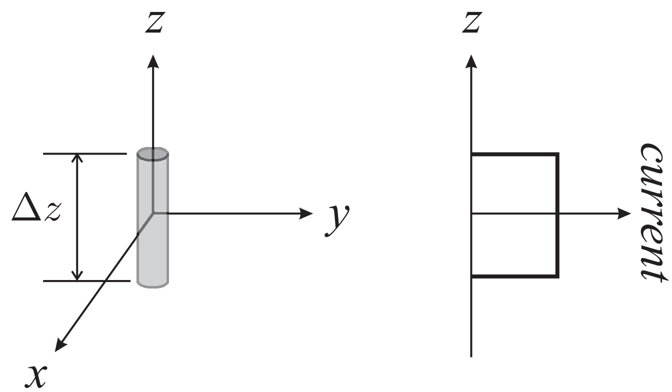
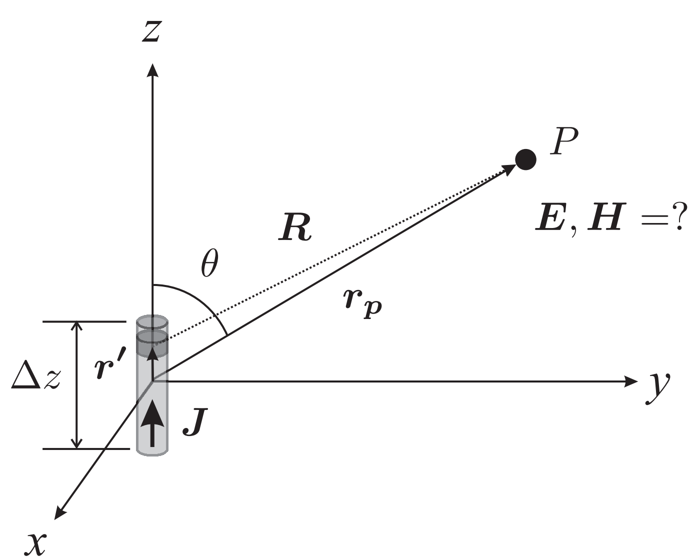
Ideal (Hertzian) Dipole

A very simple radiating element we can study is the *ideal dipole*, also known as Hertzian dipole and infinitesimal dipole. It is very short (length $\ll \lambda$), and as such has current *uniformly distributed* along its length.



Although it is difficult to implement in practice (having a current distribution that is difficult to realize since it is discontinuous), it is highly useful for helping analyze larger wire antennas which can be subdivided into short sections having uniform current (i.e., ideal dipoles). Then, much in the same way as we derived vector potential for a continuous current distribution, we can use superposition to find the fields of a long wire antenna.

Let's orient the ideal dipole along the z-axis and denote the current flowing through the dipole as I. The current has an associated surface current density $\boldsymbol{J}$.



In this illustration, R is the distance from the current element to the field point P, and r is the distance from the origin to P.

First, we need to derive the vector potential of the line source. It is a continuous current distribution over its length $\Delta\ell = \Delta z$. Since we only have a z-component of current, $\boldsymbol{A}$ will only have a z-component as well.

Recall

$$\boldsymbol{A} = \int_V \mu \boldsymbol{J} \frac{e^{-jkR}}{4\pi R} dv' = \iiint \mu \boldsymbol{J} \frac{e^{-jkR}}{4\pi R} dx' dy' dz' \tag{1}$$

in Cartesian coordinates. Here,

$$\boldsymbol{J}(\boldsymbol{r}') = \begin{cases} I_0 \delta(x')\delta(y')\hat{\boldsymbol{z}} & \Delta z/2 < z' < \Delta z/2 \\ 0 & \text{elsewhere} \end{cases} \tag{2}$$

since the dipole is infinitely thin. Therefore,

$$\begin{aligned} \boldsymbol{A} &= \hat{\boldsymbol{z}} \mu I_0 \int_{-\infty}^{\infty} \delta(x') dx' \int_{-\infty}^{\infty} \delta(y') dy' \int_{\Delta z/2}^{\Delta z/2} \frac{e^{-jkR}}{4\pi R} dz' && (3) \\ &= \hat{\boldsymbol{z}} \mu I_0 \int_{\Delta z/2}^{\Delta z/2} \frac{e^{-jkR}}{4\pi R} dz'. && (4) \end{aligned}$$

Evaluating the integral, we first notice that since Δz is small, R does not change significantly as we move along the length of the dipole, (i.e. $r \approx R$). So we can effectively say that R is not a function of z', making the integral simple to evaluate:

$$\boldsymbol{A} = \hat{\boldsymbol{z}} \mu I_0 \frac{e^{-jkr}}{4\pi r} \int_{\Delta z/2}^{\Delta z/2} dz' = \frac{\mu I_0 e^{-jkr}}{4\pi r} \Delta z \, \hat{\boldsymbol{z}}. \tag{5}$$

Now we can find the radiated magnetic field of the dipole:

$$\boldsymbol{H} = \frac{1}{\mu} \boldsymbol{\nabla} \times \boldsymbol{A} = \frac{1}{\mu} \boldsymbol{\nabla} \times A_z \, \hat{\boldsymbol{z}}. \tag{6}$$

Since we know the analysis of point sources revealed spherical wave solutions, it is best to evaluate this curl in spherical coordinates. So first we need to convert $\boldsymbol{A}$ to spherical coordinates:

$$\begin{aligned} A_r &= \boldsymbol{A} \cdot \hat{\boldsymbol{r}} = A_z \, \hat{\boldsymbol{z}} \cdot \hat{\boldsymbol{r}} = A_z \cos\theta && (7) \\ A_\theta &= \boldsymbol{A} \cdot \hat{\boldsymbol{\theta}} = A_z \, \hat{\boldsymbol{z}} \cdot \hat{\boldsymbol{\theta}} = -A_z \sin\theta && (8) \\ A_\phi &= \boldsymbol{A} \cdot \hat{\boldsymbol{\phi}} = A_z \, \hat{\boldsymbol{z}} \cdot \hat{\boldsymbol{\phi}} = 0. && (9) \end{aligned}$$

Reminder: curl in spherical coordinates is

$$\boldsymbol{\nabla} \times \boldsymbol{A} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{\boldsymbol{r}} & r\,\hat{\boldsymbol{\theta}} & r\sin\theta\,\hat{\boldsymbol{\phi}} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_r & rA_\theta & r\sin\theta A_\phi \end{vmatrix} \tag{10}$$

$$\nabla \times \boldsymbol{A} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{\boldsymbol{r}} & r\,\hat{\boldsymbol{\theta}} & r\sin\theta\,\hat{\boldsymbol{\phi}} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ \frac{\mu I_0 e^{-jkr}}{4\pi r}\Delta z\cos\theta & -\frac{\mu I_0 e^{-jkr}}{4\pi}\Delta z \sin\theta & 0 \end{vmatrix} \tag{11}$$

$$\begin{aligned} &= \frac{1}{r^2\sin\theta}\left\{\hat{\boldsymbol{r}}\left[\frac{\partial}{\partial\theta}(0) + \frac{\partial}{\partial\phi}\frac{\mu I_0 e^{-jkr}}{4\pi}\Delta z\sin\theta\right] - \right. \\ &\qquad r\,\hat{\boldsymbol{\theta}}\left[\frac{\partial}{\partial r}(0) - \frac{\partial}{\partial\phi}\frac{\mu I_0 e^{-jkr}}{4\pi r}\Delta z\cos\theta\right] + \\ &\qquad \left. r\sin\theta\,\hat{\boldsymbol{\phi}}\left[-\frac{\partial}{\partial r}\frac{\mu I_0 e^{-jkr}}{4\pi}\Delta z\sin\theta - \frac{\partial}{\partial\theta}\frac{\mu I_0 e^{-jkr}}{4\pi r}\Delta z\cos\theta\right]\right\} \qquad (12) \\ &= \frac{1}{r^2\sin\theta}\cdot r\sin\theta\,\hat{\boldsymbol{\phi}}\left[\frac{jk\mu I_0 e^{-jkr}}{4\pi}\Delta z\sin\theta + \frac{\mu I_0 e^{-jkr}}{4\pi r}\Delta z\sin\theta\right] \qquad (13) \\ &= \frac{\mu I_0 \Delta z e^{-jkr}}{4\pi}\sin\theta\left(\frac{jk}{r} + \frac{1}{r^2}\right)\hat{\boldsymbol{\phi}}. \qquad (14) \end{aligned}$$

Now,

$$\begin{aligned} \boldsymbol{H} &= \frac{1}{\mu}\nabla\times\boldsymbol{A} = \frac{I_0\Delta z}{4\pi}\left(\frac{jk}{r} + \frac{1}{r^2}\right)e^{-jkr}\sin\theta\,\hat{\boldsymbol{\phi}} \qquad (15) \\ &= \frac{I_0\Delta z}{4\pi}jk\left(1 + \frac{1}{jkr}\right)\frac{e^{-jkr}}{r}\sin\theta\,\hat{\boldsymbol{\phi}}. \qquad (16) \end{aligned}$$

Next, we find the electric field from Maxwell's curl equation:

$$\boldsymbol{E} = \frac{1}{j\omega\varepsilon}\nabla\times\boldsymbol{H} \tag{17}$$

$$\nabla\times\boldsymbol{H} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{\boldsymbol{r}} & r\,\hat{\boldsymbol{\theta}} & r\sin\theta\,\hat{\boldsymbol{\phi}} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ H_r = 0 & H_\theta = 0 & r\sin\theta\frac{I_0\Delta z}{4\pi}\left(\frac{jk}{r}+\frac{1}{r^2}\right)e^{-jkr}\sin\theta \end{vmatrix}. \tag{18}$$

Evaluating the curl in the same manner as for the magnetic field case, we arrive at the final solution for $\boldsymbol{E}$:

$$\boldsymbol{E} = \frac{I_0\Delta z}{2\pi}\eta\left(\frac{1}{r} - \frac{j}{kr^2}\right)\frac{e^{-jkr}}{r}\cos\theta\,\hat{\boldsymbol{r}} + \frac{I_0\Delta z j\omega\mu}{4\pi}\left[1 + \frac{1}{jkr} - \frac{1}{(kr)^2}\right]\frac{e^{-jkr}}{r}\sin\theta\,\hat{\boldsymbol{\theta}}. \tag{19}$$

Now, let's interpret the meaning of all these fields. The first situation we wish to consider is the so-called *far field* of the antenna, which is analytically defined as when r is large $(r \gg \lambda)$[1]. Then, all the terms with r in the denominator tend to zero, and we are left with

$$\begin{aligned} \boldsymbol{E}_{\text{ff}} &= \frac{I_0\Delta z j\omega\mu}{4\pi}\frac{e^{-jkr}}{r}\sin\theta\,\hat{\boldsymbol{\theta}} \qquad (20) \\ \boldsymbol{H}_{\text{ff}} &= \frac{I_0\Delta z}{4\pi}jk\frac{e^{-jkr}}{r}\sin\theta\,\hat{\boldsymbol{\phi}}. \qquad (21) \end{aligned}$$

[1]We will study another definition of where the far field is defined later.

Some important observations:

- $\boldsymbol{E}$ no longer has a radial component; in the far field, it is totally polarized in the $\hat{\boldsymbol{\theta}}$ direction;
- $\boldsymbol{E}$ and $\boldsymbol{H}$ are orthogonal to each other and the direction of propagation and hence the resulting wave is TEM (as we expect for a spherical wave);
- The ratio of E_θ/H_ϕ is
$$\frac{E_\theta}{H_\phi} = \frac{\omega\mu}{k} = \sqrt{\frac{\mu}{\varepsilon}} = \eta \tag{22}$$
which is also what we found for a plane wave. We shall see that this is a property of radiated fields.

What is the power radiated by the antenna? First we compute the Poynting vector of the far fields components,

$$\boldsymbol{P} = \frac{1}{2}\boldsymbol{E} \times \boldsymbol{H}^* = \frac{1}{2}E_\theta H_\phi^* \,\hat{\boldsymbol{r}}, \tag{23}$$

since $\boldsymbol{E}$ and $\boldsymbol{H}$ are orthogonal ($\hat{\boldsymbol{\theta}} \times \hat{\boldsymbol{\phi}} = \hat{\boldsymbol{r}}$). Then,

$$\begin{aligned} P_r &= \frac{1}{2}\frac{I_0\Delta z j\omega\mu}{4\pi}\frac{e^{-jkr}}{r}\sin\theta \cdot \frac{I_0\Delta z}{4\pi}(-jk)\frac{e^{jkr}}{r}\sin\theta & (24) \\ \boldsymbol{P} &= \frac{I_0^2\Delta z^2\omega\mu k}{2(4\pi r)^2}\sin^2\theta\,\hat{\boldsymbol{r}}. & (25) \end{aligned}$$

An important observation is that $\boldsymbol{P}$ rolls off as $1/r^2$, indicating that a square-law in power density with distance (i.e. double the distances gives quadruple the loss [-6 dB]). Now we surround the dipole with an imaginary sphere of radius r and compute the power by taking the surface integral of the (radiated) power density:

$$\begin{aligned} W_{rad} &= \int_S \boldsymbol{P} \cdot d\boldsymbol{s}' = \int_0^\pi \int_0^{2\pi} \boldsymbol{S} \cdot r^2 \sin\theta\,\hat{\boldsymbol{r}} d\phi d\theta & (26) \\ &= 2\pi \int_0^\pi \left(\frac{I_0\Delta z}{4\pi}\right)^2 \frac{\omega\mu k}{2}\sin^3\theta d\theta & (27) \\ &= \frac{(I_0\Delta z)^2}{12\pi}\omega\mu k, & (28) \end{aligned}$$

where we note that $\int_0^\pi \sin^3\theta d\theta = 4/3$. Since P is real, it is *dissipated* or radiated power (versus stored [imaginary] power).

Let's focus on the structure of the electric field expression in the far field for a moment, since the magnetic field is readily computed knowing the intrinsic impedance of the medium. We observe that the electric field can be expressed as follows:

$$\boldsymbol{E} = \underbrace{\frac{I_0\Delta z}{4\pi} j\omega\mu}_{\text{strength factor}} \cdot \underbrace{\frac{e^{-jkr}}{r}}_{\text{distance factor}} \cdot \underbrace{\sin\theta}_{\text{shape/element factor}} \cdot \hat{\boldsymbol{\theta}}. \tag{29}$$

The expression can be separated into the product of three components:

- Strength factor – determined solely by material parameters, magnitude of excitation current, and dipole length
- Distance factor – purely the amplitude decay and phase shift incurred with distance
- Shape factor – determined the radiation pattern of the antenna, or the part that is a function of θ, ϕ.

At this point it is worth comparing the far-field electric and magnetic fields to the vector potential in (5). Notice that in the far field, $E_\theta = -j\omega A_\theta$. The dipole only radiates a θ-polarized E-field, but it can be shown that if it radiated in the ϕ-polarization as well, in the far field, $E_\phi = -j\omega A_\phi$. Also, there is no radial component of $\boldsymbol{E}$ in the far field, nor is there a radial component in the vector potential. Hence, *just for far-field electric and magnetic fields*, we can say:

$$\begin{aligned} \boldsymbol{E}_{\text{ff}} &\approx -j\omega \boldsymbol{A} & (30) \\ \boldsymbol{H}_{\text{ff}} &\approx \frac{\hat{\boldsymbol{r}}}{\eta} \times \boldsymbol{E}_{\text{ff}} = -j\frac{\omega}{\eta}\hat{\boldsymbol{r}} \times \boldsymbol{A}. & (31) \end{aligned}$$

These equations form a fast an easy way to determine the far-field radiated electric field, without going through two curl operations as we had to do before.

We have considered the far field quantities to this point. What about the other fields? Since they are not in the far field, they are in the so-called *near field* of the antenna, or where $r \ll \lambda$. Examining the expressions for $\boldsymbol{E}$ and $\boldsymbol{H}$, under this condition the $1/r^n$ terms dominate and we have:

$$\begin{aligned} \boldsymbol{H}_{\text{nf}} &= \frac{I_0 \Delta z e^{-jkr}}{4\pi jkr^2} jk \sin\theta\, \hat{\boldsymbol{\phi}} = \frac{I_0 \Delta z e^{-jkr}}{4\pi r^2} \sin\theta\, \hat{\boldsymbol{\phi}} & (32) \\ \boldsymbol{E}_{\text{nf}} &= \frac{I_0 \Delta z}{4\pi} j\omega\mu \left[\frac{1}{jkr} - \frac{1}{(kr)^2}\right] \frac{e^{-jkr}}{r} \sin\theta\, \hat{\boldsymbol{\theta}} + \\ & \quad \frac{I_0 \Delta z}{2\pi} \eta \left[\frac{1}{r} - j\frac{1}{kr^2}\right] \frac{e^{-jkr}}{r} \cos\theta\, \hat{\boldsymbol{r}}. & (33) \end{aligned}$$

In the expression for $\boldsymbol{E}_{\text{nf}}$, the $1/r^3$ terms dominate for small r, so

$$\boldsymbol{E}_{\text{nf}} = \frac{-I_0 \Delta z}{4\pi} j\omega\mu \frac{e^{-jkr}}{k^2 r^3} \sin\theta\, \hat{\boldsymbol{\theta}} - j\frac{I_0 \Delta z}{2\pi} \eta \frac{e^{-jkr}}{kr^3} \cos\theta\, \hat{\boldsymbol{r}}. \quad (34)$$

Since $\omega\mu/k = \eta$,

$$\boldsymbol{E}_{\text{nf}} = \frac{-jI_0 \Delta z}{4\pi k} \eta \frac{e^{-jkr}}{r^3} \sin\theta\, \hat{\boldsymbol{\theta}} - j\frac{I_0 \Delta z}{2\pi} \eta \frac{e^{-jkr}}{kr^3} \cos\theta\, \hat{\boldsymbol{r}}. \quad (35)$$

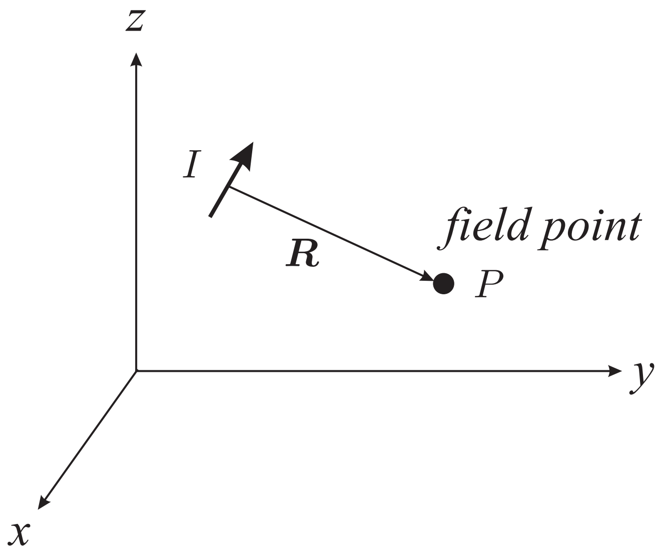
What is the practical significance of these fields? We make two important observations with comparison to familiar fields from statics:

- H_ϕ^{nf}: With the exception of the e^{-jkr} term, this expression very closely resembles the field of a static current element along the z-axis.

- E_θ^{nf}: With the exception of the e^{-jkr}/k term, this expression resembles the static field of an electric dipole.

Let's discuss these observations by examining/deriving the static fields in question. Starting with the first observation, recall from the Biot-Savart Law that for a short (infinitesimal) segment of current, the magnetic field intensity produced is

$$d\boldsymbol{H} = \frac{I_0 d\boldsymbol{\ell} \times \boldsymbol{R}}{4\pi R^3}. \tag{36}$$



For a current element at the origin, $\boldsymbol{R} = r\,\hat{\boldsymbol{r}}$ and

$$d\boldsymbol{H} = \frac{I_0 d\boldsymbol{\ell} \times \hat{\boldsymbol{r}}}{4\pi R^2} \Rightarrow \Delta\boldsymbol{H} = \frac{I_0 \Delta\boldsymbol{\ell} \times \hat{\boldsymbol{r}}}{4\pi R^2}. \tag{37}$$

Since

$$\Delta\boldsymbol{\ell} \times \hat{\boldsymbol{r}} = \Delta\ell\;\hat{\boldsymbol{z}} \times \hat{\boldsymbol{r}} = \begin{vmatrix} \hat{\boldsymbol{r}} & \hat{\boldsymbol{\theta}} & \hat{\boldsymbol{\phi}} \\ \cos\theta & -\sin\theta & 0 \\ 1 & 0 & 0 \end{vmatrix} = \sin\theta\,\hat{\boldsymbol{\phi}}, \tag{38}$$

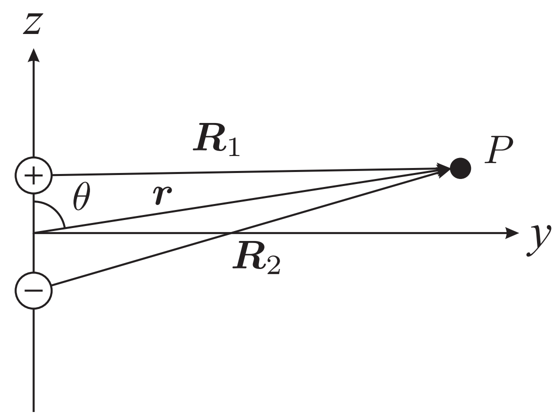
$$\Delta\boldsymbol{H} = \frac{I_0\Delta\ell}{4\pi r^2}\sin\theta\,\hat{\boldsymbol{\phi}}. \tag{39}$$

Furthermore, if $\omega \to 0$ (frequency approaches DC), $k \to 0$ and the expression for $\boldsymbol{H}_{\mathsf{nf}}$ becomes identical to the induction field derived above. Therefore, we can say that *the magnetic field in the near field of a dipole resembles that of a steady magnetic field (induction field) produced by the element.*

Now for the electric field term. From electrostatics you may recall analyzing an electric dipole:

The potential at point P is equal to

$$V = \frac{Q}{4\pi\varepsilon}\left(\frac{1}{R_1} - \frac{1}{R_2}\right) = \frac{Q}{4\pi\varepsilon}\frac{R_2 - R_1}{R_2 R_1}. \tag{40}$$



For sufficiently distant P, $R_2 - R_1 = \Delta z \cos\theta$, $R \approx r$,

$$V = \frac{Q\Delta z \cos\theta}{4\pi\varepsilon r^2}, \tag{41}$$

and

$$\boldsymbol{E} = -\boldsymbol{\nabla} V = \frac{Q\Delta z\cos\theta}{2\pi\varepsilon r^3}\,\hat{\boldsymbol{r}} + \frac{Q\Delta z\sin\theta}{4\pi\varepsilon r^3}\,\hat{\boldsymbol{\theta}}. \tag{42}$$

We notice a remarkable between this expression and $\boldsymbol{E}_{\mathsf{nf}}$ without the exponential term:

$$\frac{-jI_0\Delta z\eta}{2\pi k r^3}\cos\theta\,\hat{\boldsymbol{r}} - \frac{jI_0\Delta z\eta}{4\pi k r^3}\sin\theta\,\hat{\boldsymbol{\theta}} = \frac{1}{j\omega}\left(\frac{I_0\Delta z}{2\pi\varepsilon r^3}\cos\theta\,\hat{\boldsymbol{r}} - \frac{I_0\Delta z}{4\pi\varepsilon r^3}\sin\theta\,\hat{\boldsymbol{\theta}}\right). \tag{43}$$

Since in real-time notation $1/j\omega$ represents a time integral, and $\int I\,dt = Q$, this expression identically equals the static dipole field derived above! So, *the electric field in the near field resembles the static field of an electric dipole of the same length as the antenna dipole*.

We can make a conclusion that the fields close in to a Hertzian dipole resemble the fields produced in static conditions.

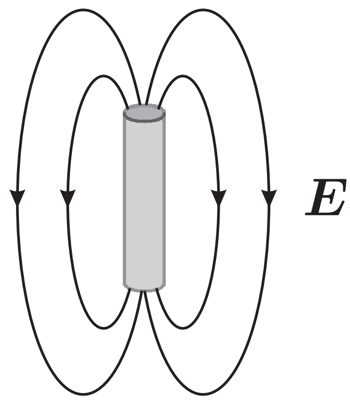
We can also show that these fields are *storing energy*: both E-field components are in *phase quadrature* with the H-field component, indicating reactive power. Explicitly evaluating the Poynting vector, we find

$$\boldsymbol{P}_{\mathsf{nf}} = \frac{1}{2}\left[E_\theta^{\mathsf{nf}} H_\phi^{nf*}\,\hat{\boldsymbol{r}} - E_r^{\mathsf{nf}} H_\phi^{nf*}\,\hat{\boldsymbol{\theta}}\right] \tag{44}$$

$$= \frac{-j\eta}{2k}\left(\frac{I_0\Delta z}{4\pi}\right)^2 \frac{1}{r^5}\left(\sin^2\theta\,\hat{\boldsymbol{r}} - \cos\theta\sin\theta\,\hat{\boldsymbol{\theta}}\right), \tag{45}$$

which is *imaginary* (power flow/dissipation is always *real*). Since the power is imaginary, is represents stored energy in the electric/magnetic (near) fields of the antenna. At one point in the cycle, all the energy is stored in charge accumulations at the ends of the antenna (like an electric dipole), and the antenna is acting very much like a capacitor with the dipole 'ends' acting as plates giving a fringing capacitance. A quarter cycle later, the two charges have produced a current through the dipole which creates the near induction field we described earlier. A quarter cycle later, the magnetic field has collapsed producing an EMF that charges the 'capacitor' back up with the charge polarity reversed, and so on.

Note that this phase quadrature of E and H reminds us of phase quadrature between V and I in a familiar component: as E *lags* H by 90° $(1/j)$, in what component does V lag I by 90°?



The answer is in a *capacitor*, $\frac{V}{I} = (j\omega C)^{-1}$. So, in the near field, this antenna is acting very much like a capacitor with the dipole 'ends' acting as plates giving a fringing capacitance.