

Small Loop Antenna and Duality Theorem

Having studied the ideal electric dipole we now turn our attention to an interesting antenna closely related to the electric dipole: the electric loop antenna. Such an antenna is shown in Figure 1. The antenna consists of a loop of thin wire wound into a circle of radius a in the xy -plane. A current \mathbf{I} flows along the wire around the loop in the $\hat{\phi}$ -direction. Also shown in the figure is an equivalent magnetic dipole at the origin, which can be ignored for now. It will be discussed later.

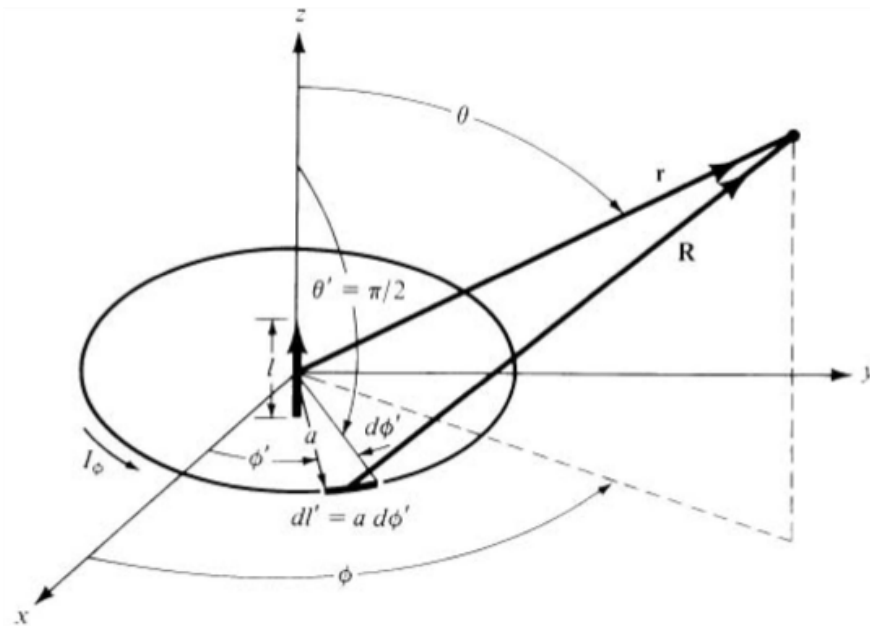


Figure 1: Small loop antenna (source: C. A. Balanis, *Antenna Theory, Analysis, and Design*, 2nd ed., John Wiley and Sons)

The vector magnetic potential in Cartesian coordinates, for a wire conductor (1D) is

$$\mathbf{A}(x, y, z) = \frac{\mu_0}{4\pi} \int_C \mathbf{I}(x', y', z') \frac{e^{-jkR}}{R} dl'. \quad (1)$$

We make the following assumptions:

1. The radius of the loop a is small compared to the wavelength, $a \ll \lambda$;
2. The current along loop flows only in the circumferential direction, that is, \mathbf{I} only has a $\hat{\phi}$ component;
3. The current along the loop can be treated as uniform.

Collectively, these imply that the current on the conducting loop is

$$\mathbf{I} = I_\phi \hat{\phi}', \quad 0 \leq \phi' \leq 2\pi. \quad (2)$$

Then, in cylindrical coordinates,

$$\mathbf{I}(\rho', \phi', z') dl' = I_\phi a d\phi' \hat{\phi}' \quad (3)$$

We can now evaluate \mathbf{A} in cylindrical coordinates. Since there is only a ϕ -component of the current, we only expect a ϕ -component of \mathbf{A} . \mathbf{A} is then given by

$$\begin{aligned} \mathbf{A}(\rho, \phi, z) &= \hat{\phi} \frac{\mu_0}{4\pi} \int_0^{2\pi} I_\phi a d\phi' \hat{\phi}' \cdot \hat{\phi} \frac{e^{-jkR}}{R} \\ &= \hat{\phi} \frac{\mu_0 a I_\phi}{4\pi} \int_0^{2\pi} \cos(\phi - \phi') \frac{e^{-jkR}}{R} d\phi' \end{aligned} \quad (4)$$

where it is important to note that the unit vectors $\hat{\phi}'$ and $\hat{\phi}$ do not point in the same direction because we are in a cylindrical coordinate system, and hence their dot product gives the $\cos()$ function seen here.

The length R can be found as

$$R = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}. \quad (5)$$

In spherical coordinates, we can express these quantities as follows,

$$x = r \sin \theta \cos \phi \quad (6)$$

$$y = r \sin \theta \sin \phi \quad (7)$$

$$z = r \cos \theta \quad (8)$$

$$x' = a \cos \phi' \quad (9)$$

$$y' = a \sin \phi' \quad (10)$$

$$z' = 0. \quad (11)$$

Then,

$$\begin{aligned} R &= \sqrt{r^2 - 2r \sin \theta \cos \phi a \cos \phi' - 2r \sin \theta \sin \phi a \sin \phi' + a^2} \\ &= \sqrt{r^2 + a^2 - 2ar \sin \theta \cos(\phi - \phi')}. \end{aligned} \quad (12)$$

The expression for A_ϕ becomes

$$A_\phi = \frac{\mu_0 a I_\phi}{4\pi} \int_0^{2\pi} \cos(\phi - \phi') \frac{\exp[-jk \sqrt{r^2 + a^2 - 2ar \sin \theta \cos(\phi - \phi')}]}{\sqrt{r^2 + a^2 - 2ar \sin \theta \cos(\phi - \phi')}} d\phi' \quad (13)$$

Since the problem has axial symmetry, A_ϕ does not depend on ϕ ; therefore, we evaluate it at an arbitrary observation angle. Let's take $\phi = 0$ for simplicity,

$$A_\phi = \frac{\mu_0 a I_\phi}{4\pi} \int_0^{2\pi} \cos(\phi') \frac{\exp[-jk \sqrt{r^2 + a^2 - 2ar \sin \theta \cos(\phi')}]}{\sqrt{r^2 + a^2 - 2ar \sin \theta \cos(\phi')}} d\phi' \quad (14)$$

Integration of this expression is challenging. We approximate the integrand by expanding the $\exp(x)/x$ term in a Taylor series about $x = 0$, i.e.,

$$f(x) = f(0) + f'(0)x + \frac{1}{2!}f''(0)x^2 + \dots + \frac{1}{(n-1)!}f^{(n-1)}(0)x^{n-1} + \dots \quad (15)$$

where in this case

$$f(a) = \frac{\exp[-jk\sqrt{r^2 + a^2 - 2ar \sin \theta \cos(\phi')}]}{\sqrt{r^2 + a^2 - 2ar \sin \theta \cos(\phi')}} \quad (16)$$

which is valid if the loop is small ($a \rightarrow 0$). Then, keeping only the first two terms of the series,

$$f(a) \approx f(0) + f'(0)a. \quad (17)$$

Here,

$$f(a=0) = \frac{e^{-jkr}}{r}, \quad (18)$$

and the second term is a bit more involved. Let us let

$$u = \sqrt{r^2 + a^2 - 2ar \sin \theta \cos(\phi')}, \quad (19)$$

where we see that $u(0) = r$ and

$$f(u) = \frac{e^{-jku}}{u}. \quad (20)$$

Then, by the chain rule,

$$f'(a) = \frac{df}{du} \frac{du}{da}. \quad (21)$$

These derivatives are

$$\frac{df}{du} = -\frac{jke^{-jku}}{u} - \frac{e^{-jku}}{u^2}, \quad (22)$$

and

$$\frac{du}{da} = \frac{1}{2} (r^2 + a^2 - 2ar \sin \theta \cos \phi')^{1/2} \cdot (2a - 2r \sin \theta \cos \phi'). \quad (23)$$

Then,

$$f'(a)|_{a=0} = \left. \frac{df}{du} \frac{du}{da} \right|_{a=0} = \left(\frac{-jke^{-jkr}}{r} - \frac{e^{-jkr}}{r^2} \right) (-\sin \theta \cos \phi') \quad (24)$$

Returning to our series approximation for $f(a)$,

$$\begin{aligned} f(a) &\approx \frac{e^{-jkr}}{r} + \left(\frac{jke^{-jkr}}{r} + \frac{e^{-jkr}}{r^2} \right) (\sin \theta \cos \phi')a \\ &\approx e^{-jkr} \left[\frac{1}{r} + a \left(\frac{jk}{r} + \frac{1}{r^2} \right) \sin \theta \cos \phi' \right] \end{aligned} \quad (25)$$

Substituting this result into (14),

$$A_\phi \approx \frac{\mu_0 a I_\phi}{4\pi} \left[\int_0^{2\pi} \frac{e^{-jkr}}{r} \cos \phi' d\phi' + a \left(\frac{jk}{r} + \frac{1}{r^2} \right) \sin \theta \frac{e^{-jkr}}{r} \int_0^{2\pi} \cos^2 \phi' d\phi' \right]. \quad (26)$$

The first integral is zero, while the second one integrates to π , so

$$A_\phi \approx \frac{\mu_0 a^2 I_\phi}{4} e^{-jkr} \left(\frac{jk}{r} + \frac{1}{r^2} \right) \sin \theta. \quad (27)$$

Next, we find the magnetic field from

$$\mathbf{H} = \frac{1}{\mu_0} \nabla \times \mathbf{A} = \frac{1}{\mu_0} \left[\frac{\hat{\mathbf{r}}}{r \sin \theta} \frac{\partial}{\partial \theta} (A_\phi \sin \theta) - \frac{\hat{\boldsymbol{\theta}}}{r} \frac{\partial}{\partial r} (r A_\phi) \right] \quad (28)$$

which gives

$$H_r = \frac{jka^2 I_\phi}{2} \left(\frac{1}{r} - \frac{j}{kr^2} \right) \cos \theta \frac{e^{-jkr}}{r} \quad (29)$$

and

$$H_\theta = -\frac{(ka)^2 I_\phi}{4} \left(1 + \frac{1}{jkr} - \frac{1}{(kr)^2} \right) \frac{e^{-jkr}}{r} \sin \theta. \quad (30)$$

Finally, the electric field is found as

$$\mathbf{E} = \frac{1}{j\omega\epsilon} \nabla \times \mathbf{H} = \frac{\eta(ka)^2 I_\phi}{4} \left(1 + \frac{1}{jkr} \right) \frac{e^{-jkr}}{r} \sin \theta \hat{\boldsymbol{\phi}} \quad (31)$$

Let us compare these fields to those produced by an ideal dipole,

$$\mathbf{H} = \frac{I\Delta z}{4\pi} jk \left(1 + \frac{1}{jkr} \right) \frac{e^{-jkr}}{r} \sin \theta \hat{\boldsymbol{\phi}}. \quad (32)$$

$$\mathbf{E} = \frac{I\Delta z}{2\pi} \eta \left(\frac{1}{r} - \frac{j}{kr^2} \right) \frac{e^{-jkr}}{r} \cos \theta \hat{\mathbf{r}} + \frac{I\Delta z j\omega\mu}{4\pi} \left(1 + \frac{1}{jkr} - \frac{1}{(kr)^2} \right) \frac{e^{-jkr}}{r} \sin \theta \hat{\boldsymbol{\theta}}. \quad (33)$$

Notice the extreme similarity between the loop magnetic field (29), (30) and the dipole electric field (33)! The same is also true of the loop electric field (31) and the dipole magnetic field (32). In fact, if we take the dipole fields created from an ideal dipole with current strength I_m and length ℓ , and make the following substitutions,

$$\mathbf{E} \Leftrightarrow \mathbf{H} \quad (34)$$

$$\mathbf{H} \Leftrightarrow -\mathbf{E} \quad (35)$$

$$\epsilon \Leftrightarrow \mu \quad (36)$$

$$\mu \Leftrightarrow \epsilon \quad (37)$$

$$\eta \Leftrightarrow \frac{1}{\eta} \quad (38)$$

$$\frac{1}{\eta} \Leftrightarrow \eta \quad (39)$$

then the following fields result,

$$\mathbf{E} = \frac{-jkI_m\ell}{4\pi r} \left(1 + \frac{1}{jkr}\right) \frac{e^{-jkr}}{r} \sin\theta \hat{\phi} \quad (40)$$

$$H_r = \frac{I_m\ell}{\eta 2\pi} \left(\frac{1}{r} - \frac{j}{kr^2}\right) \cos\theta \frac{e^{-jkr}}{r} \quad (41)$$

and

$$H_\theta = \frac{jkI_m\ell}{\eta 4\pi} \left(1 + \frac{1}{jkr} - \frac{1}{(kr)^2}\right) \frac{e^{-jkr}}{r} \sin\theta. \quad (42)$$

If we set

$$I_m\ell = j\eta k\pi a^2 I_\phi = j\omega\mu S I_\phi, \quad (43)$$

where $S = \pi a^2$ is the area of the loop, then these fields are *exactly* equal to the loop fields (29), (30), and (31)!

This is not a coincidence. The relations (34) - (39) form what is known as the Duality Theory and it is a very useful theorem associated with Maxwell's equations. Consider the source-free curl equations, which are

$$\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H} \quad (44)$$

$$\nabla \times \mathbf{H} = j\omega\epsilon\mathbf{E} \quad (45)$$

Notice the symmetry in these equations, similar to what we observed when comparing the E and H fields of the loop and dipole. Using the Duality Theorem, we can actually synthesize the second curl equation from the first, without even needing to know the second equation. Or vice versa.

Now introduce sources to the equations. Ampere's Law becomes

$$\nabla \times \mathbf{H} = j\omega\epsilon\mathbf{E} + \mathbf{J} \quad (46)$$

If we apply the Duality Theory to this equation, the dual would be

$$\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H} + \mathbf{M} \quad (47)$$

where \mathbf{M} is the *dual* of the electric current source in the first equation,

$$\mathbf{J} \Leftrightarrow \mathbf{M}. \quad (48)$$

Such a source is called a *magnetic current source*. While magnetic current sources do not exist in nature, they can be useful mathematical tools because introducing \mathbf{M} in Faraday's Law makes it a perfect dual of Ampere's Law.

How does this apply to the loop antenna we have derived? We see that by applying duality to the electric dipole problem, we can arrive at the fields of a loop antenna. Equivalently, this is the same as finding the fields from a so-called *magnetic dipole* whose current is I_m and length is ℓ , as we did above in (40)-(42). This is a lot simpler than the long procedure proposed initially to determine the fields from the loop directly from the electric current. Knowing the relationship

between the magnetic dipole strength I_m and the loop antenna parameters given by (43), the fields can be related to the geometry of the loop.

We can observe that the loop antenna / magnetic dipole antenna creates the same fields as an electric dipole but with the roles of the E and H fields reversed. So this time, it is \mathbf{E} that has a $\hat{\phi}$ -oriented component, instead of \mathbf{H} . Similarly, this time \mathbf{H} has both \hat{r} - and $\hat{\theta}$ -components, instead of \mathbf{E} . Essentially, the polarization of the antenna is orthogonal to that of an electric dipole. Hence the co-location of an electric and magnetic dipole can be used to create two orthogonally-polarized fields, which can be useful for creating CP or EP waves, for example.