\circ\right)} \quad (27)$$ which is the same as (22).

using (21) in (20)

$$\vec{B}(\vec{r},t)=\hat{a}_y \frac{5\times\cancel{10^8\pi}}{\cancel{10^8\pi}\, c}\cos\left(10^8\pi t-10^8\pi\frac{3}{c}-69.73^\circ\right)$$

$$\boxed{\begin{aligned}\vec{B}(\vec{r},t)&=\hat{a}_y\frac{5}{c}\cos\left(10^8\pi\left(t-\frac{3}{c}\right)-69.73^\circ\right)\\ \vec{B}(\vec{r},t)&=\hat{a}_y\, 1.667\times10^{-8}\cos\left(10^8\pi\left(t-\frac{3}{3\times10^8}\right)-69.73^\circ\right)\end{aligned}}\quad (22)$$

$c=3\times10^8$ m/s

c) $\vec{B}(\vec{r},t)=\mu\vec{H}(\vec{r},t)\Rightarrow \vec{H}(\vec{r},t)=\dfrac{\vec{B}(\vec{r},t)}{\mu}$ (23) For vacuum $\mu=\mu_0\Rightarrow$

$$\boxed{\begin{aligned}\vec{H}(\vec{r},t)&=\hat{a}_y\frac{5}{\mu_0 c}\cos\left(10^8\pi\left(t-\frac{3}{c}\right)-69.73^\circ\right)\\ \vec{H}(\vec{r},t)&=\hat{a}_y\, 0.013\cos\left(10^8\pi\left(t-\frac{3}{3\times10^8}\right)-69.73^\circ\right)\end{aligned}}\quad (24)$$

$\mu_0=4\pi\times10^{-7}$ H/m

• Note we could have found $\vec{H}$ from $\vec{H}=\frac{1}{\eta_0}\hat{a}_k\times\vec{E}$ where $\hat{a}_k=\hat{a}_z$ then

(25) $\vec{H}(\vec{r},t)=\dfrac{\sqrt{\varepsilon_0}}{\sqrt{\mu_0}}\hat{a}_z\times\hat{a}_x E_0\cos\left(10^8\pi\left(t-\frac{3}{c}\right)+\theta\right)=\dfrac{\sqrt{\varepsilon_0}\sqrt{\mu_0}}{\mu_0}\hat{a}_y E_0\cos\left(10^8\pi\left(t-\frac{3}{c}\right)+\theta\right)$

$$\boxed{\vec{H}(\vec{r},t)=\frac{\hat{a}_y}{\mu_0 c}5\cos\left(10^8\pi\left(t-\frac{3}{c}\right)-69.73^\circ\right)}\quad (26)$$ which is the same as (24).

Consequently, $\vec{B}=\mu_0\vec{H}\Rightarrow \vec{B}(\vec{r},t)=\dfrac{\hat{a}_y\mu_0}{\mu_0 c}5\cos\left(10^8\pi\left(t-\frac{3}{c}\right)-69.73^\circ\right)\Rightarrow$

$$\boxed{\vec{B}(\vec{r},t)=\hat{a}_y\frac{5}{c}\cos\left(10^8\pi\left(t-\frac{3}{c}\right)-69.73^\circ\right)}\quad (27)$$ which is the same as (22).