Spherical Waves

1 The Helmholtz Wave Equation in Spherical Coordinates

In the previous section we reviewed the solution to the homogeneous wave (Helmholtz) equation in Cartesian coordinates, which yielded plane wave solutions. An equally important solution to the wave equation which we will encounter many times in this course in the *spherical wave*, which is solved by considering the Helmholtz equation written in spherical coordinates. This is a much more advanced topic, but we will try to elucidate the key form of the solution here. Later in the course, we will study particular solutions to the spherical wave equation, when we solve the nonhomogeneous version of the wave equation.

Let's rewrite the wave equation here as a reminder,

$$\nabla^2\psi + k^2\psi = 0. \tag{1}$$

For the time being, we consider the wave equation in terms of a scalar quantity ψ, rather than a vector field $\boldsymbol{E}$ or $\boldsymbol{H}$ as we did before. The reason is that a vector solution will be more appropriate when we study the solution to the nonhomogenous wave equation; here we only try to get a sense of what the solutions will look like by studying the scalar homogeneous equation.

We now expand the Laplacian operator in spherical coordinates, which is found in any electromagnetics textbook,

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2} + k^2\psi = 0. \tag{2}$$

It can be quite challenging to solve this equation because there are three spatial variables, r, θ, and ϕ.

2 Solution to the Wave Equation for a Point Source

We will primarily concern ourselves with the solution to this equation assuming a *point source* at the origin. A point source at the origin should produce a solution with radial symmetry, i.e. a wave that only varies with r and not with θ or ϕ. So, let us assume that

$$\frac{\partial\psi}{\partial\theta} = \frac{\partial\psi}{\partial\phi} = 0. \tag{3}$$

This reduces the wave equation to

$$\frac{1}{r^2}\frac{d}{\partial r}\left(r^2\frac{d\psi}{\partial r}\right) + k^2\psi = 0. \tag{4}$$

Multiplying both sides by r^2 yields

$$\frac{d}{\partial r}\left(r^2\frac{d\psi}{\partial r}\right) + k^2r^2\psi = 0. \tag{5}$$

Applying the product rule,

$$r^2\frac{d^2R}{dr^2} + 2r\frac{dR}{dr} + (kr)^2R = 0. \tag{6}$$

This equation is the same form as the the *spherical Bessel equation* which is

$$r^2\frac{d^2R}{dr^2} + 2r\frac{dR}{dr} + [r^2 - n(n+1)]R = 0. \tag{7}$$

whose solutions are *spherical Bessel functions*,

$$b_n^{(m)}(r) = \sqrt{\frac{\pi}{2r}}B_{n+1/2}^{(m)}(r), \tag{8}$$

where $B_n^{(m)}$ is a Bessel function of the mth kind. Comparing the spherical Bessel equation to our wave equation gives $n = 0$ and so the solutions to our equation must be combinations of Bessel functions $B_0^{(m)}$. Bessel functions are an advanced topic, beyond the scope of this course, but one of the solutions to the spherical Bessel equation (5) is

$$R(r) = B_0(kr) - jB_0^{(1)}(kr) = -\frac{e^{-jkr}}{jkr} \tag{9}$$

The time-harmonic equation associated with this solution is

$$R(r,t) = -\frac{\sin(\omega t - kr)}{kr} \tag{10}$$

which is a wave propagating in the $+r$ direction (radially away from the origin) whose amplitude decays according to $1/r$ but whose phase and amplitude are uniform for a fixed r. This is known as a *spherical wave*, and we will see it many times when analyzing antennas and the corresponding solutions to the nonhomogeneous wave equation.

A more straightforward approach to solving (4) can be undertaken by assuming ahead of time that the solution is of the form

$$R(r) = \frac{f(r)}{r}, \tag{11}$$

which reduces the equation to

$$\begin{aligned}
\frac{1}{r^2}\frac{d}{dr}\left[r^2\left(\frac{1}{r}\frac{df}{dr} - \frac{f}{r^2}\right)\right] + k^2\frac{f}{r} &= 0 \\
\frac{1}{r^2}\left[\frac{df}{dr} + r\frac{d^2f}{dr^2} - \frac{df}{dr}\right] + k^2\frac{f}{r} &= 0 \\
\frac{d^2f}{dr^2} + k^2f = 0. & \qquad (12)
\end{aligned}$$

One solution to this equation is

$$f = C_1e^{-jkr} \tag{13}$$

so

$$R(r) = C_1\frac{e^{-jkr}}{r} \tag{14}$$

which is of the same form of (9). This is the spherical wave solution we use in this course, but bear in mind that in general the solution to (4) are the spherical Bessel functions described by (8).

3 General Solution

While beyond the scope of this course, you may wonder what happens if we relax the assumption of radial symmetry in the solution, i.e. (3) no longer holds. Then, we must solve the entire wave equation, repeated here as

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}+k^2\psi=0. \tag{15}$$

To make the solution easier, we invoke the separation of variables method and let the solution be given by

$$\psi=R(r)\Theta(\theta)\Phi(\phi). \tag{16}$$

Substituting this into (15) gives

$$\frac{\Phi\Theta}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\frac{R\Phi}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+\frac{R\Theta}{r^2\sin^2\theta}\frac{d^2\Phi}{d\phi^2}+k^2R\Theta\Phi=0. \tag{17}$$

Dividing by $R\Theta\Phi$ and multiplying by $r^2\sin^2\theta$ gives

$$\frac{\sin^2\theta}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}+k^2r^2\sin^2\theta=0. \tag{18}$$

With the ϕ dependence separated out, we let the separation constant be set so that

$$\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}=-m^2. \tag{19}$$

Subsituting this into the wave equation, and dividing by $\sin^2\theta$,

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)-\frac{m^2}{\sin^2\theta}+k^2r^2=0. \tag{20}$$

Now we see the r and θ dependence have been properly separated. For the next separation constant, we make the seemingly odd choice of separation constant n such that

$$\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)-\frac{m^2}{\sin^2\theta}=-n(n+1). \tag{21}$$

With this choice, the wave equation becomes

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)-n(n+1)+k^2r^2=0, \tag{22}$$

which completes the separation of variables. We have now formed a set of separated equations

$$\begin{aligned}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+[(kr)^2-n(n+1)]R &= 0 \qquad (23)\\ \frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+\left[n(n+1)+\frac{m^2}{\sin^2\theta}\right]\Theta &= 0 \qquad (24)\\ \frac{d^2\Phi}{d\phi^2}+m^2\Phi &= 0. \qquad (25)\end{aligned}$$

We have already solved the first equation, which is the spherical Bessel equation. The last equation is familiar to us, and we know it solutions are of the form

$$\Phi(\phi) = e^{\pm jm\phi}. \tag{26}$$

The solution to the second equation is actually known as a *Legendre function* given by $L_n^m(\cos\theta)$. This is an advanced topic and not treated here.

According to (16), the solution to the wave equation is actually a product of all the three solutions presented here. That is,

$$\psi = -\frac{e^{-jkr}}{jkr} L_n^m(\cos\theta) e^{\pm jm\phi} \tag{27}$$

This is the general solution to the homogenous wave equation in spherical coordinates.